EXPONENTIAL DECAY OF 2×2 OPERATOR MATRIX SEMIGROUPS

András Bátkai,
Klaus-Jochen Engel

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Abstract. We show the exponential decay of certain 2×2 operator matrix semigroups in Hilbert spaces. Applications are given to several damped wave equations.

0. Introduction

In [H] S.-Z. Huang studied the exponential decay of certain matrix semigroups. By applying his results to abstract second order Cauchy problems and damped wave equations he obtained estimates for the decay rate of various elastic systems.

The aim of this note is to give a new and simplified proof of Huang's main theorem with slightly different decay rate estimates. This is done in the first section.

In the second part we give sufficient conditions for the well-posedness of complete second order Cauchy problems and apply our results of Section 1 in order to study the asymptotic behavior of the solutions.

Finally, in the last section we consider some typical applications which illustrate our abstract results obtained in the previous sections.

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1. Asymptotic behaviour of solutions

We start by setting up our general framework.

Let $C : D(C) \subset H \to H$ be a densely defined and invertible linear operator on a Hilbert space H. On its domain $D(C)$ we consider the inner product $\langle x, x\rangle_C := \langle Cx, Cx\rangle$ and denote by H_1^C the Hilbert space $(D(C), \|\cdot\|_C)$ equipped with the graph norm $\|x\|_C := \langle x, x\rangle_C^{\frac{1}{2}} = \|Cx\|$ (cf. [EN], Subsection II.a).

Moreover, let $B : D(B) \subset H \to H$ and $D : D(D) \subset H_1^C \to H_1^C$ be dissipative linear operators such that

$$\mathcal{A}_0 := \begin{pmatrix} D & Id \\ -C^*C & B \end{pmatrix} \quad \text{with} \quad D(\mathcal{A}_0) := \big(D(C^*C) \cap D(D)\big) \times \big(D(C) \cap D(B)\big)$$

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is densely defined on the Hilbert space $\mathcal{W} := H_1^C \times H$ equipped with the canonical inner product. In wave equations typically $D = 0$, but we consider the more general case.

One easily verifies that $\mathcal{A}_0$ is dissipative, hence closable (see [EN], Proposition 3.14.(iv)) and we denote by $\mathcal{A} := \overline{\mathcal{A}_0}$ its closure. Here in the terminology of dissipative, accretive and sectorial operators we follow [Ka].

Our main assumptions on the operators B, C and D are the following.

Hypothesis (H). *There exist constants $\gamma \geq 0$ and $\delta > 0$ such that for all $\binom{x}{y} \in D(\mathcal{A}_0)$ we have*

$$\delta \min\left\{\|x\|_C^2, \|y\|^2\right\} \leq \mathrm{Re}\langle -Dx, x\rangle_C + \mathrm{Re}\langle -By, y\rangle \tag{$*$}$$

and

$$\left|\,\mathrm{Im}\langle Dx, x\rangle_C\right| + \left|\,\mathrm{Im}\langle By, y\rangle\right| \leq \gamma\big(\mathrm{Re}\langle -Dx, x\rangle_C + \mathrm{Re}\langle -By, y\rangle\big). \tag{$**$}$$

In the special case of $D = 0$ and B symmetric, Hypothesis (H) simplifies to the condition that B has to be negative definite.

To prove our main theorem, we need the following technical result.

1.1 Lemma. *Let $0 < \varepsilon < \frac{\delta}{2}$ and $\alpha \in (-\frac{\delta}{2} + \varepsilon, 0]$. If*

$$\inf_{\mathfrak{x} \in D(\mathcal{A}), \|\mathfrak{x}\|=1} \|(\alpha + i\beta - \mathcal{A})\mathfrak{x}\| < \varepsilon,$$

then

$$|\beta| < \frac{(\varepsilon - \alpha)\gamma + 3\varepsilon}{\delta + 2(\alpha - \varepsilon)} \cdot \delta.$$

