A multidimensional superposition principle: classical solitons I

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10 January 1999

Abstract

A new concept, the multidimensional superposition principle, is proposed for differential equations. In its frameworks classical soliton interactions can easily be explained and described, and other wave structures ('soliton structures') with more complex and intriguing properties are predicted.

The case of nonlinear PDEs associated with the Riccati equations via singular manifold expansions is considered in details. The possibility of the existence for the KdV-type soliton/kink is shown for them. Moreover, the cases with unelastic interactions are also demonstrated for the specific conditions.

The above approach can be applied to naturally construct solutions of ordinary and partial differential equations as well. As such an example, some rational, soliton-rational, and two-soliton solutions were obtained for the KdV equation.

1 Introduction

Owing to the extensive studies of integrable nonlinear models performed during the previous decades, especially of such famous equations as the KdV, MKdV and SG ones, nowadays there exists the comprehensive understanding of soliton behaviour in these systems. In many respects we owe this

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situation to the Inverse Scattering Transform. Its discovery in [1] has had a considerable impact on all following progress in nonlinear science. The IST approach yields the general solution to the Cauchy problem with reasonable boundary and initial conditions. But what is more, solitons are embedded in the method in a natural manner and emerge from the spectrum of discrete eigenvalues of related scattering problems, although some of them may be connected with non-solitonic equations as well [2]. While, despite its validity, rigour and power, the theory is very complicated frequently. A procedure for finding a suitable spectral problem and generalization in order to handle $(n+1)$ cases are also open questions up to now. Because of these circumstances a number of cunning techniques have been invented, e.g. the Hirota's technique [3], trace method [4], various linearizations [5, 6], the Weiss-Tabor-Carnevale approach [7], and so on. Most of them could be classified as *direct approaches.* On frequent occasion they allow one to recover the same results, but in the more laconic form and in a more easy way, whereas reasons for their success are in essence not clear and cannot be explained in the frameworks of themselves usually. Nevertheless, it turns out that they may be effective even when the IST does not work, because the latter is rather restrictive. For instance, some special solutions can be obtained (see, say, [8]). This does not, however, remove the main problem, and their solitonic properties have to be investigated by means of computer simulation further. So, we face the dilemma, because such methods themselves are not self-consistent from this point of view.

Thus, the primary motivation of this work is to propose *a direct concept* in some sense alternative to the IST in which the notion of solitons would be incorporated by a definition and be, perhaps, more tractable than usually. In addition, deeper understanding of a solitonic interaction could give novel aspects of nonlinear dynamics as a whole. Such an attempt will be made in the present paper.

2 Theoretical background

Consider some nonlinear partial differential equation, for the sake of definiteness in $(1+1)$ dimensions, not depending explicitly on the independent variables

$$E\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial t}; u(x,t)\right) = 0 \tag{1}$$

and construct its $(2+2)$-dimensional *adjoint equation* by the formal replacement

$$E\left(\frac{\partial}{\partial x_1}+\frac{\partial}{\partial x_2},\frac{\partial}{\partial t_1}+\frac{\partial}{\partial t_2};u(x_1,x_2,t_1,t_2)\right)=0 \tag{2}$$

keeping in mind that

$$u(x,t)=u(x_1,x_2,t_1,t_2)|_{x_1=x_2=x,\,t_1=t_2=t} \tag{3}$$

Not obvious as yet how all this can bear on the solitonic concept. Notice, however, that it is possible to impose *an additional restriction* on evolution with respect to the variables x_1 and t_1 without the loss in generality for (1) and (3). To be precise, assume there exists some operator M such that

$$\begin{aligned}
&u(x_1,x_2,t_1,t_2)=M[u_{\mathrm{s}}(x_1,x_2,t_1,t_2)]\\
&E\left(\frac{\partial}{\partial x_1},\frac{\partial}{\partial t_1};u_{\mathrm{s}}(x_1,x_2,t_1,t_2)\right)=0\\
&M[u_{\mathrm{s}}(x_1,t_1)]=u_{\mathrm{s}}(x_1,t_1)
\end{aligned} \tag{4}$$

with (2), (4) are compatible. By this means the variables x_1, t_1 and x_2, t_2 appear *to be separated*, and there can be common only constants in the related govern equations. As a result, the functions $u(x_1,x_2,t_1,t_2)$ and $u(x,t)$, respectively, are determined by the local evolutions in the (x_1,t_1)- and (x_2,t_2)-spaces. While in the view of an observer in our real world there exists only *a projection of events* in that $(2+2)$-dimensional world, and the section $x_1=x_2$, $t_1=t_2$ is only accessible for him. Such the paradigm will be called *a multidimensional principle* in contrast to a superposition principle for ordinary differential equations, which everywhere occurs in the literature (see, e.g, [9] and the references therein) and where a conventional algebraic superposition is implied. It may seen that the term superposition is not quite successful in this context. This, however, becomes transparent in a moment taking into account that we will mainly be interested such solutions that in some areas on the section $x_1=x_2$, $t_1=t_2$

$$\begin{aligned}
&u(x_1,x_2,t_1,t_2)\longrightarrow u_{\mathrm{lim}_1}(x_1,t_1)\\
&u(x_1,x_2,t_1,t_2)\longrightarrow u_{\mathrm{lim}_2}(x_2,t_2)
\end{aligned} \tag{5}$$

In these cases the above separated behaviour will explicitly be reflected in properties of the associated solutions to the original equation (1).

