Some notes on the singular manifold method: several manifolds and constrains

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Abstract

Some questions arising when various modifications of the singular manifold method are considered. The solutions are shown to lie outside the frameworks of the singular analysis. The approach is illustrated by a number of examples with ODEs, and further perspectives are discussed.

Short title: Notes on the singular manifold method

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1 Introduction

Although the singular manifold method [1] was introduced as a natural generalization of the ordinary Painlevé test to partial differential equations in the context of the necessary condition for their integrability [2] (more precisely, the connection between nonlinear PDEs associated with the linear Gelfand-Levitan-Marchenko integral equation and the Painlevé property of their scaling type self-similar solutions), nevertheless already in [1] the overwhelming majority of results concern the use of the related functional series

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for deriving various relations and structures, such as the Bäcklund transformations and Lax pairs, for the equations under consideration rather than the Painlevé property itself. In succeeding two decades several works refining the singular analysis have appeared (see [3, 4, 5, 6], e.g.), however as before the singular manifold approach owes its popularity namely to applications of those truncated expansions and the simplicity of their constructing.

The part of the original theory does not demand any assumption on the type of singularities and can successfully be applied and for 'nonintegrable' PDEs [7] and ODEs. Moreover, to expand its practicality both for integrable and nonintegrable cases a number of modifications have been made. Among them are the use of Weierstrass or elliptic functions [8], the Painlevé-Darboux transformations [9], 'double singular manifold methods' [10, 11, 12], and the various ways for introducing judiciously chosen terms [13, 14]. Some of them are compatible with the Painlevé analysis as was shown in [15], others simply break down its basic postulates, but all they turn out to be effective really.

The main goal of the present paper is to show that all the above approaches are closely related and are of the same unified algebraic nature and to point how these ideas can be developed further.

First of all it is needed to point out that the association of $f(x,t)$ with singularities in the truncated or infinite expansions

$$u(x,t) = \sum_{i=-p}^{m} w_i(x,t) \cdot f^i(x,t); \qquad p \in N;\ m = 0, +\infty$$

for solutions alone does not provide reason enough to equal the coefficients at its powers to zero after substitution into a governed equation. Genuinely, on the one hand, for such series to be correct in so doing, it is necessary so that the derivations of f and their combinations arising as factors at its powers have the zero coefficients up to some order in their related Taylor expansions and the nonzero first term. For infinite series this demand leads to ordinary Laurent ones. On the other hand, the above expansions must be valid in the whole domain for our purposes. The answer becomes especially clear taking into account the so-called invariant formalism [5]

$$u(x,t) = \sum_{i=p}^{m} w_i(x,t) \cdot V^i(x,t), \qquad p \in N;\ m = 0, -\infty \tag{1}$$

where in terms of the function f

$$V = \frac{f_x}{f} - \frac{1}{2}\left(\frac{f_{xx}}{f_x}\right)$$

such that

$$V_x = -V^2 - S/2 \tag{2}$$

$$V_t = CV^2 - C_x V + \frac{1}{2}(CS + S_{xx}) \tag{3}$$

$$S = S(x,t), \; C = C(x,t)$$

and

$$S_t + C_{xxx} + 2SC_x + CS_x = 0 \tag{4}$$

In such a way the function V depends on an additional arbitrary parameter in comparison with C, S and w_i as a consequence. In other words, the form of such series and the fact that the singularities are movable turn out to be more essential here. All the aforesaid is important for our further consideration.

2 A singular manifold with an additional constrain

Now let us presume that we have some equation and the generalized Laurent series (1) for its general solution, i.e. $m = -\infty$ and there are $n-1$ arbitrary functions in the expansion besides V, where n is the order of this differential equation. Also, assume that one has some of its reductions. What form does the above Laurent expansion take for it? Obviously, for the infinite series to pick out the solutions corresponding to the reduction it is necessary to impose some constrains to those arbitrary functions (the singular manifold function can be considered as an arbitrary one yet). For the truncated versions, however, two different situations should be distinguish. First, such constrains may do not affect the singular manifold function V. In this case the traditional algorithm is fully suitable to obtain the appropriate series for the second equation. Otherwise we are not entitle to equate the coefficients after substitution in general case. Therefore only expressions for particular solutions can be constructed in such a way. However, the relations associated with the truncated expansion of the original system can be restored uniquely if its Laurent expansion principle part is known. It is easy to see that the principle parts of the expansions for the original system and reductions are always coincident if all the arbitrary functions (resonances) place above them. But for this already another technique should be applied. Next such an approach will be presented in details, and some relations for ODEs associated with the KdV and MKdV equations and solvable in terms of Jacobi elliptic functions will be obtained in its frameworks.

