B-bounded semigroups and implicit evolution equations

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Abstract

The aim of this paper is to show an application of the recently introduced B-bounded semigroups [6] in the theory of implicit and degenerate evolution equations. The most interesting feature of this approach is its applicability to problems with non closeable operators.

Key words: B-bounded semigroups, empathy theory, implicit evolution equations
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1 Introduction

Let us consider the Cauchy problem for the implicit evolution equation

$$\begin{aligned} \frac{d}{dt}(Ku) &= Lu \\ \lim_{t\to 0^+}(Ku)(t) &= \overset{\circ}{u} \end{aligned} \tag{1.1}$$

where $K : Z \to X$, $L : Z \to X$, Z, X are, say, Banach spaces, and K, L are linear operators. There is a number of approaches to solving such problems; let us mention e.g. [11, 12], similar in spirit results of [15, 16], [18, 19] where an interesting notion of empathy is introduced, or [21] where a suitable change of space method is used. In this paper we aim neither at a comprehensive treatment of the problem (1.1), nor at an exhaustive comparison of various methods employed to solve it, but we rather describe how a new notion of B-bounded semigroups, introduced in [6, 22] and investigated in [3, 4], can be used in this field.

One of the "natural" ways of approaching (1.1) would be to factor out K and, provided it is invertible, to consider a standard Cauchy problem with the operator $K^{-1}L$ on the right-hand side. In some cases, however, the operator K is not closeable and therefore there is no way the "time derivative" or the limit at $t = 0^+$ can commute with K. Thus, it is reasonable to study (1.1) as it stands. In [19, 20] this problem is treated by introducing a pair of evolution families, called an empathy. As we shall see, the theory of B-bounded semigroups provides

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another convenient way of performing this "impossible" commutations by passing to a specially constructed space related to K. In this it is similar to the approach of [21] but some assumptions on K adopted in this paper are less restrictive.

It is worthwhile to note that the method of B-bounded semigroups does not require X to be a Banach space (in fact X is not required to have any structure but linear) and consequently the operators K and L are not assumed to have any standard topological properties when considered separately; we require, however, their good behaviour in the abstract extrapolation space X_B introduced in [4] or, equivalently, a good behaviour of the operator $K^{-1}L$ (or its suitable realization) in Z. In this the idea is similar to that of [9, 14], where the authors also seek a modification of the original space in which the given operators are e.g. closeable. Our method on one hand is less general, as the modified space is defined in a prescribed way by the operators appearing in the problem, but on the other hand this space may be much less restrictive than that stipulated in *op. cit.*

As we mentioned before, we don't give a survey of all available methods for solving (1.1); instead we shall demonstrate links between B-bounded semigroups and the empathy theory which is also focused on solving problems with possibly non closeable K, and with the method employed by Showalter in [21], which seems to be a particular case of the B-bounded semigroup method.

To keep the exposition within a reasonable length we shall focus on linear operators with K invertible. A generalization to multivalued and nonlinear cases can be done along the lines of e.g. [11, 12, 15, 21] with only minor difficulties (see also Remark 3.3). We intend to pursue this topic provided interesting applications arise.

It is also worthwhile to note that despite superficial similarities of B-bounded semigroups and C-existence families, these two notion coincide only for a very restricted class of operators. This question is addressed in detail in [5].

2 B-bounded semigroups revisited

We start with recalling basic facts from the theory of B-bounded semigroups and give generalizations relevant to the theory of implicit evolution equations.

Let us consider the standard abstract Cauchy problem in a Banach space X:

$$\begin{aligned} \frac{du}{dt} &= Au, \\ \lim_{t\to 0^+} u(t) &= \overset{\circ}{u}. \end{aligned} \tag{2.1}$$

Very often the existence of the semigroup $(\exp(tA))_{t\geq 0}$ describing the evolution of the system is established in a non-constructive way. This is especially the case when the positivity methods are employed, see e.g. [1]. Then very little quantitative information on the evolution is available. On the other hand, there may exist an operator B such that $t \to Be^{tA}$ can be calculated constructively yielding some information about the evolution. An example of this type, pertaining to the transport equation with multiplying boundary conditions, was analysed in [22] and has prompted one of the authors to define a class of evolution families which behave well if looked at through the "lenses" of another operator, and which can be thought of as generalizations of $\{Be^{tA}\}_{t\geq 0}$. Such families, called B-bounded semigroups, have been introduced in [6], and analysed and applied to various problems in a few papers [3, 4, 7, 8].

The definition of B-quasi bounded semigroups as introduced in [7] (with some modifications due to the author of this paper) reads as follows.

Definition 2.1 *Let $(A, D(A))$ be a linear operator in a Banach space X and $(B, D(B))$ be another linear operator from X to another Banach space Z with $D(A) \subset D(B)$, and let for some $\omega \in \mathbb{R}$ the resolvent set of A satisfy*

$$\rho(A) \supset]\omega, \infty[. \tag{2.2}$$

A one-parameter family of operators $(Y(t))_{t\geq 0}$ from X to Z, which satisfies

1. *$D(Y(t)) =: \Omega \supseteq D(B)$, and for any $t \geq 0$ and $f \in D(B)$*
$$\|Y(t)f\|_Z \leq M \exp(\omega t)\|Bf\|_Z, \tag{2.3}$$
2. *The function $t \to Y(t)f \in C([0, \infty[, Z)$ for any $f \in \Omega$,*
3. *For any $f \in \Omega_0 := \{f \in D(A) \cap D(B);\ Af \in \Omega\} \subset D(A) \cap D(B)$*
$$Y(t)f = Bf + \int_0^t Y(s)Afds, \qquad t \geq 0, \tag{2.4}$$

is called a B-quasi bounded semigroup generated by A.

To shorten notation, if A generates a B-bounded semigroup satisfying the above conditions, then we write $A \in B - \mathcal{G}(M, \omega, X, Z)$. We shall also shorten the name saying that the family $(Y(t))_{t\geq 0}$ defined in Definition 2.1 is a B-bounded semigroup generated by A. We will also use the standard notation $A \in \mathcal{G}(M, \omega, X)$ to express the fact that A is the generator of a C_0-semigroup in X with the Hille-Yoshida constants M and ω.

Remark 2.1 It follows [3] shows that the assumptions: $D(A) \subset D(B)$ and (2.2) can be replaced by a single assumption that for $\lambda > \omega$ the operator

$$(\lambda I - A) : D_B(A) \to D(B) \tag{2.5}$$

where

$$D_B(A) = \{x \in D(A) \cap D(B);\ Ax \in D(B)\}, \tag{2.6}$$

is bijective. Note, that this requirement is purely algebraical. ∎

We shall recall here the main results of [4] together with some recent generalizations due to [3].

The main role in the considerations of [4] is played by the space X_B which is the completion of the quotient space $D(B)/N(B)$ with respect to the semi-norm $\|\cdot\|_B = \|B\cdot\|_Z$. It is known that then $D(B)/N(B)$ is isometrically isomorphic to a dense subspace of X_B, say $\mathcal{X}$. The canonical injection of X into X_B (and onto $\mathcal{X}$) will be denoted by $\mathfrak{p}$. In a standard way B extended by density to an isometry $\mathfrak{B} : X_B \to Z$.

An important observation is that if A generates a B-bounded semigroup, then A preserves cosets of $D(B)/N(B)$ and therefore it can be defined to act from $\mathfrak{p}D_B(A) \subset \mathcal{X}$ into $\mathcal{X}$. We denote by A_B the part of A in $D(B)$, i.e. $A_B = A|_{D_B(A)}$. It can be also proved [4] that if $A \in B - \mathcal{G}(M, \omega, X, Z)$, then the shift of A to $\mathcal{X}$ is closeable in X_B; we denote its closure by $\mathfrak{A}$.

To simplify the notation we will use the same notation for the operators defined A and B and their shifts to $\mathcal{X}$, which is possible by [4]; with this convention the injection $\mathfrak{p}$ becomes the identity (or more precisely projection) and for any operator $\mathfrak{C}$ defined in X_B and $x \in D(B)$, the symbol $\mathfrak{C}x$ is to be understood as $\mathfrak{C}\mathfrak{p}x$, if the latter is defined.

Let us introduce the subspace $Z_B = \overline{R(B)}$ (the closure of the range of B in Z). The main result of [4] (Theorem 4.1) reads as follows.