The proposed scheme is general enough. First of all, there are no reasons to assert that a solution (or solutions) is physically decomposed in the (x,t)-world as above (5). For such an inference, it is necessary to know details on evolution in the (x_1, t_1)- and (x_2, t_2)-spaces as well as the form and type of the related solutions. This is why we will refer to the above-mentioned principle as *the necessary condition* for the soliton existence. Unfortunately, for the general case another part is an open problem at the present time and is outside the scope of the article. This is, however, an advantage also, because, on the other hand, the multidimensional superposition alternatively enables one to derive solution unrecognizable as solitons. It is naturally to regard them as bound states. The statement that we make here that *solitonic interactions are actually of the multidimensional essence and can arise if there exists a suitable* $(2+2)$*-dimensional space in which the related variables are splitted.*

Further, it may turn out that there are additional restrictions on the type of solutions in (1) and (4). Several situations are possible therewith. First, solutions in (4) may be limited by some class or even the only solution. (It should note here that mainly the latter corresponds to classical solitons.) Second, both solutions to (1) and to (4) may be confined. This is very interesting case, which has been elucidated never in the literature. To turn up this experimentally is very difficult due to its specificity. In this case, 'solitons' could interact with some class of perturbation elastically, but unelastically with others. Third, any restrictions may absent. It is judicious to speak about *superposition properties* for such cases. Linear equations can trivially be shown to be among that class, and any solution can be added to an arbitrary initial one via the simplest operator M. This evidently arrives at another view to solitonic equations: there are equations in which *just some fixed solutions (solitons) can be added to arbitrary ones* in opportunity to, e.g., linear PDEs.

Finally, the following questions arise: what a solution may be in (4), and how it may be modulated by the variables x_2, t_2. Indeed, if solitonic interaction takes place nevertheless, in other words, there exist two areas on the section $x_1 = x_2$, $t_1 = t_2$ such that

$$u(x_1, x_2, t_1, t_2) \longrightarrow u^{\text{before}}_{\text{soliton}}(x_1, t_1)$$
$$u(x_1, x_2, t_1, t_2) \longrightarrow u^{\text{after}}_{\text{soliton}}(x_1, t_1)$$

(and the related areas for the perturbation) how are these states connected to one another? Obviously, the only case when possible to say something definite is a traveling wave solution with the phase modulation. In other cases, the result may be determined both by an initial soliton state and by the type of a perturbation. An interaction *switches a soliton from one state to another with the different sets of the parameters.* For example, one (and, maybe, the only one in the literature to the author's knowledge) of such cases is adduced in [10, 11].

It is time to mentioned in conclusion that the introduction of the new variables is not unexpected. Really, the arbitrariness with wavenumbers in N-soliton solutions is a direct indication to determine various scales and coordinates. Moreover, the above concept, direct perturbation approach [12], and multiscale method are deeply related in substance. It is their logical prolongation. In those cases we would have

$$u(x,X,t,T) = u_0(x,X,t,T) + \varepsilon u_1(x,X,t,T) + \cdots, \qquad |\varepsilon| \ll 1$$
$$u_i = M_i[u_0]; \qquad X = \varepsilon x,\ T = \varepsilon t$$
$$\frac{\partial}{\partial x} \longrightarrow \frac{\partial}{\partial x} + \varepsilon\frac{\partial}{\partial X}, \qquad \frac{\partial}{\partial t} \longrightarrow \frac{\partial}{\partial t} + \varepsilon\frac{\partial}{\partial T}$$

The 'soliton' variables x_1, t_1 and 'perturbation' ones x_2, t_2 correspond to the fast and slow ones, respectively; and the operator M associates with the specific type of perturbation expansions. Its analog is called a canonical map or canonical transformation in the theory of perturbations [13, 14].

3 The multidimensional superposition and the singular manifold method

As seen from the foregoing, there are two parts in the above formalism. First of them is the related operator M. We do have an obvious problem in that how to construct it. A similar problem arises for deformations of nonlinear PDEs [15] and on consideration of perturbations of integrable equations [14]. By now, unhappily, it is still open for contemporary mathematics, and various ansatzes are mainly employed with this in mind. Nonetheless, with the advance of the singular manifold method, some of such mappings can be obtained, and a technique to derive these transformations was proposed on the base of truncated singular expansions in [16].

In this work our attention will be concentrated namely on such cases. As to the singular manifold approach itself, the technique will just be outline in general, with less of details. Interested readers are referred to [7, 17, 18] for other niceties. Further, we will follow its so-called invariant version due to [18].