Proof. Let $\mathfrak{x} := \binom{x}{y} \in D(\mathcal{A})$ satisfy $\|\mathfrak{x}\| = 1$ and $\|(\alpha + i\beta - \mathcal{A})\mathfrak{x}\| < \varepsilon$. Since $D(\mathcal{A}_0)$ is a core for $\mathcal{A}$ we may assume that $\mathfrak{x} \in D(\mathcal{A}_0)$ and therefore obtain

$$\|(\alpha + i\beta - \mathcal{A})\mathfrak{x}\| = \left\|\begin{pmatrix} \alpha x + i\beta x - Dx - y \\ C^*Cx + \alpha y + i\beta y - By \end{pmatrix}\right\| < \varepsilon. \tag{1.1}$$

Thus $|\langle(\alpha + i\beta - \mathcal{A})\mathfrak{x}, \mathfrak{x}\rangle| < \varepsilon$, i.e.,

$$\left|\alpha + i\beta - \langle Dx, x\rangle_C + 2i\,\mathrm{Im}\langle Cx, Cy\rangle - \langle By, y\rangle\right| < \varepsilon \tag{1.2}$$

and hence

$$\left|\,\mathrm{Re}\langle Dx, x\rangle_C + \mathrm{Re}\langle By, y\rangle - \alpha\right| < \varepsilon. \tag{1.3}$$

This estimate combined with $(*)$ implies

$$\delta \min\left\{\|x\|_C^2, \|y\|^2\right\} \leq -\mathrm{Re}\langle Dx, x\rangle_C - \mathrm{Re}\langle By, y\rangle < \varepsilon - \alpha. \tag{1.4}$$

At this point we have to divide the proof in two cases.

Case I: $\|y\|^2 < \frac{\varepsilon-\alpha}{\delta}$. Then

$$\|x\|_C^2 = 1 - \|y\|^2 > 1 - \frac{\varepsilon-\alpha}{\delta},$$

hence the assumption on α implies

$$1 - 2\|x\|_C^2 < 2\frac{\varepsilon-\alpha}{\delta} - 1 < 0,$$

and therefore

$$\left|1 - 2\|x\|_C^2\right| > 1 - 2\frac{\varepsilon-\alpha}{\delta}. \tag{1.5a}$$

On the other hand, we obtain from (1.1) the estimate

$$\| - Dx - y + (\alpha + i\beta)x\|_C < \varepsilon \tag{1.6a}$$

which gives by taking first the inner product with x

$$|\langle -Dx - y + (\alpha + i\beta), x\rangle_C| < \varepsilon,$$

and then imaginary parts

$$\left| - \operatorname{Im}\langle Dx, x\rangle_C + \operatorname{Im}\langle Cx, Cy\rangle + \beta\|x\|_C^2\right| < \varepsilon. \tag{1.7a}$$

Next, we combine (1..7a) with (1.2) and conclude

$$\begin{aligned} 3\varepsilon &> \left|\beta\left(1 - 2\|x\|_C^2\right) + \operatorname{Im}\langle Dx, x\rangle_C - \operatorname{Im}\langle By, y\rangle\right| \\ &\geq |\beta| \cdot \left|1 - 2\|x\|_C^2\right| - \left|\operatorname{Im}\langle Dx, x\rangle_C - \operatorname{Im}\langle By, y\rangle\right|. \end{aligned} \tag{1.8a}$$

Together with (1.5a), $(**)$ and (1.4) this implies

$$|\beta|\left(1 - 2\,\frac{\varepsilon-\alpha}{\delta}\right) \leq 3\varepsilon + \gamma\left(- \operatorname{Re}\langle Dx, x\rangle_C - \operatorname{Re}\langle By, y\rangle\right) < 3\varepsilon + \gamma(\varepsilon - \alpha) \tag{1.9}$$

and the assertion follows.