Theorem 2.1 *If $A \in B - \mathcal{G}(M, \omega, X, Z)$ and $\overline{B[D_B(A)]}^Z = Z_B$, then $\mathfrak{A} \in \mathcal{G}(M, \omega, X_B)$. Conversely, if there is $\mathcal{A} \supset A$ such that $\mathcal{A} \in \mathcal{G}(M, \omega, X_B)$, then $\mathcal{A} = \mathfrak{A}$ and $A \in B - \mathcal{G}(M, \omega, X, Z)$.*

The B-bounded semigroup $(Y(t))_{t\geq 0}$ for $x \in D(B)$ is given by

$$Y(t)x = \exp(t\mathfrak{B}\mathfrak{A}\mathfrak{B}^{-1})Bx = \mathfrak{B}\exp(t\mathfrak{A})x. \tag{2.7}$$

The assumption that $B[D_B(A)]$ is dense in Z_B can be discarded if Z (and consequently Z_B) are reflexive spaces ([4], Corollary 4.1). Recently Arlotti [3] proved that if the B-bounded semigroup satisfies the additional condition:

$$\forall_{x\in D(B)}\, Y(0)x = Bx, \tag{2.8}$$

then $B[D_B(A)]$ is dense in Z_B (or equivalently, $D_B(A)$ is dense in X_B). Note that (2.4) gives (2.8) only for $x \in \Omega_0$ which in most cases reduces to $D_B(A)$. It is easy to see that the converse is also true. Therefore, if (2.8) holds, then the density assumption in Theorem 2.1 can be omitted.

Since the space X_B is in many cases rather difficult to handle, Theorem 2.1 is most often used in the following version ([4], Theorem 4.3, [3], Theorem 2.1).

Theorem 2.2 *Let the operators A and B satisfy the conditions of Definition 2.1. Then A is the generator of a B-quasi bounded semigroup satisfying (2.8) iff the following conditions hold:*

1. *$B[D_B(A)]$ is dense in Z_B,*
2. *there exist $M > 0$ and $\omega \in \mathbb{R}$ such that for any $x \in D(B)$, $\lambda > \omega$ and $n \in \mathbb{N}$:*

$$\|B\,(\lambda I - A)^{-n}\,x\|_Z \leq \frac{M}{(\lambda-\omega)^n}\|Bx\|_Z. \tag{2.9}$$

If we don't require $(Y(t))_{t\geq 0}$ to satisfy (2.8), then condition 1. is sufficient but not necessary.

The main point in the proof of Theorem 2.2 is the observation that (2.9) can be extended to hold on the entire X_B. This allows a useful corollary.

Corollary 2.1 *Let operators A and B satisfy the conditions of Definition 2.1 and let also assumptions 1. and 2. of Theorem 2.2 are satisfied.*

1. *If the estimate (2.9) is satisfied for $n = 1$ with $M = 1$ and $\omega = 0$, then $\mathfrak{A}$ generates a semigroup of contractions in X_B and consequently $A \in B - \mathcal{G}(1, 0, X, Z)$.*

2. *If the estimate*
$$\|B(\lambda I - A)^{-1} x\|_Z \leq \frac{M}{|\lambda - \omega|}\|Bx\|_Z. \tag{2.10}$$
holds for $\lambda \in S_\theta = \{\lambda \in \mathbb{C};\ |Arg\lambda| \leq \frac{\pi}{2} + \theta, \theta > 0\}$, then $\mathfrak{A}$ generates an analytic semigroup in X_B and consequently $A \in B - \mathcal{G}(M', \omega, X, Z)$ for some constant M'. ■

2.1 Further improvement of the generation theorem

Let us note that it is not necessary for A to generate a semigroup in X. Thus, the existence of a B-bounded semigroup is no longer related to the existence of $(\exp(tA))_{t\geq 0}$, as was the case in the motivating example of [6]. As was mentioned before, the assumption that $[\omega, \infty[\subseteq \rho(A)$ for some $\omega \in \mathbb{R}$ was replaced by the requirement that $(\lambda - A) : D_B(A) \to D(B)$ is bijective. It follows that this assumption can be relaxed even further. A detailed discussion of this topic together with the proofs can be found in [5]. Here we shall sketch the main results.

Our aim is to replace assumption (2.2) by a weaker one which would require only the bijectivity of a suitable extension of A. In fact, in the proof of Theorem 2.1 the assumption (2.2) was used to show that $[\omega, \infty[\subset \rho(\mathfrak{A})$. Thus, what we really need is the Hille-Yoshida estimate valid on some dense subspace $\mathfrak{X}$ of X_B. Moreover, as we are using the pseudo resolvent identity, we must have that $D_\lambda = (\lambda I - A)^{-1}\mathfrak{X} \subset \mathfrak{X}$ for $\lambda > \omega$ and this yields that D_λ must be independent of λ [5]. Finally, as our starting point are X and the operators defined in it, the space $\mathfrak{X}$ must be accessible from X in the sense of the operator closure in X_B. All these indicate that we can free ourselves from any topological structure of X.

Therefore we adopt the following new assumptions on A and X.

(2.1') The space X is a linear space and the operator A_B is closeable in X_B. Denoting $\mathfrak{A} = \overline{A_B}^{X_B}$, we assume further that there exist subspaces: $\mathfrak{X}$ satisfying $D(B) \subseteq \mathfrak{X} \subseteq X_B$, and $D_B(A) \subset D \subset \mathfrak{X} \cap D(\mathfrak{A})$ such that $(\lambda - \mathfrak{A}|_D) : D \to \mathfrak{X}$ is bijective for all $\lambda > \omega$.

We have then the following theorem [5].

Theorem 2.3 *Let the operators A and B satisfy the conditions of Definition 2.1 with assumption (2.2) replaced by assumption* $2.1'$. *Then $A \in B - \mathcal{G}(M, \omega, X, Z)$ and (2.8) holds if and only if the following conditions are satisfied:*

1. *$B(D)$ is dense in Z_B,*

2. *there exist $M > 0$ and $\omega \in \mathbb{R}$ such that for any $\mathfrak{y} \in \mathfrak{X}$, $\lambda > \omega$ and $n \in \mathbb{N}$:*
$$\|\mathfrak{B}(\lambda I - \mathfrak{A}|_D)^{-n} \mathfrak{y}\|_Z \leq \frac{M}{(\lambda - \omega)^n}\|\mathfrak{B}\mathfrak{y}\|_Z. \tag{2.11}$$

If we don't assume (2.8), then the assumption 1. is sufficient but not necessary. In both cases the B bounded semigroup is given again by Eq. (2.7). ■

Example 2.1 Let us consider $X = L_2(\mathbb{R}, e^{x^2}dx)$, $Au = \partial_x u$ on the maximal domain, and $(Bu)(x) = e^{-\frac{x^2}{2}}u(x)$. Clearly, $B : X \to X$ is a continuous operator. Moreover, $\|Bu\|_X = \|u\|_{L_2(\mathbb{R})}$ and since $C_0^\infty(\mathbb{R}) \subset X$, we can identify X_B wit $L_2(\mathbb{R})$. Let us consider the closure $\mathfrak{A}$ of A, that is, we take a sequence $(u_n)_{n\in\mathbb{N}}$ of elements of $D(A)$ such that $u_n \to u$ and $\partial_x u_n \to g$ as $n \to \infty$ in $L_2(\mathbb{R})$. However, this is the same as the closure of $D(A)$ in $W_2^1(\mathbb{R})$, and as $C_0^\infty(\mathbb{R}) \subset D(A)$ is dense in $W_2^1(\mathbb{R})$, we obtain that $\mathfrak{A}u = \partial_x u$ for $u \in W_2^1(\mathbb{R})$. Thus, $\mathfrak{A}$ generates a semigroup of contractions in X_B and therefore $(Y(t)u)(x) = e^{-\frac{x^2}{2}}u(t+x)$ satisfies conditions 1-3 of Definition 2.1. By standard argument (see also [5]) one can prove however that, $\lambda I - A : D(A) \to X$ is not bijective for any λ, hence the Definition 2.1 is not applicable. ■

2.2 Special case

It is of interest to determine conditions under which X_B is not an abstract space but can be identified with a subspace of X. We have the following theorem.