First, there is no difficulty in introducing the $(2+2)$-dimensional analogous of the deterministic equations for the function χ associated with a singular manifold function. So, for its inversion

$$v = \chi^{-1} = \frac{f_{x_1}}{f} - \frac{1}{2}\left(\frac{f_{x_1x_1}}{f_{x_1}}\right), \qquad f = f(x_1, x_2, t_1, t_2)$$

more suitable for truncated series, one has the following Riccati equations

$$v_{x_1} = -v^2 - s/2 \tag{6}$$
$$v_{t_1} = cv^2 - c_{x_1}v + (cs + s_{x_1x_1})/2 \tag{7}$$
$$v_{x_2} = av^2 - a_{x_1}v + (as + s_{x_1x_1})/2 \tag{8}$$
$$v_{t_2} = bv^2 - b_{x_1}v + (bs + s_{x_1x_1})/2 \tag{9}$$
$$s = s(x_1, x_2, t_1, t_2),\ c = c(x_1, x_2, t_1, t_2),\ a = a(x_1, x_2, t_1, t_2),$$
$$b = b(x_1, x_2, t_1, t_2)$$

provide that the functions s, c, a and b satisfy the compatibility conditions

$$s_{t_1} + c_{x_1x_1x_1} + 2sc_{x_1} + cs_{x_1} = 0 \tag{10}$$
$$s_{t_2} + b_{x_1x_1x_1} + 2sb_{x_1} + bs_{x_1} = 0 \tag{11}$$
$$s_{x_2} + a_{x_1x_1x_1} + 2sa_{x_1} + as_{x_1} = 0 \tag{12}$$
$$a_{t_1} + ca_{x_1} - ac_{x_1} - c_{x_2} = 0 \tag{13}$$
$$a_{t_2} + ba_{x_1} - ab_{x_1} - b_{x_2} = 0 \tag{14}$$
$$c_{t_2} + bc_{x_1} - cb_{x_1} - b_{t_1} = 0 \tag{15}$$

Of course, we could directly investigate the adjoint (2) to an equation of interest. However, since there obviously takes place only differentiation with respect to $\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} = \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2} = \frac{\partial}{\partial t}$ in the early stage, it is convenient to perform initial calculations needed for the singular manifold method itself in our old coordinates and use the relations for the $(1+1)$-dimensional case in so doing

$$V_x = -V^2 - S/2 \tag{16}$$
$$V_t = CV^2 - C_xV + \frac{1}{2}(CS + S_{xx}) \tag{17}$$
$$S = S(x,t),\ C = C(x,t)$$

with the constrain

$$S_t + C_{xxx} + 2SC_x + CS_x = 0 \tag{18}$$

After that it is already necessary to introduce the new set of coordinates and investigate the system (16), (17) in turn taking into account the auxiliary equation (18) and the singular manifold one obtained in the previous step. So, let us return to the singular manifold method once again and consider already (16), (17) from this view point.

Expand V according to the technique as

$$V = w_1(x_1, x_2, t_1, t_2)v + w_0(x_1, x_2, t_1, t_2)$$

with w_1, w_0 being than determined and replace it in (16) and (17). Next, setting each of its coefficients at the powers of v to zero, one has from (16)

$$V = (1-a)v + \frac{a_{x_1}}{2} + \frac{a_{x_1} + a_{x_2}}{2(1-a)} \tag{19}$$

$$S = s(a-1)^2 + (a-2)a_{x_1x_1} - a_{x_1x_2} - \frac{(a_{x_1})^2}{2} - \frac{3}{2}\left(\frac{a_{x_1} + a_{x_2}}{a-1}\right)^2 \tag{20}$$
$$+\frac{a_{x_1x_1} + 2a_{x_1x_2} + a_{x_2x_2} + a_{x_1}(a_{x_1} + a_{x_2})}{a-1}$$

and from the leading term in (17)

$$C = \frac{b+c}{1-a} \tag{21}$$

Whereupon the rest of (17) vanishes.

Although a comprehensive investigation demands the knowledge of the singular manifold equation related with the equation under consideration, there is, however, a common property associated with the system (16), (17). Moreover, as will be demonstrated below, this property may play a key role for solitonic equations.

Genuinely, there always exists the trivial solution for s, c and a, b from the overdetermined system (10)–(15)

$$s, c = \mathrm{Const}$$
$$a_{t_2} - b_{x_2} = 0$$
$$a_{x_1} = a_{t_1} = b_{x_1} = b_{t_1} = 0$$