Assume that we have an ordinary (for the simplicity) differential equation $E = 0$ of the above type. Then after substitution of the Laurent principle part it can be factored

$$E(x, u, u_x, \dots) = A(V; x, S, S_x, \dots)\, G(V; x, S, S_x, \dots) \tag{5}$$

here G is the constrain to the singular function V. Obviously, the only constrain can be imposed to one function V. From this fact and the condition of its compatibility with the differential equations to V and S it follows that

$$\frac{\mathrm{d}}{\mathrm{d}x} G = L(V; x, S, S_x, \dots)\, G$$

where L has no singularities at $G = 0$. But then all the derivations of E are factorisable as before, in particular

$$\frac{\mathrm{d}}{\mathrm{d}x} E = \left(\frac{\mathrm{d}}{\mathrm{d}x} A + AL\right) G(V; x, S, S_x, \dots) \tag{6}$$

This can be used to determine the appropriate differential equation to S. In polynomial nonlinearity cases and the polynomials (5), (6) respectively this problem reduces to calculation of their resultant [16] to determine the conditions for the existence of common zeros. (For polynomial PDEs it is needed to consider the derivations $\frac{\partial}{\partial x} E$ and $\frac{\partial}{\partial t} E$ and determine the conditions to S and C so that they have common zeros together with E.)

As well known, the KdV and MKdV equations are closely allied to one another via the Bäcklund and Miura transformations within the frameworks of the singular manifold method. To be precise, the singular manifold equation

$$C - S - \lambda = 0; \qquad \lambda = Const$$

and the truncated expansion

$$u(x,t) = V^2 + \frac{C - 4S}{12} = \frac{V_x - V^2}{2} + \frac{\lambda}{12}$$

are related with the KdV

$$u_t + 12uu_x + u_{xxx} = 0$$

And in so doing, (3) and (4) take the form of the MKdV and KdV again, namely

$$V_t + \lambda V_x - 6V^2 V_x + V_{xxx} = 0$$
$$v_t + 12vv_x + v_{xxx} = 0 \qquad (S = 4v - \lambda/3)$$

Naturally, it could wait that the use of the singular manifold method to their reductions could give rise to analogous relations and maps. Paradoxical as it may seem, but it leads only to particular solutions. To overcome this in [13] the authors proposed simply to pick up suitable terms, and in [14] an ansatz was applied with the same purpose. However, the problem can be solved easily and straightforwardly.

As an example, consider the following traveling wave reduction (twice integrating)

$$u_x^2 - u^4 + cu^2 + bu + a = 0; \qquad a, b, c = Const \tag{7}$$

of the MKdV equation. Substituting the truncated 'expansion'

$$u = \pm V$$

into (7), we have the equation

$$E = 4(c+S)V^2 \pm 4bV + (4a + S^2) = 0$$

And the appropriate resultant is of the form

$$\mathrm{Res}\left(E, \frac{\mathrm{d}}{\mathrm{d}x}E\right) = 16(c+S)\Big[S^4 + 4cS^3 + 4(c^2-2a)S^2 \\ +8(b^2-2ac)S + 16a^2\Big]\Big[S_x^2 + S^3 + cS^2 + 4aS + (4ac - b^2)\Big]$$

Finally, ignoring the first trivial factors, finding the common divisor of E, $\frac{\mathrm{d}}{\mathrm{d}x}E$ and respectively the connection between V and S, one comes to the following map

$$u = \pm \frac{S_x + b}{2(S+c)}$$

between (7) and the analogous reduction of the KdV

$$S_x^2 + S^3 + cS^2 + 4aS + (4ac - b^2) = 0$$

as and might be expected.