Theorem 2.4 *Let X, Z be Banach spaces and $B : X \to Z$ be an injective operator. The following conditions are equivalent:*

(i) X_B has the following properties:

(i') each coset $\tilde{x} \in X_B$ contains a sequence $(x_n)_{n\in\mathbb{N}}$ converging in the norm of X to some $x \in X$, and x is the limit of any other X-Cauchy sequence in $\tilde{x}$,

(i") if $(x_n)_{n\in\mathbb{N}} \in \tilde{x}$, $(y_n)_{n\in\mathbb{N}}$ satisfy $\|x_n - y_n\|_X \to 0$ as $n \to \infty$ and $(y_n)_{n\in\mathbb{N}} \in \tilde{y}$ for some $\tilde{y} \in X_B$, then $\tilde{x} = \tilde{y}$,

(ii) The operator B is closeable and B^{-1} is bounded,

(iii) There is an isometric isomorphism $T : X_B \to X'_B \hookrightarrow X$ which satisfies $T|_{D(B)} = Id$.

Proof. (i) $\Leftrightarrow$ (ii). Since for each $\tilde{x} \in X_B$, there is exactly one representative $(x_n)_{n\in\mathbb{N}}$ which converges in X to, say, x, we the formula

$$T\tilde{x} = T[(x_n)_{n\in\mathbb{N}}] = \lim_{n\to\infty} x_n = x$$

defines an operator $T : X_B \to X$. Moreover, as $D(B) \ni \tilde{x} = [(x, x, \ldots)]$, we get $T|_{D(B)} = Id$. Next define,

$$X'_B = \{x \in X;\ x = \lim_{n\to\infty} x_n \text{ for some } (x_n)_{n\in\mathbb{N}} \subset D(B) \text{ such that } (Bx_n)_{n\in\mathbb{N}} \text{ is Cauchy}\}$$

By the assumption in (i"), $T : X_B \to X'_B$ is a bijection. This allows to induce a norm in X'_B by

$$\|x\|_{X'_B} = \|[(x_n)_{n\in\mathbb{N}}]\|_B = \lim_{n\to\infty} \|Bx_n\|, \tag{2.12}$$

where $x = \lim_{n\to\infty} x_n$. In fact, if $x \in X'_B$, then there must be $\tilde{x} \in X_B$ such that for some $(x_n)_{n\in\mathbb{N}} \in \tilde{x}$, $\lim_{n\to\infty} x_n = x$. By the assumption, there is only one such $\tilde{x}$, hence $x \in X'_B$ determines uniquely $\tilde{x}$. Since $\|[(x_n)_{n\in\mathbb{N}}]\|_B$ is well-defined by the last equality in (2.12), $\|\cdot\|_{X'_B}$ is well-defined. This norm turns T into an isometry and thus X'_B becomes a Banach space.

Thanks to the assumption (i") again, we can define the shift $\mathfrak{B}' : X_B \to Z$ by

$$\mathfrak{B}'x = \mathfrak{B}\widetilde{x} = \lim_{n\to\infty} Bx_n \tag{2.13}$$

where $(x_n)_{n\in\mathbb{N}}$ is as before. We have

$$B : D(B) \overset{onto}{\to} ImB$$

and

$$\mathfrak{B}' : X'_B \overset{onto}{\to} \overline{ImB}.$$

Since $X'_B \subset X$, then we can consider $(\mathfrak{B}', X'_B)$ as an unbounded operator in X. Let $(x_n)_{n\in\mathbb{N}} \subset X'_B$ converges to $x \in X$ and $(\mathfrak{B}'x_n)_{n\in\mathbb{N}} \subset X$ converges to $y \in X$, both in the norm of X. The latter implies that $(x_n)_{n\in\mathbb{N}}$ converges to some $\overline{x} = \lim_{n\to\infty} x_n$ in X'_B. From the assumption (i'), for each $n \in \mathbb{N}$ there is a sequence $(x^{(n)}{}_k)_{k\in\mathbb{N}} \in T^{-1}x_n$, $x_k^{(n)} \in D(B)$, converging to x_n in X. From the construction of cosets in X_B, $(Bx_k^{(n)})_{k\in\mathbb{N}}$ is a Cauchy sequence and as a consequence, for any $n \in \mathbb{N}$, $Bx_k^{(n)} \to Bx_n$ as $k \to \infty$. Indeed, denote $\widetilde{x_n} = T^{-1}x_n = [(x_k^{(n)})_{k\in\mathbb{N}}]$ and $\widetilde{x_l^{(n)}} = T^{-1}x_l^{(n)} = [(x_l^{(n)}, x_l^{(n)}, \ldots)]$. Since $(Bx_k^{(n)})_{k\in\mathbb{N}}$ is a Cauchy sequence, we obtain, for any sufficiently large $l \in \mathbb{N}$

$$\|\widetilde{x_n} - \widetilde{x_l^{(n)}}\|_{X_B} = \lim_{k\to\infty} \|Bx_k^{(n)} - Bx_l^{(n)}\|_Z < \epsilon,$$

that is, $\lim_{l\to\infty} \widetilde{x_l^{(n)}} = \widetilde{x_n}$ in X_B. Since T is an isomorphism, and by Eqs. (2.12), (2.13) we obtain that for any $n \in \mathbb{N}$,

$$\lim_{k\to\infty} Bx_k^{(n)} = \mathfrak{B}'x_n.$$

For any $r \in \mathbb{N}$ we can find $n_r \in \mathbb{N}$ such that

$$\|x_{n_r} - x\|_X < \frac{1}{r}, \qquad \|Bx_{n_r} - y\|_Z < \frac{1}{r} \tag{2.14}$$

and for such a fixed n_r we select $k_r \in \mathbb{N}$ satisfying

$$\|x_{k_r}^{(n_r)} - x_{n_r}\|_X < \frac{1}{r}, \qquad \|Bx_{k_r}^{(n_r)} - Bx_{(n_r)}\|_Z < \frac{1}{r}. \tag{2.15}$$

Define $\phi_r = x_{k_r}^{(n_r)}$. For any $r \in \mathbb{N}$, $\phi_r \in D(B)$. Fixing $\epsilon > 0$ we find $r_0 > 2/\epsilon$, and for any $r > r_0$ we have by estimates (2.14) and (2.15)

$$\begin{aligned} \|\phi_r - x\|_X &< \frac{2}{r} < \epsilon, \\ \|B\phi_r - y\|_Z = \|B\phi_r - \mathfrak{B}'\bar{x}\|_Z &< \frac{2}{r} < \epsilon \end{aligned}$$

which means that $(\phi_r)_{r\in\mathbb{N}}$ is a sequence defining $\overline{x}$ and from assumption (i') we obtain $\overline{x} = x \in X'_B$. Thus $(\mathfrak{B}', X'_B)$ is a closed operator in X. Therefore $X'_B = D(\mathfrak{B}')$, equipped with the norm of the graph, is a Banach space, and $\mathfrak{B}' : X'_B \overset{onto}{\to} \overline{ImB}$ is a continuous bijection. Therefore the inverse of $\mathfrak{B}'$ is bounded in the norm of the graph, and for its restriction B^{-1} we obtain

$$\forall_{y\in ImB}\ \|x\|_X \le \|x\|_X + \|Bx\|_Z = \|x\|_{D(\mathfrak{B}')} \le K\|Bx\|_Z$$

for some constant K, which gives the boundedness of $B^{-1} : Z \to X$.

Let now $B : X \to Z$ be a closeable operator with a bounded inverse. For any $\widetilde{x}$, let $(x_n)_{n\in\mathbb{N}} \in \widetilde{x}$, that is, $(Bx_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in Z. By the boundedness of B^{-1}, $(x_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in X and its limit $x \in D(\overline{B})$. Also, by the boundedness of B^{-1}, if any other Cauchy sequence $(By_n)_{n\in\mathbb{N}}$ is such that $(B(x_n - y_n))_{n\in\mathbb{N}}$ is convergent to 0, then $\lim\limits_{n\to\infty} y_n = x$. Next, if $x_n \to x$ and $y_n \to y$ in X determine two cosets $\widetilde{x}$ and $\widetilde{y}$, then by definition both $(Bx_n)_{n\in\mathbb{N}}$ and $(By_n)_{n\in\mathbb{N}}$ converge. Since B is closeable, they must have the same limit, that is, $\widetilde{x} = \widetilde{y}$.

(ii)$\Leftrightarrow$ (iii). From (ii) to (iii) the theorem follows easily, as in the previous part we constructed a required isometry, and since $X'_B = D(\overline{B})$, set-theoretically and topologically, $X'_B \hookrightarrow X$.