Case II: $\|x\|_C^2 < \frac{\varepsilon-\alpha}{\delta}$. Then

$$\|y\|^2 = 1 - \|x\|_C^2 > 1 - \frac{\varepsilon-\alpha}{\delta},$$

and the proof follows analogously. Hence we collect only the corresponding formulas

$$\left|1 - 2\|y\|^2\right| > 1 - 2\frac{\varepsilon-\alpha}{\delta}, \tag{1.5b}$$

$$\|C^*Cx - By + (\alpha + i\beta)y\| < \varepsilon, \tag{1.6b}$$

$$\left| - \operatorname{Im}\langle Bx, x\rangle + \operatorname{Im}\langle Cx, Cy\rangle + \beta\|y\|^2\right| < \varepsilon \tag{1.7b}$$

and

$$\begin{aligned} 3\varepsilon &> \left|\beta\left(1 - 2\|y\|^2\right) - \operatorname{Im}\langle Dx, x\rangle_C + \operatorname{Im}\langle By, y\rangle\right| \\ &\geq |\beta| \cdot \left|1 - 2\|y\|^2\right| - \left|\operatorname{Im}\langle Dx, x\rangle_C - \operatorname{Im}\langle By, y\rangle\right|, \end{aligned} \tag{1.8b}$$

which again imply (1.9).

Remark. If $D = 0$ the proof simplifies to Case I.

Next we observe that if $\mathcal{A}$ is invertible, it is maximal dissipative and hence by the Lumer–Phillips theorem generates a contraction semigroup on $\mathcal{W}$. Our main result gives an estimate on the growth bound (or type) $\omega_0(\mathcal{A})$ of this semigroup, cf. [EN], Definition I.5.6.

1.2 Theorem. *Assume that $\mathcal{A}$ is invertible.*

(a) If $\gamma \neq 0$, then

$$\omega_0(\mathcal{A}) \leq w,$$

where $w \in (-\frac{\delta}{2}, 0)$ is the unique solution of the equation

$$w^2 + \frac{w^2\gamma^2\delta^2}{(\delta+2w)^2} = \left\|\mathcal{A}^{-1}\right\|^{-2}.$$

(b) If $\gamma = 0$, then

$$\omega_0(\mathcal{A}) \leq w := \max\left\{-\tfrac{\delta}{2}, -\left\|\mathcal{A}^{-1}\right\|^{-1}\right\}.$$

Proof. Due to the Gearhart–Greiner–Prüß theorem, see [EN] Theorem V.1.11, it suffices to show that

$$\alpha + i\mathbb{R} \subset \rho(\mathcal{A}) \quad \text{and} \quad \sup_{\beta\in\mathbb{R}} \|R(\alpha+i\beta, \mathcal{A})\| < +\infty \quad \text{for all} \quad \alpha > w. \tag{1.10}$$

Since $\mathcal{A}$ generates a contraction semigroup, we know that $\{z \in \mathbb{C} : \operatorname{Re} z > 0\} \subset \rho(\mathcal{A})$. On the other hand, the boundary $\partial\sigma(\mathcal{A})$ is always contained in the approximate point spectrum $A\sigma(\mathcal{A})$, see e.g. [EN] Proposition IV.1.10. However, from the previous lemma it follows that for $\lambda := \alpha + i\beta$ satisfying $\alpha \in (-\frac{\delta}{2}, 0)$ and

$$|\beta| \geq -\frac{\alpha\gamma\delta}{\delta + 2\alpha},$$

we have

$$\inf_{x\in D(\mathcal{A}),\|x\|=1} \|(\lambda - \mathcal{A})x\| > 0,$$

and therefore $\lambda \notin \partial\sigma(\mathcal{A}) \subset \sigma(\mathcal{A})$. Moreover, the set $\left\{\mu \in \mathbb{C} : |\mu| < \|\mathcal{A}^{-1}\|^{-1}\right\}$ is contained in $\rho(\mathcal{A})$, hence a simple geometric consideration and the equality

$$\|R(\lambda, \mathcal{A})\| = \frac{1}{\inf\limits_{x\in D(\mathcal{A}),\|x\|=1} \|(\lambda - \mathcal{A})x\|}$$

imply (1.10).

By a simple density argument we obtain the following result.