As the next MKdV reduction, chose the popular P_{II} equation

$$u_{xx} - 2u^3 - xu + a = 0; \qquad a = Const$$

In this case,

$$E = \pm 2(S-x)V \mp S_x + 2a = 0$$

and the resultant is as follows

$$\mathrm{Res}\left(E, \frac{\mathrm{d}}{\mathrm{d}x}E\right) = 2(S-x)\Big[2(S-x)S_{xx} - S_x^2 + 2S_x + 2S^3$$
$$-4xS^2 + 2x^2S + 4a(a \mp 1)\Big]$$

From where, taking into account the Miura transformation $S = w_x - w^2$, one has the same type equation for $w(x)$

$$w_{xx} - 2w^3 - xw + a' = 0$$

and the mapping

$$u = -w \mp \frac{a \pm a'}{2w_x - 2w^2 - x}$$

such that

$$a' \mp a + 1 = 0$$

This result was also obtained in [14] via the specially chosen ansatz.

In the same manner the relations for other self-similar KdV and MKdV reductions can easily be derived. In particular, the linkage between the KdV traveling wave reductions

$$u_{xx} + 6u^2 + cu + b = 0; \qquad b, c = Const$$
$$S_{xx} + \frac{S^2}{4} + \frac{24b - c^2}{16} = 0$$

is as follows

$$u = \frac{-4S^2 + c^2 - 24b}{16S_x}$$

3 Expansions with several constrained manifolds

In the works [10, 11] the attempt to introduce truncated expansions with two singular manifolds was undertook for integrable equations with different singular branches. In doing so, to applied the traditional method and cancel the terms of the form $f_1^{-n_1} f_2^{-n_2}$ the validity of the linkage

$$\frac{1}{f_1 f_2} = \frac{a(x,t)}{f_1} + \frac{b(x,t)}{f_2}$$

was assumed. Further it was demonstrated that for the cases with two opposite branches the last problem could be solved by reducing to the one-manifold case via the new comfortable notations [12]. However, since the second manifold in such truncated expansions can present the rest of one-manifold infinite series [15], it is necessary to be able to use the above of any type and form. Besides it is also important to perceived that the case at hand is namely constrained manifolds. By this means the problem is analogous to the one from the previous section. Really, assume again that we deal with a polynomial ODE and consider the equation after substitution of an n-manifold truncated expansion with respect to some singular function V_1

$$E_1 = A(V_1, \dots ; x, S_1, \dots)\, G(V_1, \dots , V_n; x, S_1, \dots , S_n, \dots) = 0$$

All the above-mentioned is valid in this case, and the appropriate resultant can be constructed

$$\mathrm{Res}\left(E_1, \frac{\mathrm{d}}{\mathrm{d}x}E_1\right) = E_2(V_2, \dots , V_n; x, S_1, \dots , S_n, \dots) = 0$$

to determine already the constrain to some V_2. This process can be continued up to some m.

$$E_m(V_m, \dots , V_n; x, S_1, \dots , S_n, \dots) = 0 \tag{8}$$

If $m \le n$, there will be the $m-1$ constrains to the n functions $\{V_1, \dots , V_n\}$, the functions $\{V_m, \dots , V_n\}$ remain 'free' with the arbitrary parameters due to (2), (3), and we should further equate to zero the related coefficients at their products in (8). The case $m = n + 1$

$$E_{n+1}(x, S_1, \dots , S_n, \dots) = 0$$

remains us to the problem studied previously.