First of all consider the choice $s \neq 0$, corresponding to a kink solution for v in the (x_1, t_1)-space ((6), (7)). Denoting the kink wavenumber, frequency, and phase by k, ω and φ_1, one has

$$v = \frac{k}{2}\left[\frac{\exp\left(k\theta_1 + \theta(x_2, t_2)\right) - 1}{\exp\left(k\theta_1 + \theta(x_2, t_2)\right) + 1}\right], \quad \theta_1 = x_1 + \frac{\omega}{k}t_1 + \varphi_1 \tag{22}$$
$$\lim_{\theta_1 \to \pm\infty} v = \pm\frac{|k|}{2}$$

with $s = -k^2/2$, $c = -\omega/k$. From which, taking into account (8), (9) and

(19), it is easy to deduce for limited values of x_2, t_2 the following relations

$$\frac{\partial\theta}{\partial x_2} = -ka \tag{23}$$

$$\frac{\partial\theta}{\partial t_2} = -kb \tag{24}$$

$$V_{\pm\infty}(x_2,t_2) = \lim_{\theta_1\to\pm\infty} V(\theta_1,x_2,t_2) = \pm\frac{|k|(1-a)}{2} + \frac{a_{x_2}}{2(1-a)} \tag{25}$$

$$a = \frac{V_{-\infty} - V_{+\infty}}{|k|} + 1 \tag{26}$$

Therewith underline that $V_{-\infty}$, $V_{+\infty}$ and V are associated with the same functions S and C.

As a consequence, since x_1, t_1 and x_2, t_2 are independent of one another, and φ_1 is arbitrary, (25) and (26) enable us to construct some new solution in the (x_1, x_2, t_1, t_2)-space adding together the above kink and two particular solutions $V_{-\infty}(x_2,t_2)$, $V_{+\infty}(x_2,t_2)$ of the original system (16), (17). In doing so (23) and (24), in view of the relations (21), (26), determine the phase function θ for v in (22). While in the (x,t)-space the situation is not so unambiguous. The relations can as before be applied to derive the general solution to (16), (17), and we have the superposition law for the Riccati equations which may, in contradiction to the previously known ones [9, 19], explicitly introduce (or exhibit) the new functional dependence in the general solution, but the latter will be true not in all cases. This is the crucial fact associated with non-uniqueness of transitions from more dimensional spaces. Indeed, in going to the (x,t)-space, it is necessary to remember that θ in (22) may contain addends that cancel θ_1 completely or partially. The question is solved simply however. The functions c and s directly appear in (23), (24) through the expressions (26), (21) and can (and must) be chosen so that such constituents would be excluded from θ whenever possible.

It is time to make the remark here. It should be pointed out that all our reasonings hold true for any values of the above s and c. So far additional restrictions have been made neither to them nor to the function θ, and all this is valid even when there absents physical separation in the (x,t)-space. On the other hand, such limitations are needed in order that the end-states $V_{-\infty}$ and $V_{+\infty}$ correspond to real asymptotic expressions for a perturbation before and after an interaction, and v does the real kink solution.

First and foremost, $s, c = \mathrm{Const}$ must obviously satisfy the singular manifold equation of the same form as for S and C, because (1), (4) are of the same type as well. This is always possible, e.g., if the above-mentioned equation does not include the independent variables explicitly and is purely differential. Further, for physical separation of the kink from a localized perturbation the following conditions to the asymptotes of θ are necessary

$$\lim_{x_2 \to -\infty} \frac{\partial \theta^{\text{before}}}{\partial x_2} = 0, \quad \lim_{x_2 \to +\infty} \frac{\partial \theta^{\text{after}}}{\partial x_2} = 0$$

(or

$$\lim_{x_2 \to +\infty} \frac{\partial \theta^{\text{before}}}{\partial x_2} = 0, \quad \lim_{x_2 \to -\infty} \frac{\partial \theta^{\text{after}}}{\partial x_2} = 0$$

according to the direction of kink moving) as well as, of course, the linkage conditions

$$\lim_{x_2 \to \infty} V_{-\infty} = -\frac{|k|}{2}, \quad \lim_{x_2 \to \infty} V_{+\infty} = \frac{|k|}{2} \tag{27}$$

In doing so during the interaction there will take place the following linkage between the perturbation end-states

$$V_{+\infty} = S_{-\infty}^{+\infty} T_{t_{\text{interaction}}} V_{-\infty}$$

The essence of the switch operator $S_{-\infty}^{+\infty}$ is obvious from (25). Also, (25) and (23) determine θ and its evolution taking into account that the evolution of $V_{-\infty}$ (or $V_{+\infty}$) is governed by the original nonlinear equation under consideration. (It is easy to verify that (24) is satisfied identically in this case.) In other words, the evolution of a localized perturbation during an interaction is fully determined by *its evolution without the kink* (the operator $T_{t_{\text{interaction}}}$)!

As follows from the aforesaid, it is necessary to know explicitly the initial nonlinear equation or the singular manifold one to describe the result of a kink-perturbation interaction in details. (Its linearized version seems to be enough to analyse the evolution of asymptotes.) However, if changing in the perturbation form can be neglected or, differently,

$$t_{\text{characteristic}} \gg t_{\text{interaction}}$$

for it, the answer is uniquely determined, and in view of one of the relations (27) corresponding to the state before the collision there are only two cases for the θ asymptotes from the analysis of (25): first

$$\lim_{x_2 \to \infty} \frac{\partial \theta}{\partial x_2} = 0$$

and second

$$\lim_{x_2 \to +\infty} \frac{\partial \theta}{\partial x_2} = 0$$

$$\frac{\partial \theta_2}{\partial x_2} \sim -k + \text{Const}\, e^{|k| x_2}, \quad x_2 \longrightarrow -\infty$$

provided that, however,

$$V_{-\infty} \sim \mathrm{Const}\,\mathrm{sign}(k) e^{|k| x_2}, \quad x_2 \longrightarrow -\infty$$

(Here, for definiteness, the kink has been positioned to the right from the perturbation before the collision and moved to the left). And as this takes place, another of the relations (27) is satisfied identically as well.