1.3 Corollary. *Let $\mathcal{A}$ be maximal dissipative and assume that $\gamma \neq 0$. If there exists $\kappa > 0$ such that*

$$\|\mathcal{A}_0 x\| \geq \kappa\|x\| \quad \textit{for all } x \in D(\mathcal{A}_0),$$

then

$$\omega_0(\mathcal{A}) \leq w,$$

where $w \in (-\frac{\delta}{2}, 0)$ is the unique solution of the equation

$$w^2 + \frac{w^2\gamma^2\delta^2}{(\delta+2w)^2} = \kappa^2. \tag{1.11}$$

If B and D are symmetric operators we can choose $\gamma = 0$ in Theorem 1.2 obtaining the following result.

1.4 Corollary. *If B and D are symmetric and $\mathcal{A}$ is maximal dissipative then*

$$\omega_0(\mathcal{A}) \leq \max\left\{-\kappa, -\tfrac{\delta}{2}\right\}.$$

We now turn our attention to complete second order Cauchy problems.

2. Exponential energy decay of complete second order Cauchy problems

In this section we consider the complete second order abstract Cauchy problem

$$(\mathrm{ACP}_2) \qquad \begin{cases} u''(t) - Bu'(t) + Au(t) = 0, \ t \geq 0, \\ u(0) = u_0, \quad u'(0) = u_1 \end{cases}$$

on a Hilbert space H. Using the standard reduction $\mathfrak{u} := \binom{u}{u'}$ (see [F] Section VIII or [Ko] Chapter 7.2) we can transform (ACP_2) into a first order system

$$(\mathrm{ACP}) \qquad \begin{cases} \mathfrak{u}'(t) = \mathcal{A}\mathfrak{u}(t), \ t \geq 0, \\ \mathfrak{u}(0) = \binom{u_0}{u_1} \end{cases}$$

for the operator matrix $\mathcal{A} := \left(\begin{smallmatrix} 0 & Id \\ -A & B \end{smallmatrix}\right)$ on some product space $\mathcal{W}$.

An equivalent and commonly used reduction can be found in [XL] Chapter 6.4. In this sense, our results will be generalizations of the stability results in [XL].

In order to apply our results from Section 1, we assume that $A = C^*C$ for a densely defined and invertible operator C on H. Note that this implies that A is self-adjoint and positive definite, see [W] Theorem 5.39. Moreover, we suppose that B is dissipative and sectorial such that $D(C) \cap D(B)$ is dense in H.

We then consider the operator $\mathcal{A}$ defined (as in Section 1 for $D = 0$) by the closure of

$$\mathcal{A}_0 := \begin{pmatrix} 0 & Id \\ -A & B \end{pmatrix} \quad \text{with} \quad D(\mathcal{A}_0) := D(A) \times \big(D(C) \cap D(B)\big)$$

on the Hilbert space $\mathcal{W} := H_1^C \times H$.

Finally, we assume that there exist constants $\gamma \geq 0$ and $\delta > 0$ such that

$$(2.1) \qquad |\operatorname{Im}\langle By, y\rangle| \leq \gamma \operatorname{Re}\langle -By, y\rangle \quad \text{and} \quad \delta \|y\|^2 \leq \operatorname{Re}\langle -By, y\rangle$$

for all $y \in D(B)$ which implies that $\mathcal{A}_0$ satisfies the conditions $(*)$ and $(**)$ in Hypothesis (H).

If $\mathcal{A}$ is invertible, we can represent its inverse in matrix form

$$\mathcal{A}^{-1} = \begin{pmatrix} U & V \\ W & S \end{pmatrix} \in \mathcal{L}(\mathcal{W})$$

and (formally) conclude from $\mathcal{A}^{-1}\mathcal{A}_0 = \mathrm{Id}|_{D(\mathcal{A}_0)}$ that its entries satisfy

$$V = -A^{-1}, \ S = 0, \ W = Id, \text{ and}$$
$$Uz = A^{-1}Bz \quad \forall z \in D(A^{-1}B).$$

These operators give rise to a bounded inverse of $\mathcal{A}$ if and only if $A^{-1}B$ has a bounded closure in (H_1^C) or, equivalently, if the set $C(D(B) \cap D(C))$ is dense in H and

$$(2.2) \qquad Q := \overline{(C^*)^{-1}BC^{-1}} \in \mathcal{L}(H).$$

In particular, these conditions are satisfied if B is C-bounded. In this case solutions of (ACP) even give strong solutions of (ACP_2) (in the sense of Fattorini, cf. [F] Section VIII. for the definition).