Although this scheme is simple enough and good from the theoretical point of view but brings to very huge algebra. Furthermore, it will finally be necessary to do away from trivial relations like, e.g., $V_1 = V_2$. If we, however, have the set of some constrains $\{F_1, \dots , F_k\}$, they should satisfy the compatibility conditions

$$\frac{\mathrm{d}}{\mathrm{d}x}F_i = \sum_{j=1}^{k} L_{ij}(V_1, \dots , V_n; x, \dots)\, F_j; \qquad i = \overline{1, k}$$

as and before in the one-manifold case. Respectively, we can calculate the k differential consequences ($E = E_1$)

$$E = \sum_{j=1}^{k} A_{0j}(V_1, \ldots, V_n; x, \ldots)\, F_j = 0$$

$$\frac{\mathrm{d}}{\mathrm{d}x}E = \sum_{j=1}^{k} A_{1j}(V_1, \ldots, V_n; x, \ldots)\, F_j = 0$$

$$\ldots$$

$$\frac{\mathrm{d}^k}{\mathrm{d}x^k}E = \sum_{j=1}^{k} A_{kj}(V_1, \ldots, V_n; x, \ldots)\, F_j = 0$$

and apply the technique of Gröbner bases [17] to find $\{F_1, \ldots, F_m\}$ and the differential equations to S_i. In so doing the form of F_i is clearly not unique and depends on an ordering method for V_i. In effect the above direct algorithm corresponds to the LEX ordering mode. In conclusion it is necessary to say that, of cause, all the above is easily generalized to a PDE case taking into account additional derivations with respect to t.

Now consider from this point of view the ODE

$$cu_x - u_{xx} + 2u^3 + bu + a = 0; \qquad a, b, c = Const \tag{9}$$

corresponding to the following equation [18]

$$u_t - u_{xx} + 2u^3 + bu + a = 0 \tag{10}$$

and show that new results can be obtained.

The equation (9) has two opposite singular branches. As a preliminary, let us dwell on the next two-manifold truncated expansion

$$u = V_1 - V_2 + w(x, t) \tag{11}$$

In other words, one assumes that the infinite series can be presented in that manner. Therewith the first manifold function V_1 is associated with the principle part as usually, while the second one does to the rest of the series and could in turn be expanded into terms of V_1.

Inserting (11) into (9), one has

$$\begin{aligned} E = &-12V_1^2V_2 + 2(6w - c)V_1^2 + 12V_1V_2^2 - 24wV_1V_2 \\ &+(2b - S^- - S^+ + 12w^2)V_1 + 2(6w + c)V_2^2 \\ &+(-2b - S^- + S^+ - 12w^2)V_2 \\ &+S_x^- - 2w_{xx} + 2cw_x + 2a + 2bw - cS^- + 4w^3 = 0 \end{aligned}$$

here

$$S^- = S_1 - S_2$$
$$S^+ = S_1 + S_2$$

and $\frac{\mathrm{d}}{\mathrm{d}x}E$ respectively. Taking into account the former equation to eliminate the terms $V_1^3V_2$ and $V_1^2V_2$ from $\frac{\mathrm{d}}{\mathrm{d}x}E = 0$, the latter is written down as

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}x}E = 2(18w_x + 3b + c^2 - 6cw - 6S^- + 3S^+ + 18w^2)V_1^2 \qquad (12)\\
&+24(cw - 3w_x + S^-)V_1V_2 + (3S_x^- - 3S_x^+ - 12w_{xx} + 12cw_x + 72ww_x\\
&+12a - 2bc + 12bw - 2cS^- + 4cS^+ - 12cw^2 - 36S^-w + 24w^3)V_1\\
&+2(18w_x - 3b - c^2 - 6cw - 6S^- - 3S^+ - 18w^2)V_2^2 + (3S_x^- + 3S_x^+\\
&-12w_{xx} + 12cw_x - 72ww_x + 12a + 2bc + 12bw - 2cS^- - 4cS^+\\
&+12cw^2 + 36S^-w + 24w^3)V_2 + 3S_{xx}^- - 4cS_x^- - 6w_{xxx} + 8cw_{xx}\\
&+6bw_x - 2c^2w_x + 36w^2w_x - 2ac - 2bcw - 3bS^- + c^2S^- - 4cw^3\\
&+3S^-S^+ - 18S^-w^2 = 0
\end{aligned}$$

