LINEAR CHAOS AND APPROXIMATION

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Abstract. Let $\{T_n(t)\}_{n=1}^{\infty}$ be a sequence of strongly continuous linear semigroups on Banach spaces X_n converging in the sense of Kato to a semigroup $T(t)$ on the Banach space X. We discuss under what conditions the chaoticity of $T_n(t)$ is inherited by $T(t)$. We apply our results to a discrete parabolic equation.

1. Introduction

During the last twenty years or so, several definitions of "chaos" have been proposed (see [Dev], [K–S],[L–Y], and [Wig]). One of the most used among these definitions can be formulated as follows [Dev]:

Definition 1.1. Let X be a metric space. A continuous map $f: \quad X \mapsto X$ is said to be chaotic on X, if

(1) f is topologically transitive;
(2) the periodic points of f are dense in X; and
(3) f has sensitive dependence on initial conditions.

Condition (1) means that for all non-empty open subsets U and V of X there exists a natural number n such that $f^n(U) \cap V \neq \emptyset$. In the sequel we shall dispense with condition (3). Indeed, it has been proven in [Ba-al] that if f is transitive and has dense periodic points then f has sensitive dependence on initial conditions. One of the most appealing properties of Definition 1.1 is that it extends rather naturally to linear systems([McC], [P-A], [D-S-W]). More precisely, let X be an infinite-dimensional separable Banach space and T a linear bounded operator on X. The operator T is called *hypercyclic* if there

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exists $x^* \in X$ such that its orbit under the operator T, $\{T^n x^* \mid n \in \mathbb{N}\}$, is dense in X. The operator T is hypercyclic if and only if it is topologically transitive (see for instance [D-S-W, Theorem 2.2]). Thus, a hypercyclic operator T is chaotic if and only if

$$X_p \equiv \{x \in X \mid T^n x = x, \text{ for some } n \in \mathbb{N}\}$$

is dense in X. Let us now restate Definition 1.1 in the context of linear semigroups defined on a Banach space.

Definition 1.2. A C_0-semigroup $\{T(t)\}_{t \geq 0}$ on the Banach space X is called *chaotic* provided that $T(t)$ satisfies the following two conditions:

(H) (Hypercyclicity) There exists $x^* \in X$ such that $\{T(t)x^* \mid t \geq 0\}$ is dense in X.
(P) $X_p \equiv \{x \in X \mid \exists\, t > 0 : T(t)x = x\}$ is dense in X.

Our objective in this paper is to investigate the chaoticity of the approximating semigroups. In order to analyze this question, we define the approximation in the sense of Kato, which is one of the most general frameworks in approximation theory for linear semigroups(see [Kat]). Let $(X_n, \|.\|_n)$ be a sequence of Banach spaces such that on each X_n there is defined a C_0-semigroup $T_n(t)$. Assume that $T_n(t)$ converges in the sense of Kato to a C_0-semigroup $T(t)$, i.e.

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|T_n(t)P_n f - P_n T(t) f\|_n = 0 \tag{1.1}$$

for some approximating operator P_n(the precise definition is given in Section 3).

To approximate a chaotic semigroup $T(t)$ by a sequence of C_0-semigroups $\{T_n(t)\}$, one should first settle the following questions:

(Q1) If $\{T_n(t)\}$ is a sequence of linear chaotic C_0-semigroups which converges to $T(t)$ in the sense of Kato, can we assert that $T(t)$ is chaotic?
(Q2) Assuming that $T(t)$ is chaotic and $\{T_n(t)\}$ converges to $T(t)$ in the sense of Kato, under which conditions can we assert that at least one of $T_n(t)$ is chaotic?

The relevance of these questions can be seen in the two examples of the next section. The first example is constructed to show that in general the statement:

(S1) $$T_n(t) \quad \text{is chaotic} \quad \forall n \in \mathbb{N} \quad \Longrightarrow \quad T(t) \quad \text{is chaotic},$$

is not true, which gives a negative anwer to question **(Q1)**. The converse statement:

(S2) $$T(t) \quad \text{is chaotic} \quad \Longrightarrow \quad T_n(t) \quad \text{is chaotic} \quad \forall n \in \mathbb{N},$$

is not true either. But we can show that **(S2)** is true for those n's such that instead of (1.1) we have

$$P_n T(t) f = T_n(t) P_n f \qquad \forall f \in X. \tag{1.2}$$

To apply the above result to the chaotic semigroup which will be introduced in Example 2.4 of the following section, we explicitly construct in Section 4 the approximating spaces and we prove that if we want to approximate a chaotic problem by a numerical scheme, then for some specific mesh sizes the discretized problem may become chaotic. We mention that this feature has already been observed and studied in the context of nonlinear chaos (see [T-W-P]).