Conversely, let T be the stipulated isomorphism and define for $x \in X'_B$

$$\mathbb{B}x = \mathfrak{B}T^{-1}x$$

Since $X'_B \hookrightarrow X$, then $T : X_B \to X$ is continuous, and therefore T^{-1} is closed as an operator defined in X. Since $\mathfrak{B}$ is an isomorphism, then $\mathbb{B} : X \to \overline{ImB}$ is closed. But since $T|_{D(B)} = Id$, $\mathbb{B}$ is an extension of B and therefore B is closeable. Moreover, $\mathbb{B}^{-1} = T\mathfrak{B}^{-1} : \overline{ImB} \to X$ is continuous, therefore $B^{-1} = T\mathfrak{B}^{-1}|_{ImB}$ is also bounded. ■

Remark 2.2 It is interesting to note that there exist operators which are injective with bounded inverse but which are not closeable [18]. ■

If B is not invertible, then we can still obtain a similar result, however in a less compact form. Firstly we note that it is impossible to follow the way leading to Theorem 2.1 as already the quotient space $D(B)/N(B)$ is not a subspace of X.

We start with some preliminaries. Following e.g. [23], p. 766-7 we can write split $D(B)$ as an algebraic direct sum

$$D(B) = N(B) \oplus D_B \tag{2.16}$$

for some linear space D_B such that $B : D_B \to R(B)$ is a bijection (clearly D_B is not unique, but $R(B)$ is independent of the choice of D_B). We denote by $B_R^{-1} : R(B) \to D_B$ the right inverse to B having D_B as its range. Conversely, given any right inverse B_R^{-1} to B defined on $R(B)$ we can always split $D(B)$ as in (2.16) with $D_B = R(B_R^{-1})$.

We can now construct a number of extrapolation spaces $\widetilde{X}_B$ completing various D_B's in the norm $\|B \cdot \|_Z$. It is important to note all these spaces are isometrically isomorphic to each other and also to the space X_B. Indeed, let $\widetilde{X}_B$ be a completion of D_B with respect to the abovementioned norm. Denote by $\widetilde{\mathfrak{B}} : \widetilde{X}_B \to \overline{R(B)}^Z$ the extension by continuity of (B, D_B). As in the original case, this is an isometric isomorphism. However, also $\mathfrak{B} : X_B \to \overline{R(B)}^Z$ is an isometric isomorphism, therefore $\mathfrak{T} = \widetilde{\mathfrak{B}}^{-1}\mathfrak{B} : X_B \to \widetilde{X}_B$ is an isometric isomorphism. Let $x \in D_B \subset D(B)$, then $\mathfrak{B}x = Bx \in R(B)$, but $\widetilde{B}^{-1}$ is here the extension of the inverse of $B|_{D_B}$, hence we obtain

$$\mathfrak{T}|_{D_B} = Id. \tag{2.17}$$

Now we can formulate a generalization of Theorem 2.4

Corollary 2.2 *Let X, Z be Banach spaces, $X = N(B) \oplus D_B$ and D_B be the range of some right inverse B_R^{-1}. The following are equivalent* *(i) The operator (B, D_B) is closeable and B_R^{-1} is bounded,*

(ii) There is an isomorphism $T : X_B \to X'_B \hookrightarrow X$ which satisfies $T|_{D_B} = Id$.

If either of the above condition holds and additionally both $N(B)$ and D_B are closed, then $(B, D(B))$ is also closeable.

Proof. (i)$\Leftrightarrow$ (ii). The proof is straightforward by repeating the proof of the equivalence (ii)$\Leftrightarrow$(iii) of Theorem 2.4 with $(B, D(B))$ replaced by (B, D_B) and applying Eq. (2.17).

To prove the last statement let us consider $x_n \to 0$ in X, $x_n \in D(B), n = 1, \ldots,$ such that $Bx_n \to y$ in Z. We write $x_n = x'_n + x''_n$ with $x'_n \in N(B)$ and $x''_n \in D_B$ for $n = 1, \ldots,$. Since $Bx_n = Bx''_n$, we see that $(Bx''{}_n)_{n\in\mathbb{N}}$ is also convergent, and by the boundedness of the right inverse, $(x''{}_n)_{n\in\mathbb{N}}$ is also convergent in X. Therefore $(x'{}_n)_{n\in\mathbb{N}}$ converges and by the closedness of both subspaces we get $x' = \lim_{n\to\infty} x'_n \in N(B)$ and $x'' = \lim_{n\to\infty} x''_n \in D_B$. On the other hand $x' + x'' = 0$ which yields $x' = x'' = 0$ by Eq. (2.16). Thus $x''_n \to 0$ and $Bx''_n \to y$ implies $y = 0$ by closeability of (B, D_B) and therefore $(B, D(B))$ is also closed. ■

If we have the case described in Theorem 2.4, the operator $\mathfrak{A}$ also becomes much simpler.

Theorem 2.5 *If B is a closeable operator such that B^{-1} is bounded, and A is closeable in X with $\lambda I - \overline{A}$ injective for some $\lambda > \omega$, and moreover $A \in B - \mathcal{G}(M, \omega, X, Z)$, then*

$$\mathfrak{A} = \overline{A}|_{D_{\overline{B}}(\overline{A})} \tag{2.18}$$

where

$$D_{\overline{B}}(\overline{A}) = \{x \in D(\overline{A}) \cap D(\overline{B});\ \overline{A}x \in D(\overline{B})\}$$

Proof. Let $x \in D(\mathfrak{A})$. Then there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in $D_B(A)$ such that $Bx_n \to \overline{B}x$ and $BAx_n \to \overline{B}\mathfrak{A}x$ as $n \to \infty$. From the proof of Theorem 2.4 it follows that also $\overline{B}^{-1}$ is bounded, that is, $x_n \to x$ and $Ax_n \to \mathfrak{A}x$ in X as $n \to \infty$. From this formulae we get that $x \in D(\overline{B})$, $x \in D(\overline{A})$ with $\mathfrak{A}x = \overline{A}x$ and $\overline{A}x \in D(\overline{B})$, that is, $D(\mathfrak{A}) \subset D_{\overline{B}}(A)$.

Conversely, for $\lambda > \omega$, $\lambda I - \mathfrak{A} : D(\mathfrak{A}) \overset{onto}{\to} D(\overline{B})$, and for $x \in D_{\overline{B}}(A)$, $\lambda x - \overline{A}x \in D(\overline{B})$. Thus for some $y \in D(\mathfrak{A})$ we obtain

$$\lambda y - \mathfrak{A}y = \lambda x - \overline{A}x,$$

but $y \in D(\mathfrak{A}) \subset D_{\overline{B}}(A)$ yields $\mathfrak{A}y = Ay$, and by injectivity of $\lambda I - \overline{A}$ we get $x = y \in D(\mathfrak{A})$. ■

Example 2.2 Let us consider $X = L_2(\mathbb{R}, e^{-x^2}dx)$, $Au = \partial_x u$ on the maximal domain, and $(Bu)(x) = e^{\frac{x^2}{2}}u(x)$. $B : X \to X$ is an unbounded operator and since $\|Bu\|_X = \|u\|_{L_2(\mathbb{R})}$, we see that $D(B) = L_2(\mathbb{R})$. Since $B(D(B)) = X$, we obtain that $X_B = D(B) = L_2(\mathbb{R})$ by Theorem 2.4. Then $D_B(A) = W_2^1(\mathbb{R})$ and A generates a contraction semigroup, say $(T(t))_{t\geq 0}$, in $L_2(\mathbb{R})$. Thus $Y(t)u = BT(t)u = e^{\frac{x^2}{2}}u(t + x)$ is the B-bounded semigroup generated by (A, B).

Note that here neither $D(A) \subset D(B)$, nor $\rho(A) \supset [\omega, \infty[$, but the (2.5) is satisfied. ■

3 B-bounded semigroups and implicit evolution equations

3.1 X_B-solutions of implicit evolution equations

Let us consider again the original Cauchy problem (1.1). It is often the case that the original spaces X and Z are not the most convenient spaces from the mathematical point of view. We are usually interested to keep the values of the solution in the original space which may be related to some physical properties like, finite total energy space, finite mass, etc., but for (1.1) to hold in the strict sense may be too restrictive and often it is enough that it holds in some other Banach (or even linear topological) space $\widetilde{X}$ with K and L replaced by appropriate extensions $\widetilde{K}$ and $\widetilde{L}$ acting from Z to $\widetilde{X}$. This concept is similar to the differentiation in the sense of distributions; a related concept for semigroups is sometimes called the Haraux extrapolation [13].

To be able to link K and L with $\widetilde{K}$ and $\widetilde{L}$ we restrict these extensions to the closures of respective operators. In other words, $D(L)$ and $D(K)$ are required to be cores for $\widetilde{L}$ and $\widetilde{K}$, respectively.