The first case satisfies all the conditions involved and, as a result, corresponds to an elastic interaction with asymptotic separation. The second case is specific. The matter is that asymptotic separation after a collision is absent here (the related $\frac{\partial \theta}{\partial x_2}$ asymptote), but in this case according to the dispersion relation of the appropriate linearized equation the back perturbation tail must propagate to the same direction and with the same speed as the kink. Hence in reality either the interaction will last infinitely or the kink will never leave behind the perturbation.

Both of the above scenarios are depicted in figures 1 and 2, for simplicity in assumption that the perturbation are immovable. Figure 1 with

$$k\theta_1 = x_1 + t_1 + 4$$
$$\theta = \frac{-8}{e^{x_2} + 1}$$

demonstrates the classical elastic collision, whereas figure 2 where

$$k\theta_1 = x_1 + t_1$$
$$\theta = \ln\left(e^{x_2} + 1\right) - x_2 - \frac{7}{e^{x_2} + 1}$$

could be described as the 'capture' of the kink, because the end stage is absent. Although in most of known nonlinear equations such a situation is impossible, however from the theoretical viewpoint it may take place in models without or with complicated dispersion relations.

The analogous analysis can be performed and for $s = 0$. In this case we have for v the pole solution

$$v = \frac{1}{\theta_1 + \theta(x_2, t_2)}, \quad \theta_1 = x_1 + \omega t_1 + \varphi_1 \tag{28}$$
$$\lim_{\theta_1 \to \infty} v = 0$$

with $c = -\omega$ and the relations

$$a = -\frac{\partial \theta}{\partial x_2}$$
$$b = -\frac{\partial \theta}{\partial t_2}$$
$$V_\infty(x_2, t_2) = \frac{a_{x_2}}{2(1-a)}$$

which directly lead to the expression for $\frac{\partial \theta}{\partial x_2}$

$$\frac{\partial \theta}{\partial x_2} = \left(1 + \left.\frac{\partial \theta}{\partial x_2}\right|_{x_2=+\infty}\right) \exp\left(2\int_{x_2}^{+\infty} V_\infty dx_2\right) - 1$$

From whence it immediately follows that an elastic interaction, or

$$\lim_{x_2 \to \infty} \frac{\partial \theta}{\partial x_2} = 0$$

is possible when

$$\int_{-\infty}^{+\infty} V_\infty dx_2 = 0$$

(Note that the integral is a conservation quantity for many nonlinear equations.) Figure 3 where

$$\theta_1 = x_1 + t_1 - 5$$
$$\theta = \frac{e^{x_2 - t_2}}{2\left[e^{2(x_2 - t_2)} + 1\right]}$$

demonstrates such a collision, while figure 4 with

$$\theta_1 = x_1 + t_1$$
$$\theta = -\ln\left(e^{-4x_2} + 1\right)$$

corresponds to the opposite situation. (The shelf appears between the pole and perturbation after the interaction.)

In closing one degenerate case should singly be mentioned for completeness. Up to now, we have been omitting (see (6)) that v does not need to depend on x_1. And although obviously this specific case is of no interest from the viewpoint of solitonic interactions, but it, for instance, corresponds to rational solutions. All the aforesaid pertains equally to it with minimal modifications, and, already leaving aside the details, we only gather the related relations together here

$$
\begin{aligned}
& S = ca^2 + aa_{t_1t_1} - a_{t_1t_2} - \frac{a_{t_1}^2}{2} + \frac{a_{t_1}a_{x_2} + a_{x_2x_2}}{a} - \frac{3}{2}\left(\frac{a_{x_2}}{a}\right)^2 \\
& C = \frac{1-b}{a} \\
& V = -av - \frac{aa_{t_1} - a_{x_2}}{2a} \\
& c = \mathrm{Const} \\
& a_{t_2} - b_{x_2} = 0 \\
& a_{t_1} = b_{t_1} = 0
\end{aligned} \tag{29}
$$

These formulas are imposed in the same manner as before taking into account the original system for v

$$
\begin{aligned}
v_{t_1} &= -v^2 - c/2 \\
v_{x_2} &= av^2 - a_{t_1}v + (ac + c_{t_1t_1})/2 \\
v_{t_2} &= bv^2 - b_{t_1}v + (ac + c_{t_1t_1})/2 \\
& c = c(t_1),\ a = a(x_2, t_1, t_2),\ b = b(x_2, t_1, t_2)
\end{aligned}
$$

Although all the above is simple enough, nevertheless it can naturally result in the existence of soliton interactions for nonlinear partial differential equations as well as in various types of solutions as demonstrated below.