2. Two examples

In this section we give two examples of linear chaotic semigroups.

Example 2.1 [P-A]**.** Let $X \equiv \ell^1$ and $A \equiv -\alpha I + \beta B$, where B is the backward shift in ℓ^1, i.e. if $\mathbf{f} = (f_1, f_2, \cdots) \in \ell^1$, then $(B\mathbf{f})_n = f_{n+1}$. Then the linear evolution problem in X

$$\begin{cases} \frac{d\mathbf{f}}{dt} = -\alpha I\mathbf{f} + \beta B\mathbf{f} \equiv A\mathbf{f}, \\ \mathbf{f}(0) = \mathbf{f}_0 \in \ell^1, \end{cases}$$

generates a chaotic semigroup $T(t) = e^{tA}$, provided that $\beta > \alpha > 0$.

For the reader's convenience we briefly sketch the proof by using a Lemma which gives a sufficient condition that a C_0-semigroup be hypercyclic. For the original proof see [P-A].

Lemma 2.2 (see [D-S-W, Theorem 2.3]). *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on the Banach space X. Define*

$$X_0 = \left\{x \in X \mid \lim_{t\to\infty} T(t)x = 0\right\}$$

and

$$X_\infty = \left\{x \in X \mid \forall \epsilon > 0, \exists\, u \in X \text{ and } t > 0 \text{ such that } \|u\| < \epsilon \text{ and } \|T(t)u - x\| < \epsilon\right\}.$$

If X_0 and X_∞ are both dense in X, then $\{T(t)\}_{t\geq 0}$ is hypercyclic.

Thus, in order that $\{T(t)\}_{t\geq 0}$ be a chaotic semigroup in X, it suffices that X_0, X_∞ and X_p be simultaneously dense in X.

Proof of Example 2.1. The spectrum of A is $\sigma(A) = \{-\alpha + \beta\mu \mid |\mu| \leq 1\}$. Indeed for any $|\mu| < 1$, the vector

$$\mathbf{h}_\mu = \{\mu^n\}_{n\in\mathbf{N}}$$

satisfies the eigenvalue equation $A\mathbf{h} = \gamma\mathbf{h}$ with eigenvalue $\gamma = -\alpha + \beta\mu$.

We assume that $0 < \frac{\alpha}{\beta} < 1$.

(1) $\underline{X_0 \text{ is dense in } X.}$ Indeed, for $0 < \mu < \frac{\alpha}{\beta}$, one has $e^{tA}\mathbf{h}_\mu = e^{-\alpha t + \beta\mu t}\mathbf{h}_\mu \to 0$ as $t \to \infty$. Hence h_μ and any finite linear combination of these vectors belongs to X_0. Since this set of vectors is dense in X, X_0 is also dense in X.

(2) X_∞ is dense in X. Since e^{tA} is a C_0-group, for $1 > \mu > \frac{\alpha}{\beta}$, one has $e^{-tA}\mathbf{h}_\mu = e^{\alpha t - \beta\mu t}\mathbf{h}_\mu \to 0$ as $t \to \infty$. Any such $\mathbf{h}_\mu \in X_\infty$, since for any $\epsilon > 0$ one can choose t large enough such that for $\mathbf{g}_\mu = e^{-tA}\mathbf{h}_\mu$, $\|\mathbf{g}_\mu\| < \epsilon$ and $\|e^{tA}\mathbf{g}_\mu - \mathbf{h}_\mu\| = 0$. The same argument shows that X_∞ is dense in X.