The process could, of cause, be continued. However, if we demand that V_1 and V_2 are not trivial and do be of singularities and namely different singularities, this is enough already. Really, from the line of reasoning we should here equal to zero the coefficient at V_1^2 and V_2^2. After that (12) takes such a form that further simplifications are possible. As a result, after finding S_1 and S_2 one has the Bäcklund transformation (one parameter) for the equations of the type (9)

$$u = V_1 - V_2 + \frac{v(x,t)}{2}$$

where $(v = w/2)$

$$cv_x - v_{xx} + 2u^3 + \left(4b + \frac{2}{3}c^2\right)u + (8a + 4bc) = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}x}V_{1,2} = -V_{1,2}^2 - \frac{1}{24}\Big(\pm 9v_x - 6b - 2c^2 \mp 3cv - 9v^2\Big)$$

provided that

$$c = 0 \quad \text{or} \quad a = 9b + 2c^2 = 0$$

and there exists the one constrain to V_1 and V_2

$$\begin{aligned}
&-216V_1^2V_2 + 36(3v - c)V_1^2 + 216V_1V_2^2 - 216vV_1V_2\\
&+3(-9v_x + 18b + 2c^2 + 3cv + 27v^2)V_1 + 36(c + 3v)V_2^2\\
&+3(-9v_x - 18b - 2c^2 + 3cv - 27v^2)V_2\\
&-9cv_x + 108a + 36bc + 54bv + 8c^3 + 15c^2v + 27v^3 = 0
\end{aligned}$$

In the latest choice of the parameters, one has the auto-Bäcklund transformation.

The natural question arises now, however. Whether the use of namely the opposite branches was essential for the success? To reply consider another truncated expansion with two manifolds but belonging to the same branch

$$u = V_1 + V_2 + w(x,t) \tag{13}$$

Substituting this into (9)

$$\begin{aligned} E &= 12V_1^2V_2 + 2(6w - c)V_1^2 + 12V_1V_2^2 + 24wV_1V_2 \\ &+2(b - S_1 + 6w^2)V_1 + 2(6w - c)V_2^2 + 2(b - S_2 + 6w^2)V_2 + S_{1x} + S_{2x} \\ &-2w_{xx} + 2cw_x + 2a + 2bw - cS_1 - cS_2 + 4w^3 = 0 \end{aligned}$$

and proceeding as described before, we leads to the following relations ($w = v + c/6$)

$$\begin{aligned} &u = V_1 + V_2 + v(x) + \frac{c}{6} \\ &27a - 9bc - 2c^3 = 0 \\ &\frac{\mathrm{d}}{\mathrm{d}x}V_{1,2} = -V_{1,2}^2 - \frac{1}{48}\Big(36v_x + 6b + c^2 - 36v^2\Big) \\ &48V_1V_2 + 48v(V_1 + V_2) - 12v_x + 60v^2 + (6b + c^2) = 0 \end{aligned}$$

With the equation for v

$$v_{xx} - 6vv_x + 4v^3 + \left(2b + \frac{c^2}{3}\right)v = 0$$

is linearizable, but it is sufficiently more interesting that it itself can be consider successfully in the same manner

$$v = -\frac{(V_3 + V_4)}{2} \tag{14}$$

$$\frac{\mathrm{d}}{\mathrm{d}x}V_{3,4} = -V_{3,4}^2 - \frac{6b + c^2}{12} \tag{15}$$

The technique can be applied with success to a number of nonlinear ordinary and partial differential equations ($\frac{\partial}{\partial x}E$ and $\frac{\partial}{\partial t}E$ are needed to determine both the x- and t-evolution of S_1, S_2, C_1, C_2, and w) considering as hopeless for the original singular manifold method. Some time-depending solutions of the initial equation (10) could be obtained as well.

4 Some remarks and further perspectives

In the previous sections it has been demonstrated that some problems arising in modern applications of the singular manifold method can rigorously be solved if one treats to this technique more attentively. A critical reader, however, has questions: Series of what form could be used in light of the preceding? Could one introduce terms like $V_1^{n_1} V_2^{n_2}$ there? What can the maximal order of such series be with respect to V_i if the related coefficient depends on other functions V_j $(j \neq i)$? How many manifolds are needed to describe the general solution? There is no constructive answer within the frameworks of the singular analysis. Moreover, the technique itself could be considered from another point of view.