4 One simple but instructive example

In this section the simple example will be considered. To be precise, we will consider the well-known KdV equation

$$
u_t + 6uu_x + u_{xxx} = 0 \tag{30}
$$

and some of its special solutions, including two-soliton and rational ones. And although in this case the present technique only gives back the known results, however such reproducing is particular useful and important for illustrating the meaning of the multidimensional principle, and the KdV would be the best choice for it, namely because it is well studied.

As known [17, 20], the truncated singular expansion for the KdV equation is of the form

$$
u - -2V^2 + \frac{C - 4S}{6}
$$

or taking account the related singular manifold equation

$$C - S - \lambda = 0 \tag{31}$$

the following form

$$u = V_x - V^2 + \frac{\lambda}{6} \tag{32}$$

As is seen, the latter constitutes the x-part of the Riccati system (16), (17). Another t-part is trivially derived from the equation (30) itself. It can be shown that the link

$$u \overset{\text{Riccati}}{\longleftrightarrow} V \overset{\text{Riccati}}{\longleftrightarrow} S$$

with the above formalism and relations in hand fully explain the existence of the rational, N-soliton and soliton-rational hierarchies for the KdV. In so doing, although any straightforward way to generate the required particular solutions $V_{\pm\infty}$ or V_∞ is absent, in most cases this does not represent a problem. Since namely solutions without the new functional dependence are needed for it, an ansatz according to the S or u type is enough practically.

As usually we start from some trivial solution $u_0 = \text{Const}$. In accordance with its value and λ (λ_0) three branches are possible and should be divided. The first one brings about rational solutions. Choosing $u_0 = \lambda_0/6$ and $V_\infty = 0$ respectively, one immediately has the simplest pole (28)

$$V_0 = -\frac{1}{\theta_0}, \quad \theta_0 = x - \lambda_0 t + \varphi_0$$

In its turn this solution can be considered as the limited one V_∞ for

$$S_1 = -\frac{4}{\theta_0^2}$$

and the t_2-pole can already be added via (29)

$$V_1 = \frac{2\theta_0^3 - \theta_1}{\theta_0(\theta_1^2 + \theta_0)}, \quad \theta_1 = 12t + \varphi_1$$

such that for the KdV equation the following rational solution with two phases is obtained

$$u_1 = \frac{6\theta_0(2\theta_1 - \theta_0^3)}{(\theta_1 + \theta_0^3)^2} + \frac{\lambda_0}{6}$$

The scheme 'hither and thither' can be continued, of course, to generate new and new solutions.

Next, beginning from $u_0 = \lambda_0/6 - k_0^2/4$ and $V_{\pm\infty} = \pm|k|/2$, one results in the kink solution analogous to (22)

$$V_0 = -\frac{|k_0|}{2}\left[\frac{\exp(k_0\theta_0) - 1}{\exp(k_0\theta_0) + 1}\right], \quad \theta_0 = x + \left(\frac{k_0^2}{2} - \lambda_0\right)t + \varphi_0$$

which can also be considered as the degenerated one V_∞ with the related S_1, and the pole (28) can be added

$$V_1 = \frac{k_0}{2}\left[\frac{e^{3k_0\theta_0} + e^{2k_0\theta_0}(7 - 2k_0\theta_1) + e^{k_0\theta_0}(7 + 2k_0\theta_1) + 1}{e^{3k_0\theta_0} + e^{2k_0\theta_0}(2k_0\theta_1 + 1) + e^{k_0\theta_0}(2k_0\theta_1 - 1) - 1}\right],$$
$$\theta_1 = x - \left(\frac{3k_0^2}{2} + \lambda_0\right)t + \varphi_1$$

leading to the rational-soliton solution for the KdV

$$u_1 = \frac{2\lambda_0 - 3k_0^2}{12} + 4k_0^2\frac{e^{k_0\theta_0}\left(e^{k_0\theta_0} + 1\right)\left[e^{k_0\theta_0}(2k_0\theta_1 - 2) - (2k_0\theta_1 + 2)\right]}{\left(e^{2k_0\theta_0} + 2k_0\theta_1 e^{k_0\theta_0} - 1\right)^2}$$

These steps could be continued again, but we will pass to the N-soliton hierarchy. In reality more complicated solutions, N-soliton-rational ones, appear in general. Pure N-soliton solutions are lightly evolved in the following way. Taking $u_0 = \lambda_0/6 - k_0^2/4$, we will be confined to the special solution $V_0 = k_0/2$ with $S_1 = -k_0^2/2$. The latter gives the kink for V_1 as before and further the one-soliton solution

$$u_1 = \frac{2\lambda_0 - 3k_0^2}{12} + 2k_0^2\frac{e^{k_0\theta_0}}{(e^{k_0\theta_0} + 1)^2}, \quad \theta_0 = x + \left(\frac{k_0^2}{2} - \lambda_0\right)t + \varphi_0$$