(3) X_p is dense in X. Indeed, since $\beta > \alpha$, the spectrum of the operator A, $\sigma(A)$, contains a nonempty segment of the imaginary axis. On that segment, the complex numbers $\kappa = i\frac{m}{n}, (m,n) \in \mathbb{Z}^2$ form a dense set. Each such κ is an eigenvalue of A with the eigenvector $\mathbf{h}_{\frac{\kappa+\alpha}{\beta}} = \left(\frac{\kappa+\alpha}{\beta}, (\frac{\kappa+\alpha}{\beta})^2, \cdots\right)$. Each $\mathbf{h}_{\frac{\kappa+\alpha}{\beta}} \in X_p$ and linear combinations of such vectors are dense in ℓ^1. Indeed, if they were not dense, there would exist a non-identically zero functional $\Phi = (\phi_1, \phi_2, ...) \in \ell^\infty$ such that $\langle \Phi, \mathbf{h}_{\frac{\kappa+\alpha}{\beta}} \rangle = 0$ for all $\mathbf{h}_{\frac{\kappa+\alpha}{\beta}}$ constructed as above. This would imply that the analytic function represented by the series $\phi_1 \frac{\kappa+\alpha}{\beta} + \phi_2(\frac{\kappa+\alpha}{\beta})^2 + \phi_3(\frac{\kappa+\alpha}{\beta})^3 + \cdots$ is zero on a set of points κ with an accumulation point, which implies it is identically zero, in contradiction with the assumption that $\Phi \neq 0$.

$\square$

Lemma 2.3. *If in the above example $\alpha = \beta$, then the generating C_0-semigroup e^{tA} is not chaotic.*

Proof. For $\alpha = \beta$, $-\alpha + \beta B$ has no purely imaginary eigenvalues and the uniform boundedness

$$\|T(t)\| = \|e^{\alpha t(B-1)}\| = e^{-\alpha t}\|e^{t\alpha B}\| \leq e^{-\alpha t}e^{\|t\alpha B\|} = 1,$$

contradicts the hypercyclicity of the semigroup. Thus $T(t)$ cannot be chaotic. $\square$

Example 2.4 [D-S-W, Example 4.12]**.** The linear evolution problem is a convection–diffusion type equation of the form:

$$\begin{cases} f_t = af_{xx} + bf_x + cf \equiv Af, \\ f(0,t) = 0, \\ f(x,0) = f_0(x) \end{cases}$$

in the Hilbert space $L^2(\mathbb{R}_+, \mathbb{C})$. The operator A generates an analytic chaotic semigroup $T(t) = e^{tA}$, provided that $a, b, c > 0$ and $c < \frac{b^2}{2a} < 1$.

If in the above example we replace df/dx by its finite difference approximation $[\delta f]_j = (f_j - f_{j-1})/h_n$ and d^2f/dx^2 by

$$[\delta^2 f]_j = \frac{f_{j+1} - 2f_j + f_{j-1}}{h_n^2},$$

then the generator A of Example 2.4 becomes

$$[Af]_j = a\frac{f_{j+1} - 2f_j + f_{j-1}}{h_n^2} + b\frac{f_j - f_{j-1}}{h_n} + cf_j.$$

By taking $h_n = a/b$ one gets $[Af]_j = \left(-\frac{b^2}{a} + c\right) f_j + \left(\frac{b^2}{a}\right) f_{j+1}$ and this is the generator of Example 2.1, namely $Af \equiv -\alpha I + \beta B$ with $\alpha = \frac{b^2}{a} - c$ and $\beta = b^2/a$. Since $c < b^2/2a$, this implies $0 < \alpha < \beta$, which is a sufficient condition for e^{tA} in Example 2.1 to be chaotic. By disregarding the fact that the underlying spaces are different in these examples, the condition $h_n = a/b$ answers the question **(Q2)**. In Section 4 we construct the precise spaces to answer this question.

3. Approximation in the sense of Kato

Definition 3.1. We say that a sequence of Banach spaces $\{(X_n, \|.\|_n) \ : \ n = 1, 2, \cdots\}$ converges to a Banach space $(X, \|.\|)$ in the sense of Kato and we write

$$X_n \xrightarrow{K} X$$

if for any n there is a linear operator $P_n \in \mathcal{L}(X, X_n)$ (called an approximating operator) satisfying the following two conditions:

(K1) $\lim_{n\to\infty} \|P_n f\|_n = \|f\| \quad \forall f \in X$;

(K2) $\forall f_n \in X_n, \exists f^{(n)} \in X$ such that $f_n = P_n f^{(n)}$ with $\|f^{(n)}\| \leq C\|f_n\|_n$ (C is independent of n).