As we mentioned in Introduction, in general, thanks to Theorem 2.3, we don't need any topological structure in X and therefore there is no need to introduce any topological assumptions on K and L separately - as we shall see, these will be replaced by appropriate assumption imposed on either LK^{-1} or $K^{-1}L$.

Let us introduce the following definition.

Definition 3.1 *Let $X \subset \widetilde{X}$ and $\widetilde{L} = \overline{L}^{\widetilde{X}}$, $\widetilde{K} = \overline{K}^{\widetilde{X}}$. A Z-valued function $t \to u(t)$ is called an $\widetilde{X}$-solution of the problem (1.1) if it is a classical solution of the problem*

$$\begin{aligned} \frac{d}{dt}(\widetilde{K}u) &= \widetilde{L}u \\ \lim_{t\to 0^+}(\widetilde{K}u)(t) &= \mathring{u}, \end{aligned} \tag{3.1}$$

that is, $t \to \widetilde{K}u(t)$ is continuously differentiable in $\widetilde{X}$, the differential equation holds for all $t > 0$ in $\widetilde{X}$, and the initial condition holds as a limit in the topology of $\widetilde{X}$.

With this definition we can formulate the following theorem.

Theorem 3.1 *Let us suppose that we are given operators $K : D(K) \to X$ and $L : D(L) \to X$ with $D(L), D(K) \subset Z$, where Z is a Banach and X is a linear space. Assume that K is a densely defined, one-to-one operator. Define $A = LK^{-1}$ with the natural domain $D(A) = K(D(L))$ and $B = K^{-1}$. If $A \in B - \mathcal{G}(M, \omega, X, Z)$ (in the sense of Subsection 2.1) , then, for any $x \in D_B(A) = \{x \in K(D(L) \cap D(K));\ LK^{-1}x \in ImK\}$, the function $t \to Y(t)x$, where $(Y(t))_{t\geq 0}$ is the B-bounded semigroup generated by A, is an X_B-solution of the problem (1.1).*

Proof. Since $Y(t)x = \mathfrak{B}\exp(t\mathfrak{A})x$, where $\mathfrak{B} : X_B \to Z$ is an isomorphism and $x \in D(\mathfrak{A})$, we obtain

$$\frac{d}{dt}Y(t)x = \frac{d}{dt}\mathfrak{B}\exp(t\mathfrak{A})\mathfrak{p}x = \mathfrak{B}\frac{d}{dt}\exp(t\mathfrak{A})\mathfrak{p}x = \mathfrak{B}\mathfrak{A}\exp(t\mathfrak{A})\mathfrak{p}x$$

which can be rewritten as

$$\frac{d}{dt}\mathfrak{B}^{-1}Y(t)x = \mathcal{A}\mathfrak{B}^{-1}Y(t)x \tag{3.2}$$

Similarly,

$$\lim_{t\to 0^+} \mathfrak{B}^{-1}Y(t)x = x \tag{3.3}$$

To complete the proof we must show that $\mathfrak{B}^{-1}$ is the closure of $K = B^{-1}$ and that $\mathfrak{A}\mathfrak{B}^{-1}$ is the closure of $L = AB^{-1}$. The first statement follows from the fact that $\mathfrak{B}$, being the extension by density, is also the closure of B and that the operation $(x, y) \to (y, x)$, which transforms graph of an operator onto the graph of its inverse, is an isomorphism.

Let us consider the second operator. By construction $\mathfrak{A}$ is the closure of LK^{-1} defined on

$$D_B(A) = D(LK^{-1}) = \{x \in K(D(L) \cap D(K));\ LK^{-1}x \in ImK\} \tag{3.4}$$

in X_B. Indeed, $D_B(A) = \{x \in D(A) \cap D(B);\ Ax \in D(B)\}$. In our case, $x \in D(A) = K(D(L))$ if and only if $x = Ky', y' \in D(L)$. On the other hand, $x \in D(B) = ImK$ if and only if $x = Ky", y" \in D(K)$. Since K is injective, $y" = y' \in D(L) \cap D(K)$. Moreover, the second requirement reads $LK^{-1}x = Ky, y \in D(K)$, which gives $K^{-1}LK^{-1}x \in D(K)$. We know that $\mathfrak{B}$ is the closure of K^{-1} in X_B. Consider the operator $(\mathfrak{A}\mathfrak{B}^{-1}, \mathfrak{B}D(\mathfrak{A}))$; it is well-defined and closed, as $\mathfrak{A}$ is closed and $\mathfrak{B}^{-1}$ is continuous. Also, since $D(\mathfrak{A})$ is dense in X_B and $\mathfrak{B}$ is an isomorphism, this is a densely defined operator.

We shall prove that the operator $(\mathfrak{A}\mathfrak{B}^{-1}, \mathfrak{B}D(\mathfrak{A}))$ is the closure of $(L, D(L))$ in X_B. Let $x \in D(L) \cap D(K)$, then $Kx \in K(D(L) \cap D(K)) \subset D(A)$. For such x we have

$$\mathfrak{A}\mathfrak{B}^{-1}x = \mathfrak{A}Kx = LK^{-1}Kx = Lx,$$

hence $\mathfrak{A}\mathfrak{B}^{-1}$ is an extension of $(L, D(L))$.

We know that $\mathfrak{A}$ is the X_B-closure of LK^{-1} from $D_B(A)$, that is for any $y \in D(\mathfrak{A})$ there is a sequence $(y_n)_{n\in\mathbb{N}}$ of elements belonging to $K(D(L) \cap D(K))$ and such that $LK^{-1}y_n \in ImK$, which satisfies $y_n \to y$ in X_B and

$$\mathfrak{A}y = X_B - \lim_{n\to\infty} LK^{-1}y_n.$$

Let us take arbitrary $x \in \mathfrak{B}(D(\mathfrak{A}))$; then $x = \mathfrak{B}y$ for some $y \in D(\mathfrak{A})$ or $y \in \mathfrak{B}^{-1}x$. Now

$$\mathfrak{A}y = \mathfrak{A}\mathfrak{B}^{-1}x = X_B - \lim_{n\to\infty} LK^{-1}y_n,$$

but $y_n \in K(D(L) \cap D(K))$, that is, $y_n = Kx_n$ for some $x_n \in D(L) \cap D(K)$. However, on $ImK = D(B^{-1})$ we have $\mathfrak{B}x = Bx = K^{-1}x$ and

$$\mathfrak{B}y_n = K^{-1}Kx_n = x_n$$

and consequently

$$y_n = \mathfrak{B}^{-1}x_n = Kx_n.$$

Because $(y_n)_{n\in\mathbb{N}}$ converges in X_B by construction, $(x_n)_{n\in\mathbb{N}} = (\mathfrak{B}y_n)_{n\in\mathbb{N}}$ converges in Z to x (by continuity of $\mathfrak{B}$) and

$$\mathfrak{A}\mathfrak{B}^{-1}x = \mathfrak{A}y = X_B - \lim_{n\to\infty} LK^{-1}y_n = X_B - \lim_{n\to\infty} Lx_n.$$

This shows that $\mathfrak{A}\mathfrak{B}^{-1} \subset \overline{LK^{-1}}^{X_B}$. However, since $\mathfrak{A}\mathfrak{B}^{-1}$ is a closed extension of LK^{-1}, the theorem is proved. ■

Remark 3.1 The assumption that K is densely defined in Z is not essential. If it doesn't hold, then in the considerations the space Z should be replaced by $\overline{D(K)}^Z = Z_B$. ■

Combining Theorem 3.1 with Corollary 2.1 we obtain the following result.

Theorem 3.2 *Assume that*

1. *the set* $\{y \in D(L) \cap D(K);\ K^{-1}Ly \in ImK\}$ *is dense in* $\overline{D(K)}^Z$,
2. *for* $x \in D(K)$ *either*

$$\| \left(\lambda I - K^{-1}L\right)^{-1} x\|_Z \leq \lambda^{-1}\|x\|_Z, \qquad \lambda > 0, \tag{3.5}$$

or

$$\| \left(\lambda I - K^{-1}L\right)^{-1} x\|_Z \leq \frac{M}{|\lambda - \omega|}\|x\|_Z, \qquad \lambda \in S_\theta, \tag{3.6}$$

then, for any $x \in \{x \in K(D(L) \cap D(K));\ LK^{-1}x \in ImK\}$, *the function* $t \to Y(t)x$ *is an* X_B*-solution to (1.1).*

In reflexive spaces assumption 1. is superfluous.