We are now in the position to state our first stability result.

2.1 Corollary. *Let C and B be densely defined linear operators on a Hilbert space H satisfying the conditions* (2.1) *and* (2.2). *If C is invertible, then the matrix $\mathcal{A}$ generates a uniformly exponentially stable C_0-contraction semigroup in $\mathcal{W}$. More precisely, if*

$$\kappa^{-1} := \left\|\overline{A^{-1}B}\right\|_C + 2\left\|C^{-1}\right\| = \|Q\| + 2\left\|C^{-1}\right\|, \tag{2.3}$$

then the following estimates for the growth bound $\omega_0(\mathcal{A})$ of $\mathcal{A}$ hold.

(a) If $\gamma \neq 0$, then

$$\omega_0(\mathcal{A}) \leq w,$$

where $w \in (-\frac{\delta}{2}, 0)$ is the unique solution of equation (1.11).

(b) If $\gamma = 0$, then

$$\omega_0(\mathcal{A}) \leq \max\left\{-\kappa, -\tfrac{\delta}{2}\right\}.$$

Proof. We define

$$\begin{aligned} \widetilde{\mathcal{A}} &:= \begin{pmatrix} C^{-1} & 0 \\ 0 & C^* \end{pmatrix} \begin{pmatrix} 0 & Id \\ -Id & Q \end{pmatrix} \begin{pmatrix} C & 0 \\ 0 & C \end{pmatrix}, \\ D(\widetilde{\mathcal{A}}) &:= \left\{ \tbinom{x}{y} \in D(C) \times D(C) : Cx - QCy \in D(C^*) \right\}. \end{aligned} \tag{2.4}$$

Then $\mathcal{A}_0 \subseteq \widetilde{\mathcal{A}}$ and $\widetilde{\mathcal{A}}D(\mathcal{A}_0)$ is dense in $\mathcal{W}$. Moreover, $\widetilde{\mathcal{A}}$ is invertible and hence $D(\mathcal{A}_0)$ is a core for $\widetilde{\mathcal{A}}$, i.e. $\mathcal{A} = \overline{\mathcal{A}_0} = \widetilde{\mathcal{A}}$. Since the assumptions on B and C imply that $\mathcal{A}_0$ is dissipative we conclude by the Lumer–Philips theorem that $\mathcal{A}$ generates a contraction semigroup. By Theorem 1.2 this semigroup is uniformly exponentially stable. The final statements follow from Corollaries 1.3 and 1.4, resp., and the estimate

$$\begin{aligned} \left\|\mathcal{A}^{-1}\tbinom{x}{y}\right\| &= \left(\left\|\overline{A^{-1}B}x - A^{-1}y\right\|_C^2 + \|x\|^2\right)^{\frac{1}{2}} \\ &\leq \left\|\overline{A^{-1}B}x - A^{-1}y\right\|_C + \|x\| \\ &\leq \|\overline{A^{-1}B}\|_C \cdot \|x\|_C + \left\|A^{-1}\right\|_{H \to H_1^C} \cdot \|y\| + \|Id\|_{H_1^C \to H} \cdot \|x\|_C \\ &\leq \left(\left(\left\|\overline{A^{-1}B}\right\|_C + \|Id\|_{H_1^C \to H}\right)^2 + \left\|A^{-1}\right\|_{H \to H_1^C}^2\right)^{\frac{1}{2}} \cdot \left(\|x\|_C^2 + \|y\|^2\right)^{\frac{1}{2}} \\ &\leq \left(\left\|\overline{A^{-1}B}\right\|_C + \|Id\|_{H_1^C \to H} + \left\|A^{-1}\right\|_{H \to H_1^C}\right) \cdot \left(\|x\|_C^2 + \|y\|^2\right)^{\frac{1}{2}} \\ &\leq \left(\left\|\overline{A^{-1}B}\right\|_C + 2\left\|C^{-1}\right\|\right) \cdot \left(\|x\|_C^2 + \|y\|^2\right)^{\frac{1}{2}} \end{aligned}$$

for $\binom{x}{y} \in D(\mathcal{A}_0)$.