Let $X_n \xrightarrow{K} X$, $B_n \in \mathcal{L}(X_n)$ and $B \in \mathcal{L}(X)$. We say that B_n converges to B in the sense of Kato and we write $B_n \xrightarrow{K} B$ if $\lim_{n\to\infty} \|B_n P_n f - P_n B f\|_n = 0$. Let A_n and A be the generators of the C_0–semigroups $T_n(t) \in \mathcal{L}(X_n)$ and $T(t) \in \mathcal{L}(X)$, respectively. Consider the following three conditions:

(A) (Consistency). There is a complex number λ contained in the resolvent sets $\bigcap_{n\in\mathbb{N}} \rho(A_n)$ and $\rho(A)$, respectively, such that

$$(\lambda - A_n)^{-1} \xrightarrow{K} (\lambda - A)^{-1}.$$

(B) (Stability). There exists a positive constant M and a real number ω such that

$$\|T_n(t)\| \leq Me^{\omega t}, \quad \forall t \geq 0 \quad \text{and} \quad \forall n \in \mathbb{N}.$$

(C) (Convergence). For any finite T

$$T_n(t) \xrightarrow{K} T(t)$$

uniformly on $[0, T]$, i.e.

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|T_n(t)P_nf - P_nT(t)f\|_n = 0 \tag{3.1}$$

In [Ush] one can retrieve the standard version of the Lax equivalence theorem which says that the conditions **(A)** and **(B)** hold if and only if **(C)** holds.

The following Theorem gives a negative answer to question **(Q1)**.

Theorem 3.2. *There exists a sequence of chaotic semigroups which converges in the sense of Kato to a non-chaotic semigroup.*

Proof. By using the semigroup of Example 2.1, one can construct such a sequence. Let $X \equiv \ell^1$ and $A \equiv -\alpha I + \beta B$. From Lemma 2.3 the C_0-semigroup $T(t) = e^{tA}$ is chaotic for $\beta > \alpha > 0$ and non-chaotic for $\beta = \alpha > 0$. Hence, if $\{\beta_n\}$ is a sequence such that $\beta_n > \alpha$ and $\beta_n \to \alpha$, then the sequence of chaotic semigroups $T_n(t)$ converges uniformly to the semigroup $T(t)$. By taking $X_n \equiv X$ and $P_n \equiv I$ the identity operator, the sequence $T_n(t)$ converges also in the sense of Kato to $T(t)$, which is not chaotic. $\square$

Theorem 3.3. *Let $\{(X_n, \|.\|_n) \mid n = 1, 2, \cdots\}$ be a sequence of Banach spaces such that $X_n \xrightarrow{K} X$. Suppose $T(t)$ is a chaotic semigroup on $(X, \|.\|)$ and on each $(X_n, \|.\|_n)$ one defines a C_0-semigroup $T_n(t)$. Now, if for some $n \in \mathbb{N}$ one has*

$$P_nT(t)f = T_n(t)P_nf \qquad \forall f \in X, \tag{3.2}$$

then $T_n(t)$ is also chaotic.

Proof. For the hypercyclicity, take $g_n \in X_n$; from **(K2)** there exists $g^{(n)} \in X$ such that $g_n = P_ng^{(n)}$. From the hypercyclicity of $T(t)$ it follows that for any $\epsilon > 0$ there exist $f^* \in X$ and $t > 0$ such that $\|g^{(n)} - T(t)f^*\| < \epsilon$. The assumption **(K1)** implies the uniform boundedness of $\{P_n\}$, hence for $f_n^* = P_nf^*$ we have

$$\begin{aligned}\|g_n - T_n(t)f_n^*\|_n &= \|g_n - T_n(t)P_nf^*\|_n = \|P_n(g^{(n)} - T(t)f^*)\|_n \\ &\le M\|g^{(n)} - T(t)f^*\| \le M\epsilon.\end{aligned}$$

To prove **(P)** for T_n, take $g_n \in X_n$ and denote $g^{(n)} \in X$ such that $g_n = P_ng^{(n)}$. For $\epsilon > 0$ we take $f^{(p)} \in X_p$ such that $\|g^{(n)} - f^{(p)}\| < \epsilon$, then for $f_n^{(p)} = P_nf^{(p)}$ we have

$$\|g_n - f_n^{(p)}\|_n = \|P_n(g^{(n)} - f^{(p)})\|_n \le M\epsilon.$$

To finish the proof it suffices to see that $f_n^{(p)}$ is a periodic vector for T_n. Indeed,

$$T_n(t)f_n^{(p)} = T_n(t)P_n f^{(p)} = P_n T(t) f^{(p)} = P_n f^{(p)} = f_n^{(p)}.$$