This u_1 and (30), (32) with $\lambda_1 = 3(k_1^2 - k_0^2)/2 + \lambda_0$ $(k_1 \neq k_0)$ gives the limited solutions

$$V_{\pm\infty} = -\frac{k_1}{2} - \frac{2k_0^2 e^{k_0\theta_0}}{(e^{k_0\theta_0} + 1)\left[(k_0 - k_1)e^{k_0\theta_0} - (k_0 + k_1)\right]}$$

They could give rise to the soliton-pole solution for V as previously, but we again restrict ourselves to one of these particular solutions, its pair of the analogous form, the related S_2 for the next step. Using (25) and (26), finally the pure two-soliton solution is obtained. Setting $\lambda_0 = 3k_0^2/2$ for simplicity, the result can be rolled up into the well-known expression

$$u_2 = 2\frac{\partial^2}{\partial x^2}\ln\left[1 + e^{k_0\theta_0} + e^{k_1\theta_1} + \left(\frac{k_0 - k_1}{k_0 + k_1}\right)^2 e^{k_0\theta_0}e^{k_1\theta_1}\right],$$
$$\theta_0 = x - k_0^2 t + \varphi_0, \; \theta_1 = x - k_1^2 t + \varphi_1$$

The N-soliton solutions could be derived in the same way further. Also, choosing $\lambda_1 = \lambda_0$ or applying not only the above particular solutions, N-soliton-rational solutions could be generated as has been indicated. However our aim here is not to construct new and new solutions, but to show that all they are of some unified structure caused by the multidimensional superposition and can easily be reproduced within its frameworks. In due time various cunning techniques were popular for it (the limits of N-soliton solutions, Hirota's ansatz, invariance $f_x \to 1/f_x$ of the KdV singular manifold equation, and so on), whereas even more complicated solutions like soliton-pole and rational ones with several phases naturally appear here.

In conclusion let us come back to the sufficient condition for the existence of solitons and investigate the system (10)–(15) with the KdV singular manifold equations for S, C and s, c from that point of view. We should, however, restrict ourselves to the case $s = s(x_1, t_1)$ and $c = c(x_1, t_1)$ here. In other words, modulation of only one parameter associated with the Riccati equations themselves will be considered. The fact is that, unfortunately, even for simple singular manifolds the problem involved lies on the verge of the modern theory of involution systems and computer facilities, whereas the above assumption allows one to find permissible s and c via a differential Gröbner basis [21, 22, 23] with relative easy.

First of all, supposing $s_{x_1} \neq 0$, and with regard to (20), (21) and (31) to S, C and s, c the overdetermined set of equations (10)–(15) reduces to the following one

$$s_{t_1} = -(s_{x_1x_1x_1} + \lambda s_{x_1} + 3ss_{x_1})$$

$$a_{t_1} = as_{x_1} - (\lambda + s)a_{x_1} \tag{33}$$

$$\begin{aligned} a_{t_2} = \frac{1}{2(a-1)^2}\Big\{&-2(a-1)^2(a_{x_1x_1x_1} + 3a_{x_1x_1x_2} + 3a_{x_1x_2x_2} + a_{x_2x_2x_2}) \\ &+6(a-1)(a_{x_1} + a_{x_2})(a_{x_1x_1} + 2a_{x_1x_2} + a_{x_2x_2}) - 6(a-1)^3 a_{x_2x_2}(a_{x_1} + a_{x_2}) \\ &+(a-1)^2(a_{x_1} + a_{x_2})[3a_{x_1}^2 - 2\lambda - 6s(a-1)^2] - 3(a_{x_1} + a_{x_2})^3\Big\} \end{aligned} \tag{34}$$

$$\begin{aligned} &s_{x_1}(a_{x_1} + a_{x_2})^2 - 2s_{x_1x_1}(a-1)(a_{x_1} + a_{x_2}) \\ &\qquad + s_{x_1}(a-1)^2\left[2a_{x_1x_1}(a-1) - a_{x_1}^2 + 2ss_{x_1}a(a-2)\right] = 0 \\ &a_{x_1x_1x_1} + 2sa_{x_1} + as_{x_1} = 0 \\ &(a-1)(a_{x_1x_1} + a_{x_1x_2}) - a_{x_1}(a_{x_1} + a_{x_2}) = 0 \end{aligned}$$

Next, applying a finite number of cross-differentiations and excluding some derivatives from one equations by means of other equations, the system for the function a solved with respect to its first derivatives, a_{t_1} — (33), a_{t_2} — (34), a_{x_1} and a_{x_2} —

$$a_{x_1} = \frac{a(a-1)}{s_{x_1}^3}\Big[s_{x_1}s_{x_1x_1x_1x_1x_1} - s_{x_1x_1}s_{x_1x_1x_1x_1} + 2ss_{x_1}s_{x_1x_1x_1} \\ -2ss_{x_1x_1}^2 + 3s_{x_1}^2 s_{x_1x_1}\Big]$$

$$a_{x_2} = \frac{a(a-1)}{s_{x_1}^3}\Big[(s_{x_1}s_{x_1x_1x_1x_1x_1} - s_{x_1x_1}s_{x_1x_1x_1x_1} + 2ss_{x_1}s_{x_1x_1x_1} \\ -2ss_{x_1x_1}^2)(2a-1) + 6(s_{x_1x_1}s_{x_1}^2)a\Big],$$

is derived. Their compatibility conditions and other constrains form the original system to the derivatives of s with respect to x_1 can be readily and effectively simplifies and are compatible only for $s_{x_1} = 0$! As a result, only the well-known soliton solution remains.