We close this section by a particular important case where B is a scalar multiple of a fractional power of the positive definite operator A.

2.2 Corollary. *Let $C = A^{\frac{1}{2}}$ and $B = bA^{\vartheta}$ for $0 < \vartheta \leq 1$ and $b \in \mathbb{C}$ with* $\operatorname{Re} b < 0$. *Moreover, let $\lambda > 0$ denote a lower bound of A, i.e. $\|Ax\| \geq \lambda\|x\|$ for all $x \in D(A)$. If*

$$\kappa^{-1} := |b|\lambda^{\vartheta-1} + 2\lambda^{-\frac{1}{2}},$$

then the following estimates for the growth bound $\omega_0(\mathcal{A})$ of $\mathcal{A}$ hold.

(a) If $b \notin \mathbb{R}$, then

$$\omega_0(\mathcal{A}) \leq w,$$

where $w \in (-\frac{\delta}{2}, 0)$ is the unique solution of equation (1.11).

(b) If $b \in \mathbb{R}$, *then*

$$\omega_0(\mathcal{A}) \leq \max\left\{-\kappa, \frac{b}{2}\lambda^{\vartheta}\right\}.$$

The proof follows from the spectral mapping theorem for self-adjoint operators and Corollary 2.1.

3. Applications

3.1 Example. (*Internally damped wave equation*, see [Go]) Let Ω be a bounded connected domain in $\mathbb{R}^n$ and let Γ be its boundary. We suppose Γ to be piecewise smooth and consisting of two closed parts Γ^0 and Γ^1 such that $\Gamma = \Gamma^0 \cup \Gamma^1$ and $\Gamma^0 \cap \Gamma^1 = \emptyset$. Moreover, we denote by ν the outer unit normal of Γ.

We shall be concerned with the internally damped wave equation

$$\text{(IDW)} \qquad \begin{cases} y''(x,t) = \mu\Delta y'(x,t) + \Delta y(x,t), & (x,t) \in \Omega \times (0,\infty), \\ y(x,0) = y_0(x), \quad y'(x,0) = y_1(x), & x \in \Omega \end{cases}$$

with mixed boundary conditions

$$\begin{aligned} y(x,t) &= 0 \quad \text{on} \quad \Gamma^0 \times (0,\infty), \\ \frac{\partial y(x,t)}{\partial \nu} &= 0 \quad \text{on} \quad \Gamma^1 \times (0,\infty), \end{aligned}$$

where $\mu > 0$ is a constant.

Then the energy of a solution y of (IDW) is given by

$$E(t) = \frac{1}{2}\int_\Omega \left(|y'(x,t)|^2 + |\nabla y(x,t)|^2\right) dx.$$

In order to apply our abstract results we have to reformulate this problem as an abstract second order Cauchy problem. Here for the notations of Sobolev spaces we follow [A].

First we define the following operators on the Hilbert space $H := L^2(\Omega)$:

$$A := -\Delta, \quad B := -\mu\Delta, \quad D(B) := D(A) := \left\{y \in H^2 : y|_{\Gamma^0} = 0,\ \frac{\partial y}{\partial \nu}|_{\Gamma^1} = 0\right\}.$$

It is easy to see that $A^{-1}Bf = \mu f$ for all $f \in D(A) \subset H^1_{\Gamma^0}$ and hence we obtain $\|Q\| = \mu$, where we used that $D(A^{\frac{1}{2}}) = H^1_{\Gamma^0} := \{f \in H^1 : f|_{\Gamma^0} = 0\}$, cf. [Tr]. Hence, if we denote the first eigenvalue of A by $\lambda_1 > 0$, we can apply Corollary 2.2.(b) with

$$\kappa = \frac{\sqrt{\lambda_1}}{\mu\sqrt{\lambda_1} + 2}$$

obtaining the exponential stability of (IDW). This is equivalent to the decay rate derived in [Go].