□

To prove the converse we have to impose extra conditions on P_n. Let us denote by $(X_n)_p$ the set of all periodic vectors of T_n in X_n and for any constant $C > 0$ let us define

$$\mathcal{C}(f_n) \equiv \{ f^{(n)} \in X \mid P_n f^{(n)} = f_n \quad \text{with} \quad \|f^{(n)}\| \leq C \|f_n\|_n \}.$$

Theorem 3.4. *Suppose that* (3.2) *holds for some* $n \in \mathbb{N}$. *Suppose* $T_n(t)$ *is chaotic and* P_n *is a bijective mapping between* X_p *and* $(X_n)_p$. *If for any* $\epsilon > 0$ *and* $f \in X$ *there exists a constant* C *and* $f^{(n)} \in \mathcal{C}(P_n f)$ *such that* $\|f - f^{(n)}\| < \epsilon$, *then* $T(t)$ *is also chaotic.*

Proof. Let $g \in X$ and $g_n = P_n g$. For any $\epsilon > 0$ there exist $f_n^* \in X_n$ and $t > 0$ such that $\|g_n - T_n(t) f_n^*\|_n < \epsilon$. Let $f_*^{(n)} \in X$ be such that $f_n^* = P_n f_*^{(n)}$. Then for $h = g - T(t) f_*^{(n)}$ there exists $h^{(n)} \in \mathcal{C}(P_n h)$, with $\|h - h^{(n)}\| < \epsilon$ and $P_n h = P_n h^{(n)} = g_n - T_n(t) f_n^*$. Thus the inequality

$$\begin{aligned} \|g - T(t) f_*^{(n)}\| &\leq \|h - h^{(n)}\| + \|h^{(n)}\| \\ &\leq \epsilon + C \|g_n - T_n(t) f_n^*\|_n \\ &\leq (1 + C)\epsilon \end{aligned} \tag{3.3}$$

implies the hypercyclicity of $T(t)$. To prove **(P)** for $T(t)$, take $g \in X$ and $g_n = P_n g$. For $\epsilon > 0$ we choose $f_n^{(p)} \in (X_n)_p$ and $f^{(p)} \in X$ such that $\|g_n - f_n^{(p)}\|_n < \epsilon$ and $f_n^{(p)} = P_n f^{(p)}$. The bijectivity of P_n on X_p implies that we may choose $f^{(p)} \in X_p$. By taking $h = g - f^{(p)}$ and $h^{(n)} \in \mathcal{C}(P_n h)$, an inequality similar to (3.3) implies the Theorem. □

4. Chaoticity of the discrete parabolic equation

In this section we construct approximating operators for $X \equiv L^1([0, \infty))$ in the sense of Kato. To this end, we define for each $n \in \mathbb{N}$ the step size h_n such that $h_n \to 0$ as $n \to \infty$. The corresponding space would be the space of step functions generated by the characteristic functions $\chi_j^n \equiv \chi_{[(j-1)h_n, jh_n)}$ of the intervals $[(j-1)h_n, jh_n)$. That is, any function $f_n \in X_n$ can be expressed by

$$f_n = \sum_{j=1}^{\infty} a_j^n \chi_j^n.$$

Each X_n is endowed with the norm induced by X. Now we define P_n as a map from X into X_n by

$$f_n = P_n f = \sum_{j=1}^{\infty} f(jh_n)\chi_j^n.$$

Riemann's sum theorem implies that

$$\lim_{n\to\infty} \|P_n f\|_n = \lim_{h_n\to 0} h_n \sum_{j=1}^{\infty} |f(jh_n)| = \|f\|.$$

Hence we have **(K1)**; for **(K2)** it suffices to remark that the injection of X_n into X is continuous. It follows that the operator P_n is an approximating operator for X.

Now let us define the operator A_n by

$$A_n f_n = \beta \sum_{j=1}^{\infty} f((j+1)h_n)\chi_j^n - \alpha \sum_{j=1}^{\infty} f(jh_n)\chi_j^n.$$

Lemma 4.1. *For $\beta > \alpha > 0$ the operator A_n generates a chaotic semigroup, e^{tA_n}, on X_n.*

Proof. One can repeat the proof of Example 2.1, by taking $\mathbf{h}_\mu = \sum_{j=1}^{\infty} \mu^j \chi_j^n$ as the eigenfunction of A_n. □

Now we define the operator A such that

$$P_n A f = A_n P_n f \qquad \forall f \in D(A). \tag{4.1}$$

Actually the operator A is not exactly $aD^2 + bD + cI$ of Example 2.4, but a discrete version thereof, namely

$$A_h \equiv a\left((\frac{1}{h}\chi_{h/2}) * D\right)^2 - b\left(\frac{1}{h}\chi_h * D\right) + cI, \tag{4.2}$$

where χ_h and $\chi_{h/2}$ are the characteristic functions of $[0, h)$ and $[-\frac{h}{2}, \frac{h}{2})$.