Proof. Assumption 1. of Theorem 2.2 requires $B[D_B(A)]$ to be dense in Z_B. With $B = K^{-1}$ and $D_B(A)$ defined by (3.4) we obtain

$$\begin{aligned} B[D_B(A)] &= K^{-1}\{x \in K(D(L) \cap D(K));\ K^{-1}LK^{-1}x \in D(K)\} \\ &= \{y \in D(L) \cap D(K);\ K^{-1}Ly \in ImK\} \end{aligned}$$

so our assumption 1. is simply rephrasing of that of Theorem 2.2.

Equation (2.9) specified to our case reads

$$\|K^{-1}\left(\lambda I - LK^{-1}\right)^{-1} f\|_Z \leq \lambda^{-1}\|K^{-1}f\|_Z, \qquad \lambda > 0, \tag{3.7}$$

for $f \in ImK$. Since K is invertible, putting $x = K^{-1}f$ gives an arbitrary element of $D(K)$. Hence we have

$$\|K^{-1}\left(\lambda I - LK^{-1}\right)^{-1} Kx\|_Z = \|(K^{-1}\left(\lambda I - LK^{-1}\right) K)^{-1}x\|_Z = \|(\lambda I - K^{-1}L)^{-1}x\|_Z,$$

inequality (3.7) follows from (3.10). Analogous arguments prove the statement for analytic case. Assumption 1 is needed for the density of the domain of $\mathfrak{A}$ in X_B, and it is well-known, that for reflexive space this follows once the Hille-Yoshida estimates are satisfied. ■

An important role in the theory of B-bounded semigroups is played by the operator BAB^{-1} which gives rise to another semigroup which can be used to define the B-bounded semigroup, the advantage of which stems from acting in the space Z rather than in the abstract space X_B. In our case formally we have $BAB^{-1} = K^{-1}LK^{-1}K = K^{-1}L$. Precisely, by Eq. (3.4) we must define $K^{-1}L$ on the domain

$$D(K^{-1}L) = \{y \in D(L) \cap D(K);\ Ly \in ImK\}, \tag{3.8}$$

which has already appeared in Theorem 3.2. Then we have the following proposition.

Proposition 3.1 *The operator $(LK^{-1}, D(LK^{-1}))$ is closeable in X_B iff $(K^{-1}L, D(K^{-1}L))$ is closeable in Z and the following equality holds*

$$\mathfrak{A}x = \mathfrak{K}\overline{K^{-1}L}^{Z}\mathfrak{K}^{-1}x, \tag{3.9}$$

where $\mathfrak{K}$ is the X_B-closure of K. Consequently, the operator $\mathfrak{B}\mathfrak{A}\mathfrak{B}^{-1}$ of Theorem 2.1 is equal to $\overline{K^{-1}L}$.

Proof. Let $x \in D(\mathfrak{A})$, then

$$y = \mathfrak{K}^{-1}x = \lim_{n\to\infty} K^{-1}x_n = \lim_{n\to\infty} y_n$$

in Z, with $x_n \in D_B(A)$, hence $y_n \in D(K^{-1}L)$, and

$$\mathfrak{K}^{-1}\mathfrak{A}x = \lim_{n\to\infty} K^{-1}LK^{-1}x_n = \lim_{n\to\infty} K^{-1}Ly_n.$$

This shows that $y \in D(\overline{K^{-1}L})$ and $\overline{K^{-1}L}y = \mathfrak{K}^{-1}\mathfrak{A}x$ and consequently

$$\mathfrak{A} \subset \mathfrak{K}\overline{K^{-1}L}\mathfrak{K}^{-1}.$$

Conversely, if $y \in D(\overline{K^{-1}L})$, then for $(y_n)_{n\in\mathbb{N}} \subset D(K^{-1}L)$ we have

$$\lim_{n\to\infty} y_n = y, \qquad \lim_{n\to\infty} K^{-1}Ly_n = \overline{K^{-1}L}y.$$

Putting $x_n = Ky_n$, we obtain that $(x_n)_{n\in\mathbb{N}}$ converges in X_B and by continuity of $\mathfrak{K}$ we have $X_B - \lim\limits_{n\to\infty} x_n = x = \mathfrak{K}y$. Thus, $(K^{-1}LK^{-1}x_n)_{n\in\mathbb{N}}$ converges in Z, and therefore $(LK^{-1}x_n)_{n\in\mathbb{N}}$ converges in X_B to $\mathfrak{A}x$ and $\mathfrak{K}\overline{K^{-1}L}\mathfrak{K}^{-1} \subset \mathfrak{A}$. ■

With this proposition, Theorem 2.3 yields a stronger version of Theorem 3.2

Theorem 3.3 *Assume that*

1. *$D(K^{-1}L)$ is dense in $\overline{D(K)}^{Z}$,*
2. *the operator $(K^{-1}L, D(K^{-1}L))$ is closeable in Z and there exist spaces: $\mathfrak{X}$ satisfying for $D(K) \subseteq \mathfrak{X} \subseteq Z$, and D satisfying $D(K^{-1}L) \subseteq D \subseteq D(\overline{K^{-1}L})$ such that for $x \in \mathcal{X}$ either*

$$\| \left(\lambda I - \overline{K^{-1}L}|_D\right)^{-1} x\|_Z \leq \lambda^{-1}\|x\|_Z, \qquad \lambda > 0, \tag{3.10}$$

or

$$\| \left(\lambda I - \overline{K^{-1}L}|_D\right)^{-1} x\|_Z \leq \frac{M}{|\lambda - \omega|}\|x\|_Z, \qquad \lambda \in S_\theta, \tag{3.11}$$

then, for any $x \in D(LK^{-1})$, the function $t \to Y(t)x$ is an X_B-solution to (1.1). For $x \in D(LK^{-1})$ the classical solution is given by

$$u(t,x) = e^{t\overline{K^{-1}L}}K^{-1}x. \tag{3.12}$$

In reflexive spaces assumption 1. is superfluous. ■

Remark 3.2 An X_B-solution to (1.1) exists for a larger class of initial values, namely for all $x \in \Re D(\overline{K^{-1}L})$ ■

Remark 3.3 Using arguments similar to that preceding Corollary 2.2 we can provide analogous solvability results even when K is not invertible. In fact, all the theorems above are valid if we replace $(K, D(K))$ by (K, D_K) where $Z = N(K) \oplus D_K$, and Z by $Z_B = \overline{D_K}^Z$ as noted in Remark 3.1. However, in such a case the uniqueness is an open question, as different choices of D_K lead in general to different spaces X_B, and to different B-bounded semigroups. ■

3.2 B-bounded semigroups and empathy

Let us start with a brief outline of the empathy theory as presented in [20]. Let X and Z be Banach spaces and consider two families of operators $\mathcal{E} = \{E(t) : X \to X\}_{t>0}$ and $\mathcal{S} = \{S(t) : X \to Z\}_{t>0}$ such that the Laplace transforms: $R(\lambda)y = \mathcal{L}(E(t)y)(\lambda)$ and $P(\lambda)y = \mathcal{L}(S(t)y)(\lambda)$ exist for any $y \in X$ and $\lambda > 0$. The pair $< \mathcal{S}, \mathcal{E} >$ is called an *empathy* if for any $s, t > 0$

$$S(t+s) = S(t)E(s)$$

and $P(\xi)$ is invertible for some $\xi > 0$. It can be proved that then $P(\lambda)$ is invertible for any $\lambda > 0$, and the same is valid for $R(\lambda)$. Also, the subspaces $\mathcal{D}_E = R(\lambda)X$ and $\mathcal{D} = P(\lambda)Z$ are independent of λ. Let us define the operators: $K : \mathcal{D} \to \mathcal{D}_E$ by $K = R(\lambda)P^{-1}(\lambda)$ and $L : \mathcal{D} \to X$ by $L = (\lambda R(\lambda) - I)P^{-1}(\lambda)$; both can be proved to be independent of λ and

$$P(\lambda) = (\lambda K - L)^{-1}.$$

The pair $< L, K >$ is called the generator of empathy $< \mathcal{S}, \mathcal{E} >$. In this case, for any $y \in \mathcal{D}_E$, the function $t \to S(t)y$ is a solution of the Cauchy problem (1.1). This result is not satisfactory, as it allows to recognize which Cauchy problem is solvable by a given empathy. The inverse requires an additional assumption. To explain its meaning, let us first note that it can be proved that $\mathcal{E}$ is a semigroup, but not necessarily a C_0-semigroup, in X. In particular, the function $t \to E(t)x$ can be unbounded at $t = 0$ for some $x \in X$. To be able to prove the main generation result for empathy in [20], the author introduces the assumption that the empathy $< \mathcal{S}, \mathcal{E} >$ is uniformly bounded, that is, there exist constants N', M' such that $\sup\limits_{t>0} \|S(t)\| \leq M', \sup\limits_{t>0} \|E(t)\| \leq N'$. Then the following theorem is valid.