3.2 Example. (*Damped wave equation*, see [Tc], [Ko] Section 1.3., [CZ1], [CZ2], [Lo]) Let Ω be an open, bounded domain in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$. Consider the damped wave equation

$$\text{(DWE)} \qquad \begin{cases} u_{tt}(t,x) + q(x)u_t(t,x) - \Delta u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \Omega, \\ u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), & x \in \Omega, \\ u(t,x) = 0, & (t,x) \in \mathbb{R}_+ \times \partial\Omega \end{cases}$$

for initial values $u_0, u_1 \in L^2(\Omega)$. Here we assume that $q : \Omega \to \mathbb{C}$ is measurable and satisfies

$$|\operatorname{Im} q(x)| \le -\gamma \operatorname{Re} q(x) \quad \text{and} \quad \delta \le -\operatorname{Re} q(x) \text{ a.e.}$$

for some constants $\gamma, \delta > 0$.

In order to reformulate (DWE) as an abstract second order Cauchy problem we take $H := L^2(\Omega)$, $A := -\Delta$ with domain $D(A) := H^2(\Omega) \cap H_0^1(\Omega)$ and $B = M_q$ the multiplication operator with the function q and maximal domain, see [EN] Proposition I.4.10 or [W] 4.1.Example 1.

Then $\mathcal{A}_0$ satisfies $(*)$ and $(**)$. Hence in order to apply our results we have to look for conditions implying the boundedness of (the closure of) $A^{\frac{1}{2}}B$.

To this end we use Sobolev imbedding theorems as in [A] Theorem 5.4.C, that is, $H_0^1(\Omega) \hookrightarrow C_b(\Omega)$ in the one dimensional case, $H_0^1(\Omega) \hookrightarrow L^p(\Omega)$ $(p \ge 2)$ in two dimensions and $H_0^1(\Omega) \hookrightarrow L^q(\Omega)$ $(q \ge 2n/(n-2))$ in higher dimensions. If we assume that $q \in L^r(\Omega)$ with $r \ge 2$ if $n = 1$, $r > 2$ if $n = 2$ and $r \ge n$ if $n \ge 3$, these imbeddings combined with Hölder's inequality imply that B is $A^{\frac{1}{2}}$-bounded. Hence, by Corollary 2.1 we obtain the exponential decay of the energy

$$E(t) = \frac{1}{2} \int_\Omega \left(|u'(t,x)|^2 + |\nabla u(t,x)|^2 \right) dx$$

of the solution u of (DWE).

In the simplest case $\Omega = (0,1)$ and a function $q \in L^2(0,1)$ we arrive in this way at the estimate $\|Q\|_{\frac{1}{2}} \le \|q\|_2/\pi$, cf. [EN] Example VI.3.19. Moreover, the first eigenvalue of A is $\lambda_1 = \pi^2$ and the conclusions of Corollary 2.1 hold for

$$\kappa := \frac{\pi}{\|q\|_2 + 2}.$$

For more complicated cases one can obtain decay estimates by using the constants of the Sobolev imbedding theorem.

In the one dimensional case, Cox and Zuaza derived the exact decay rates for bounded damping. For unbounded damping the only work known by us is [Tc] where other type of conditions ensure polynomial decay rates. He obtains similar conditions for the well-posedness.