Lemma 4.2. *For $h = h_n = a/b$, $\alpha = (\frac{b^2}{a} - c)$, and $\beta = \frac{b^2}{a}$, we have*

$$P_n A_h f = A_n P_n f. \tag{4.3}$$

Proof. We can write

$$[\frac{1}{h}\chi_h * D]f(x) = \frac{1}{h}\int_0^h f'(x-y)dy = \frac{f(x-h) - f(x)}{h} \tag{4.4}$$

and for $h = h_n$

$$P_n[\frac{1}{h}\chi_h * D]f(x) = \frac{1}{h}\sum_{j=1}^{\infty}[f((j-1)h) - f(jh)]\chi_j^n(x).$$

In the same manner

$$[\frac{1}{h}\chi_{\frac{h}{2}} * D]^2 f(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \tag{4.5}$$

and

$$P_n[(\frac{1}{h}\chi_{h/2} * D)^2]f(x) = \frac{1}{h^2}\sum_{j=1}^{\infty}[f((j+1)h) - 2f(jh) + f((j-1)h)]\chi_j^n(x).$$

Thus, if $A_h \equiv a\left((\frac{1}{h}\chi_{h/2}) * D\right)^2 - b\left(\frac{1}{h}\chi_h * D\right) + cI$, we obtain (4.3) provided that $h = a/b, \alpha = (\frac{b^2}{a} - c)$ and $\beta = \frac{b^2}{a}$. $\square$

Lemma 4.3. *Suppose that e^{tA} and e^{tA_n} are the C_0-semigroups generated by A and A_n, which are related by relation (4.1). Then*

$$P_n e^{tA} f = e^{tA_n} P_n f \qquad \forall f \in X. \tag{4.6}$$

Proof. It is sufficient to differentiate, at least for $f \in D(A)$:

$$\frac{d}{dt}\left[P_n e^{tA} f\right] = P_n \frac{d}{dt}\left[e^{tA} f\right] = P_n A e^{tA} f = A_n\left[P_n e^{tA} f\right],$$

for all $t \geq 0$, and identify

$$\left[P_n e^{tA} f\right]|_{t=0} = P_n f,$$

so that $\left[P_n e^{tA} f\right]$ must equal $e^{tA_n} P_n f$. Since $D(A)$ is dense, this is true for all $f \in X$. $\square$

From (4.4) and (4.5) it follows that A_h is a bounded operator on $L^1([0,\infty))$. Consequently it generates a uniformly continuous semigroup e^{tA_h}. Furthermore, if f is a step function, then $A_h f$ is also a step function hence the subspace X_n is invariant under A_h and e^{tA_h}. Then we can prove:

Theorem 4.4. *For $h = a/b$, e^{tA_h} is a chaotic semigroup.*

Proof. For the proof we will use Theorem 3.4. Lemma 4.3 implies (3.2) and Lemma 4.1 asserts that e^{tA_n} is chaotic. To see that P_n is a bijective mapping between X_p and $(X_n)_p$, one takes $f \in X_p$ and writes $e^{tA_n} P_n f = P_n e^{tA_h} f = P_n f$. Thus, $P_n f \in (X_n)_p$. Conversely, if $f_n \in (X_n)_p$ by considering f_n as an element of X (since X_n is a subset of X) we have

$$P_n e^{tA_h} f_n = e^{tA_n} P_n f_n = e^{tA_n} f_n = f_n,$$

since P_n acts as the identity on X_n. On the other hand, $f_n \in X_n$ implies $e^{tA_h} f_n \in X_n$, thus $P_n e^{tA_h} f_n = e^{tA_h} f_n$ and consequently $e^{tA_h} f_n = f_n$.

Finally, for $f \in X$, any function which interpolates f on the points $\{(jh, f(jh)), j = 1, 2, \cdots\}$ belongs to $\mathcal{C}(P_n f)$, thus for any $\epsilon > 0$ we can find an interpolating function $f^{(n)}$ such that $\|f^{(n)}\| < \epsilon$. This concludes the theorem. $\square$

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