It can be shown that the common case of s and c gives rise to the same inference as well. This is highly complicated and hard procedure, however, and we plan to dedicate an individual research to such evidences for known soliton models.

5 Some remarks

In spite of the fact that many questions remain open yet, and first of all whether this concept enables one to describe all known soliton models or there are other mechanisms, nevertheless it is already possible to make some deductions within the frameworks of this paper:

(1) In the general case it is necessary to say on change-over of a soliton from one state to another rather then simply on an elastic interaction and phase shift. The computer simulation performed in [10, 11] counts in favour of this point of view. (In some sense the same is true and for a perturbation taking part in an interaction as well.)

(2) There are no indication that solitonic properties are directly connected with the notion of integrability or other properties of a system of interest, including the Painlevé one (see also [10, 11, 24]), although it is now commonly thought that these concepts are closely related.

(3) A soliton solution may be unique (again, see [24]). The existence of N-soliton solutions is no relevance for solitonic properties. The KdV N-soliton ones are also consequence of its truncated singular expansion.

(4) The soliton concept proposed here is rather common and not specific for some type of PDEs, e.g. conservative or reaction-diffusion systems.

(5) Paradoxical as it may seem, but in great part solitonic properties are associated with the ordinary differential equations.

(6) And finally, *complex and intricate dynamics in* $(1+1)$ *dimension can be simply explained and described in terms of* $(2+2)$ *dimension.*

6 Conclusion

In the present paper we have only introduced the general concept and briefly illustrated it. Despite its simplicity this theory can explain the complicated behaviour of nonlinear systems. While possibilities of the contemporary mathematical technique plague its application. First, there exists no straightforward approach for finding an appropriate operator M. The singular manifold method is rather restrictive in this regard. For instance, the MKdV equation already demands the two-manifold version [18, 25]. (The point is that to isolate namely the soliton from the general solution to the Riccati equations (16), (17) with the same S and C an additional constrain between them and V is needed. But the latter is, however, inconsistent with that techniques. The two-manifold modification removes the question.) Second, while still the formal theory of overdetermined PDEs is not effective for many nonlinear systems.

Nonetheless, the further researches may bring essential results and explained wave structures more rare that classical solitons ('soliton structures', as in [11]). The case when such structures are not separated physically is intriguing as well and may find a place in signal processing, e.g. Unfortunately, in contrast to the IST by the way, the approach itself proposes no procedure for analysis in such cases. Finally, and that is really very important, it can equally readily be applied to ordinary differential equations.

In the future, we plan to consider some nonlinear models also associated with the Riccati equations, both the well-known ones and others, because there are some specific details (e.g. the MKdV, Burgers cases). It is remarkable fact that a large class are naturally embedded to them. However, with the except of integrable models such equations do not play a role in applications and have attracted no attention up to now. The use of computer simulation is very conjectural for them, because most of them are strong nonlinear and very huge in terms of the function V, and suitable transformation like the KdV truncated expansion are unknown. For the moment, the relations between time and space steps needed for it are unacceptable for computations.

Another important step is N-soliton solutions both from the viewpoint of an extension to more high dimensions and from the viewpoint of truncated expansions as before for the KdV. It should be noted that there is the radical difference between N-soliton solutions and solitons with several parameters. Since each soliton modulates the parameters of each other, they for N-soliton solution will be different from the set of the individual parameters. That is the reason why the one-soliton has only been obtained for the KdV directly. For N-soliton solutions $(N+N)$-dimensional spaces should be investigated.

The research was initiated in [26] and was partially presented at the International Congress of Mathematicians, 18-27 August 1998, Berlin, FRG [27].

Acknowledgments

The author is very grateful to Professor A Jeffrey for reading of the manuscript and encouragement and expresses his gratitude to Professor W Neun, Dr T Wolf, Dr E Mansfield for their consultations on computer algebra and to Professor E Vorob'ev for the consultation on overdetermined systems of PDEs. Also, it would like to thanks to Professor H A Erbay for the interest to the work.

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Figure captions

Figure 1. The typical elastic interaction.
Figure 2. The ‘capture’ case.
Figure 3. The elastic interaction with the pole. The chain curve indicates its position in the absence of perturbations.
Figure 4. The typical unelastic interaction with the pole. The shelf appears as a result. The chain curve indicates the pole position in the absence of perturbations.