Theorem 3.4 *Suppose that the space X has the Radon-Nikodym property. The operator pair $< L, K >$ is the generator of a uniformly bounded empathy $< \mathcal{S}, \mathcal{E} >$ if and only if the operators $P(\lambda)$ and $R(\lambda)$ are bounded for every $\lambda > 0$ and there exist positive numbers M, N such that for every $\lambda > 0$ and $k = 1, 2, \ldots$*

$$\|P(\lambda)\| \leq \lambda^{-1}M, \qquad \|R^k(\lambda)\| \leq \lambda^{-k}N \tag{3.13}$$

■

To compare empathy with B-bounded semigroups we first note that for $< L, K >$ to be the generator of an empathy, K must be an injective operator. Moreover, by definition, the solution family $(S(t))_{t\geq 0}$ is a family of bounded operators in X. Since the B-bounded semigroup $(Y(t))_{t\geq 0}$

is supposed to give solutions for the same problem, it must be also defined on the whole space, which requires $B = K^{-1}$ to be a bounded operator.

We have the following theorem.

Theorem 3.5 *Let $(Y(t))_{t\geq 0}$ be a B-bounded semigroup of generated by A where $A : D(A) \to X$, $B : D(B) \to Z$, $D(A), D(B) \subset X$, A satisfies (2.2) with $\omega = 0$, and B is a bounded, one-to-one operator. The pair $< (Y(t))_{t\geq 0}, (e^{t\mathfrak{A}}|_X)_{t\geq 0} >$ is an empathy generated by $< AB^{-1}, B^{-1} >$ if and only if*

$$\forall_{t\geq 0}\; e^{t\mathfrak{A}} X \subset X \tag{3.14}$$

and

$$\forall_{x\in X, \lambda>0}\; t \to e^{-\lambda t}(e^{t\mathfrak{A}}|_X)x \in L_1(0, \infty, X). \tag{3.15}$$

Proof. Since B is a bounded operator, $(Y(t))_{t\geq 0}$ is a family of bounded operators by property 1. of Definition 2.1. Moreover, by [6] we have

$$P(\lambda)x = B(\lambda I - A)^{-1}x = \int_0^\infty e^{-\lambda t} Y(t) x dt$$

for any $x \in X$ and since B is invertible, $P(\lambda)$ exists and is invertible.

By (3.15) the operator $R(\lambda)$ is well-defined. Let us fix $t \geq 0$ and consider $x_n \to x$ and $e^{t\mathfrak{A}}x_n \to y$ in X as $n \to \infty$. Since by the boundedness of B, $X \hookrightarrow X_B$, $y = e^{t\mathfrak{A}}x$ and $e^{t\mathfrak{A}}$ is a closed operator, and being defined on the whole X, it is a bounded operator. Next, for any $x \in X$ we have by (3.14)

$$Y(t+s)x = \mathfrak{B}\exp((t+s)\mathfrak{A})x = \mathfrak{B}\exp(t\mathfrak{A})\exp(s\mathfrak{A})x = Y(t)\exp(s\mathfrak{A})x,$$

therefore $< (Y(t))_{t\geq 0}, (e^{t\mathfrak{A}}|_X)_{t\geq 0} >$ is an empathy.

Since B is bounded, we obtain for any $x \in X$

$$BR(\lambda)x = B\int_0^\infty e^{-\lambda t} e^{t\mathfrak{A}}|_X x dt = \int_0^\infty e^{-\lambda t} Y(t) x dt = B(\lambda I - A)^{-1}x, \tag{3.16}$$

where again we used $X \hookrightarrow X_B$ to obtain the equivalence of the integrals. Hence $R(\lambda) = (\lambda I - A)^{-1}$, by a simple calculation

$$K = R(\lambda)P^{-1}(\lambda) = B^{-1},$$

and

$$L = (\lambda(\lambda I - A)^{-1} - I)(\lambda I - A)B^{-1} = AB^{-1}.$$

Conversely, the properties (3.14) and (3.15) follow from the original definition of empathy. ■

A better characterization of $\left(e^{t\mathfrak{A}}|_X\right)_{t\geq 0}$ can be obtained when the empathy is uniformly bounded. Clearly, then the statement of Theorem 3.5 is valid provided conditions (3.14) and (3.15) are replaced by the requirement that $\left(e^{t\mathfrak{A}}|_X\right)_{t\geq 0}$ is a strongly measurable and uniformly bounded semigroup in X. Depending on the structure of X, we can prove some additional properties of this semigroup.

Proposition 3.2 *Let the assumptions of Theorem 3.5 be satisfied and* $< (Y(t))_{t\geq 0}, \left(e^{t\mathfrak{A}}|_X\right)_{t\geq 0} >$ *be a uniformly bounded empathy. Then* A *satisfies the Hille-Yoshida estimates in* X*, and consequently*

1. $\left(e^{t\mathfrak{A}}|_{\overline{D(A)}}\right)_{t\geq 0}$ *is a* C_0*-semigroup in* $\overline{D(A)}$*,*
2. *if* X *has the Radon-Nikodym property, then additionally* $\left(e^{t\mathfrak{A}}|_X\right)_{t>0}$ *is a bounded strongly continuous semigroup of bounded operators in* X *(but in general not a* C_0*-semigroup),*
3. *if* X *is reflexive or* $\overline{D(A)} = X$*, then* $\left(e^{t\mathfrak{A}}|_X\right)_{t\geq 0}$ *is a* C_0 *semigroup generated by* A*.*

Proof. If $< (Y(t))_{t\geq 0}, (e^{t\mathfrak{A}}|_X)_{t\geq 0} >$ is a uniformly bounded empathy, then by Theorem 7.1 of [20] there is N such that for any $\lambda > 0, k = 1, 2\ldots$ we have $\|\lambda^k R^k(\lambda)\| \leq N$. From Eq. (3.16) we obtain that

$$\|(\lambda I - A)^{-k}\| \leq \lambda^{-k} N,$$

hence A satisfies the Hille-Yoshida estimates in X. The rest of the proof follows from the well-known theorem by Arendt [2] (see also [5]). ∎

Corollary 3.1 *The pair* $< (Y(t))_{t\geq 0}, (\exp(t\mathfrak{A}))_{t\geq 0} >$ *is an empathy in the pair* Z, X_B*.* ∎

This corollary suggests that if one could find a space $X' \subset X$ such that X is a completion of X' with respect to the norm $\|K^{-1} \cdot \|_Z$, then the notion of B-bounded semigroup generated in the pair X', Z would coincide with the notion of empathy in the pair of spaces X, Z. However, we have also the following proposition.

Proposition 3.3 *The pair* $< (Y(t))_{t\geq 0}, (\exp(t\mathfrak{A}))_{t\geq 0} >$ *is an empathy in* Z, X *if and only if* K *is a bounded operator, with a bounded densely defined inverse* K^{-1}*.*

Proof. The pair $< (Y(t))_{t\geq 0}, (\exp(t\mathfrak{A}))_{t\geq 0} >$ is an empathy in Z, X if and only if X_B can be identified with X. By Theorem 2.4 this is possible if and only if $B = K^{-1}$ is closeable and $B^{-1} = K$ is bounded. X_B is then identified with $D(\overline{B}) = D(\overline{K^{-1}})$ and by Theorem 2.4 again we must have $D(\overline{K^{-1}}) = X$.

Next, the operator $\overline{K^{-1}}$ is invertible (because it equals $\mathfrak{B}$). It follows that $\overline{K} = (\overline{K^{-1}})^{-1}$. In fact, $y = \overline{K}x$ if and only if $x = \lim_{n\to\infty} x_n, x_n \in D(K)$ and $y = \lim_{n\to\infty} Kx_n$. Putting $y_n = Kx_n$ we obtain that this is equivalent to $\lim_{n\to\infty} y_n = y, y_n \in D(K^{-1})$ and $\lim_{n\to\infty} K^{-1}y_n = x$, thus $\overline{K^{-1}}y = x$ which due to the invertibility of $\overline{K^{-1}}$ proves the statement. Since for bounded operators the closure is equal to the extension by continuity, denoted for a moment by $\widetilde{K}$, we obtain that

$$\widetilde{K} : \overline{D(K)} \stackrel{onto}{\to} X$$

is invertible and bounded operator. Since $\overline{D(K)}$ is a Banach space, $\widetilde{K}^{-1} = \overline{K^{-1}}$ is a bounded operator and clearly K^{-1} is bounded. Using again the fact that the closure of a bounded operator is equal to its extension by continuity, we obtain that $X = D(\overline{K^{-1}}) = \overline{D(K^{-1})} = \overline{ImK}$.