3.3 Example. (*Damped Kirchhoff plate*, see [H] Example 4.3, [L] Chapter 6., [LLP]) Let Ω be an open, bounded domain in $\mathbb{R}^2$ with smooth boundary $\partial\Omega$. The displacement w of the Kirchhoff plate is described by the following equation.

$$\text{(DKE)} \qquad \begin{cases} \varrho h w_{tt} - \varrho \frac{h^3}{12} \Delta w_{tt} + \alpha \Delta^2 w_t + D \Delta^2 w = 0, & (t,x) \in \mathbb{R}_+ \times \Omega, \\ w(0,x) = w_0(x), \quad w_t(0,x) = w_1(x), & x \in \Omega, \\ w = 0, & (t,x) \in \mathbb{R}_+ \times \partial\Omega. \end{cases}$$

Here the constant ϱ is the mass density per unit volume, h represents the thickness of the plate and D is the modulus of flexual rigidity given by $D := Eh^3/12(1-\mu^2)$ with μ the Poisson's ratio and E Young's modulus. Moreover, $\alpha > 0$ is the damping constant.

To formulate the corresponding abstract Cauchy problem in the Hilbert space $H := L^2(\Omega)$ we consider the Dirichlet-Laplacian Δ_D with its usual domain. Moreover, we define the following two operators on H:

$$A := D\Delta_D^2 \left(\varrho h - \frac{h^3}{12}\Delta_D\right)^{-1}, \qquad B := \frac{\alpha}{D}A,$$

$$D(B) := D(A) := D(\Delta_D) = H^2(\Omega) \cap H_0^1(\Omega).$$

If we denote the first eigenvalue of Δ_D by $\lambda_1 < 0$, then the first eigenvalue of A is

$$\mu_1 = D\frac{\lambda_1^2}{\varrho h - \frac{h^3}{12}\lambda_1}.$$

Hence we can apply Corollary 2.2.(b) with

$$\kappa = \frac{\sqrt{\mu_1}}{\frac{\alpha}{D}\sqrt{\mu_1} + 2}$$

and obtain the exponential stability of (DKE).

3.4 Example. (*Euler–Bernoulli beam with Kelvin–Voigt damping*, see [R1], [R2]) We consider the Euler–Bernoulli model with Kelvin–Voigt damping for a beam of length L clamped at its ends, cf. [R1]. We will denote the second moment of the modulus of elasticity about the elastic axis by $EI(x)$, the damping constant by $\alpha > 0$ and the mass density per unit length by $\varrho(x)$. The equation for the displacement w is

$$w_{tt} + 2\alpha A w_t + Aw = 0, \tag{EBKV}$$

where

$$Aw = \frac{1}{\varrho}\frac{\partial^2}{\partial x^2}\left(EI\frac{\partial^2 w}{\partial x^2}\right),$$

$$D(A) = \left\{w \in H^4(0,L) : w(0) = w''(0) = w(L) = w''(L) = 0\right\}.$$

Under these conditions one can calculate that $\|Q\| = 2\alpha$. Denoting the first eigenvalue of A by $\lambda_1 > 0$, we apply Corollary 2.2.(b) with

$$\kappa^{-1} = 2\alpha + \frac{2}{\sqrt{\lambda_1}}.$$

In the case of constant coefficients, i.e. if EI and ϱ is constant, we can deduce

$$A^{\frac{1}{2}}w = -\left(\frac{EI}{\varrho}\right)^{\frac{1}{2}}\frac{\partial^2 w}{\partial x^2}$$

and

$$\kappa^{-1} = 2\alpha + \frac{2\sqrt{\varrho}}{\sqrt{EI}\pi^2},$$

hence obtain again from Corollary 2.2.(b) the exponential stability of (EBKV).

We note that similarly we can treat other boundary conditions implying the invertibility of A.

We conclude this paper by the observation that many other equations, for example beam equations with structural damping as in [CR] or with shear diffusion as in [LZ], can be treated with the above method as well.

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Mathematisches Institut, Universität Tübingen, Auf der Morgenstelle 10, D-72076 Tübingen, Germany

E-mail address: anba@michelangelo.mathematik.uni-tuebingen.de

Università degli Studi di L'Aquila, Facoltà di Ingegneria, Dipartimento di Energetica, Località Monteluco, I-67040 Roio Poggio (AQ), Italy
E-mail address: engel@ing.univaq.it