Assume that there is a set M of functions $\{\dots, e_i(x, t; \dots, \tau_k, \dots), \dots\}$ finite or infinite such that

$$\frac{\partial e_i}{\partial x} = \sum_l a_{il}(x,t) e_l \tag{16}$$

$$\frac{\partial e_i}{\partial t} = \sum_l b_{il}(x,t) e_l \tag{17}$$

$$e_i e_j = \sum_l c_{ijl}(x,t) e_l \tag{18}$$

The variables τ_k correspond to the arbitrary parameters associated with the differential equations (16), (17); and e_{il}, b_{il}, c_{ijl} satisfy the compatibility conditions for (16)–(18). Then a polynomial nonlinear differential operator $E(x, t, u, u_x, u_t, \dots)$ maps the linear space

$$L\{\dots, e_i, \dots\} = \left\{ \sum_j w_j(x,t) e_j(x, t; \dots, \tau_k, \dots) \right\}$$

into itself. Let there exist finite subsets $W, W_E \subset M$ with the related linear subspaces L_W and L_{W_E} such that

$$E[L] \subseteq L_{W_E}$$

If inserting an element $u \in L$ into the equation $E = 0$ and separating the variables, one has an over-determined maybe but compatible system of equations to $w_j(x,t)$, then such sets can be used for finding solutions of this equation. The fact that for special type matrices $\{a_{ij}\}$, $\{b_{ij}\}$ and $\{c_{ijk}\}$ an infinite linear system (16)–(17) can be a representation of a finite dimensional dynamic system [19, 20]. (For instance, the Riccati equations (2), (3) correspond to the set $\{1, V, V^2, \dots\}$.) It is clear now that the above-mentioned constrains correspond to the cases when one or several vectors e_i appears to be linear dependent on others. In terms of nonlinear systems they correspond to special integrals or invariant manifolds [21].

All these are not new in effect. First, the analogous concept, namely linear subspaces or sets on linear subspaces invariant under a nonlinear differential operator [22, 23], has been applying with success and quickly developing already several years. (See [24] on finding the form of such sets.) Second, in our latest example has finally led to such a system. Really, the solution obtained there can be rewritten in the following form

$$u = V_1 + V_2 - \frac{V_3}{2} - \frac{V_4}{2} + \frac{c}{6}$$
$$2V_1V_2 - V_1V_3 - V_1V_4 - V_2V_3 - V_2V_4 + \frac{3}{8}V_3^2 + \frac{5}{4}V_3V_4 + \frac{3}{8}V_4^2 = 0$$

taking into account (14) and (15), and the nonlinear system is as follows

$$\frac{\mathrm{d}}{\mathrm{d}x}V_{1,2} = -V_{1,2}^2 - \frac{3}{16}V_3^2 + \frac{3}{8}V_3V_4 - \frac{3}{16}V_4^2 - \frac{6b + c^2}{12}$$
$$\frac{\mathrm{d}}{\mathrm{d}x}V_{3,4} = -V_{3,4}^2 - \frac{6b + c^2}{12}$$

In action it is not so tedious to find a suitable system and construct relations as well as in the case of truncated singular expansions. For example, for the traveling wave reduction

$$-cu_x + u_{xx} + u^2 - u = 0$$

of the Fisher's equation [25], the auto-Bäcklund transformation can be found

$$u = -6(V_1 - V_2)^2 + u'$$

and

$$\frac{\mathrm{d}}{\mathrm{d}x}V_{1,2} = -V_{1,2}^2 + 5V_1V_2 + \left(\frac{c + 3u'}{5}\right)V_{1,2} + u'V_{2,1}$$

However, in so doing the main problem is whether such a system is integrable? While in the work [26] it was shown that these systems could be highly important from another viewpoint for understanding of behaviour of nonlinear waves in PDEs, because special superposition properties and solitonic solutions may be associated with them.

In conclusion it should be pointed out once more that the availability of the parameters τ_k is essential for all the aforesaid, because this separates the functions e_i from other ones. In principle, proceeding in the manner demonstrated in the second section, it is possible to obtain some formulas for any ansatz. However, if such a substitution is not justified theoretically, its usefulness depends only on the fortune of a researcher and case.

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