The proof in the reverse direction is obvious. ∎

It is instructive to compare Hille-Yoshida type estimates required for generation of empathy and B-bounded semigroup. For simplicity, we assume that the estimates are valid with $k = 1$ and $N = 1$ for the empathy and with $k = 1, M = 1, \omega = 0$ for B-bounded semigroup. In such a case Eqs. (3.13) can be transformed to

$$\begin{aligned} \|(\lambda I - K^{-1}L)^{-1} f\|_Z &\leq \lambda^{-1}\|Kf\|_X, \quad f \in D(K) \subset Z && (3.17)\\ \|(\lambda I - LK^{-1})^{-1} u\|_X &\leq \lambda^{-1}\|u\|_X && (3.18)\end{aligned}$$

The second inequality is simply the Hille-Yoshida estimate for the operator LK^{-1} in X (and thus if K and L commute and $X = Z$, this condition is equivalent to (3.10)).

On the other hand, for the generation a K^{-1}-bounded semigroup of contractions we must have by Theorem 3.2

$$\|(\lambda I - K^{-1}L)^{-1} x\|_Z \leq \lambda^{-1}\|x\|_Z, \tag{3.19}$$

for all $x \in D(K) \subset Z$.

Summarizing, for the generation of K^{-1}-bounded semigroup of contractions, the operator $K^{-1}L$ must generate a semigroup of contractions in Z. On the other hand, for generation of an empathy, we must have a generation of a semigroup of contractions in X by LK^{-1} and additionally, the resolvent of $K^{-1}L$ must be a continuous operator from $D(K)$ with graph norm to Z. Explanation of the different order in which the operators K^{-1} and L enter the Hille-Yoshida estimates comes from Corollary 3.1. If we think of empathy as a B-bounded semigroup with $X = X'_B$, then the estimate (3.18) follows directly from (2.9). Now the estimate (3.19) is obtained from (3.7) by passing from X to a new space X_B. See also the last section where we discuss a suitable example.

3.3 Showalter's approach

The approach presented in [21] is very similar in spirit to the ideas leading to construction of a B-bounded semigroup. The theory presented there covers, however, much broader class of problems including nonlinear evolution inclusions. On the other hand, a number of assumptions on K are more restrictive than those appearing in the theory of B-bounded semigroups. For simplicity we outline here the approach of [21] in the case of linear evolution equations.

It is assumed that $K : E \to E^*$ is a linear, symmetric and non-negative operator from a linear space E to its algebraic dual. This allows to introduce a semi-scalar product on E; the corresponding semi-normed space is denoted by E_b. Its continuous dual E'_b is a Hilbert space. The operator L is assumed to act from a domain in E_b into E'_b.

In a particular case when K generates a scalar product under which E_b is complete, K becomes the Riesz isomorphism onto E'_b. Thanks to this one can proceed as in Theorem 3.1 converting the problem (1.1) into the standard Cauchy problem

$$\begin{aligned} \frac{du}{dt} &= K^{-1}Lu \\ \lim_{t\to 0^+} u(t) &= K^{-1}\overset{\circ}{u} && (3.20)\end{aligned}$$

in E_b. Hence, the question to be solved is whether $K^{-1}L$ generates a semigroup in E_b.

Using the B-bounded semigroup approach we obtain the same problem under the following particular assumptions. Let $D(K) \subset X$ and $(K, D(K))$ be a symmetric, positive operator in X. Assume that E_b is the completion of $D(K)$ under the scalar product (Kx, y); then its continuous dual E'_b is exactly the space X_B $(B = K^{-1})$ as K is the Riesz isomorphism between E_b and E'_b.

Here we see that B-bounded semigroups offer a greater flexibility, as e.g., if we keep $Z = X$, then as X_B we obtain a space such that a suitable extension of K is an isomorphism between $\overline{D(K)}^X$ and X_B. We shall see an example of this kind in the next section.

The approach of [21] has been used also when K is not invertible. Again, the idea is similar to that used in the construction of B-bounded semigroup with non-invertible B–one passes to the quotient space $D(K)/N(K)$. The problem is that the second operator involved, L, not necessarily can be made to act in such a quotient space. In B-bounded semigroups theory the properties of B-bounded semigroup ensured that this was indeed possible. In [21] the author introduces a suitable multivalued relation which fortunately preserves the essential properties of the original operator. This seems to be a complementary approach to that sketched in Remark 3.3, but the comparison of these methods is still an open question.

3.4 The Sobolev-Galpern equation

Let us consider a model problem which can be dealt with using all three approaches:

$$\begin{aligned} \partial_t(\Delta u) &= \Delta^2 u, \\ u|_{\partial\Omega} = 0 & \qquad \partial_\nu u|_{\partial\Omega} = 0, \\ \Delta u|_{t=0} &= \mathring{u}, \end{aligned} \tag{3.21}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded sufficiently smooth set.

The empathy theory (see [18] where, in fact, an early version of empathy theory, called the B-evolution theory was used), the solution operators $t \to S(t)$ act from $L_2(\Omega)$ into $W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega)$, the related semigroup $(E(t))_{t\geq 0}$, generated by a suitable realization of $LK^{-1} = \Delta^2\Delta^{-1}$ acts in $L_2(\Omega)$ and the equation is also satisfied in $L_2(\Omega)$.

Using Showalter's approach we obtain $E_b = \mathring{W}_2^1(\Omega)$ with $E'_b = W_2^{-1}(\Omega)$; it follows that the solution is a $\mathring{W}_2^1(\Omega)$-valued function and the equation is satisfied in $W_2^{-1}(\Omega)$.

Using B-bounded semigroups we are able to construct even weaker solutions. Let $Z = X = L_2(\Omega)$, $K = \Delta$ with $D(K) = W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega)$ and $L = \Delta^2$ with $D(L) = W_2^4(\Omega) \cap \mathring{W}_2^2(\Omega)$. From the general theory of PDE's we know that K is an isomorphism onto $L_2(\Omega)$. From the elliptic regularity it follows that $X_B = (W_2^2(\Omega) \cap \mathring{W}_2^1(\Omega))'$ (the continuous dual). Since all the occurring spaces are reflexive, we have to check only the Hille-Yoshida estimates. Let $u \in D(L)$, then we have

$$(\lambda u - K^{-1}Lu, u)_X = \lambda \|u\|_X^2 - (K^{-1}Lu, u)_X,$$

next

$$(K^{-1}Lu, u)_X = (LKK^{-1}u, K^{-1}u)_X = (LKf, f)_X = \int_\Omega \Delta^2 \Delta f \overline{f} dx$$

$$= -\int_{\Omega} |\nabla \Delta f|^2 dx \leq 0,$$

hence

$$\lambda \|u\|_X^2 \leq (\lambda u - K^{-1}Lu, u)_X \leq \|(\lambda u - K^{-1}Lu\|_X \|u\|_X.$$

Now, let $x \in D(K)$ and consider the equation $x = \lambda u - K^{-1}Lu$. We can transform this equation to

$$Kx = \lambda Ku - Lu \tag{3.22}$$

with $Kx \in X$. Since we have

$$(Ku, u)_X \geq c_1 \|u\|_1^2, \qquad (-Lu, u)_X \geq c_2 \|u\|_2^2,$$

we obtain for $u \in \overset{\circ}{W}{}_2^2 \subset D(K)$

$$(\lambda Ku - Lu, u) \geq c_1 \|u\|_1^2 + c_2 \|u\|_2^2 \geq c_2 \|u\|_2^2,$$

hence $\lambda K - L$ is coercive on $\overset{\circ}{W}{}_2^2$ and for each $Kx \in X$ there is $u \in \overset{\circ}{W}{}_2^2$ (in fact, in $D(L)$ from the elliptic regularity) satisfying Eq. (3.22). Thus, we see that the Hille-Yoshida estimates (3.10) is satisfied and problem (3.21) has a unique X_B-solution.

It is interesting to note that in this case also the first two solutions can be obtained by B-bounded semigroup method by changing a proper choice of the space X.

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