AN EXISTENCE FAMILY FOR HUSIMI EQUATION

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ABSTRACT. We prove the well-posedness of the quantum Liouville equation in $L^1(\mathbf{R}^{2n})$, provided that the potential of the Schrödinger equation lies in some $H^s(\mathbf{R}^n)$. By regularizing the Wigner function we obtain the Husimi equation which is ill-posed in any $L^p(\mathbf{R}^{2n})$ spaces. We show that for the Husimi operator the maximal and minimal extensions coincide and we construct a C-existence family in the sense of R. deLaubenfels which is a new tool for studing this operator.

1. INTRODUCTION

Suppose that in the theory of quantum mechanics the potential V is chosen in such a way that the hamiltonian

$$H = \frac{-\hbar^2}{2}\Delta + V \tag{1.1}$$

($\hbar$ being the Planck's constant which is equal 1 in the sequel) is self-adjoint in the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^n)$. Let φ be the solution of the corresponding Schrödinger equation, i.e.

$$i\,\frac{\partial \varphi}{\partial t} = H\varphi, \tag{1.2}$$

and φ^* its complex conjugate. The correspondent density matrix $\rho(x,y) = \varphi(x)$ $\varphi^*(y)$ defines a positive self-adjoint trace class operator, which we designate it again by ρ:

$$\rho f = \int_{\mathbb{R}^n} \rho(x,y) f(y)dy, \qquad f \in \mathcal{H}. \tag{1.3}$$

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Furthermore , if $\mathcal{T}(\mathcal{H})$ is the Banach space of the trace class operators, endowed with the norm

$$\|T\|_{tr} = \sum_{i\in I} | < \psi_i, T\psi_i > |, \qquad T \in \mathcal{T}(\mathcal{H}),$$

where $\{\psi_i; i \in I\}$ is an ONB of $\mathcal{H}$, then

$$\|\rho\|_{tr} = \int_{\mathbb{R}^n} |\varphi(x)|^2 \, dx. \tag{1.4}$$

It is well-known that ρ governs the classical Liouville equation

$$i\,\dot\rho = [H, \rho], \qquad \rho(0) = \rho_0 \tag{1.5}$$

on which the quantum theory of open system is based (see [A-L] and [Da]). The solution of (1.5) can be expressed by so called *quantum dynamical semigroup* $S(t)$

$$\rho(t) = S(t)\rho_0 = e^{-itH} \rho_0 \, e^{itH} \tag{1.6}$$

which is a unitary group on $\mathcal{T}(\mathcal{H})$.

There is also another link between quantum theory of the open systems and the theory of statistical mechanics. Let us consider the *Wigner equation* (also called *quantum Liouville equation*)

$$\frac{\partial w}{\partial t} + \xi \cdot \nabla_x w - P(x, \nabla_\xi) w = 0, \tag{1.7}$$

where $P \equiv P(x, \nabla_\xi)$ is a pseudo-differential operator, which is symbolically

$$i[V(x + \frac{i}{2}\nabla_\xi) - V(x - \frac{i}{2}\nabla_\xi)],$$

or equivalently

$$\begin{aligned} Pw &= (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{-i(\xi-\eta)y} \, \frac{1}{i} \left[V(x + \frac{y}{2}) - V(x - \frac{y}{2}) \right] w(x, \eta) dy d\eta \\ &= k \underset{\xi}{*} w, \end{aligned} \tag{1.8}$$

where $k(x, \xi) = (2\pi)^{-n/2} \mathcal{F}_y \big[\frac{1}{i} \left(V(x + \frac{y}{2}) - V(x - \frac{y}{2}) \right) \big](\xi)$. This equation follows from (1.2) or (1.5) via the Wigner transform [Wi]. In fact, a solution of (1.7) can be expressed by

$$W(x, \xi, t) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{-i\xi y} \rho(x + \frac{y}{2}, x - \frac{y}{2}, t) \, dy. \tag{1.9}$$

If ρ is the density matrix of φ, then we call W the Wigner transform of φ and we write: $W = W(\varphi)$. This function provides a phase-space 'view' of the quantum state. e.g. if the potential $V \equiv 0$, then W satisfies the *advection equation*

$$\frac{\partial w}{\partial t} + \xi \cdot \nabla_x w = 0 \tag{1.10}$$

(is called also *collisionless transport equation* in the statistical mechanics).

For any $g \in L^2(\mathbb{R}^n \times \mathbb{R}^n)$, let us define $\widetilde{g}(x, y) = g(r, s)$, where

$$r = x + \frac{y}{2}, \qquad s = x - \frac{y}{2}.$$

Hence, the Wigner function can be expressed as

$$W(x, \xi, t) = (2\pi)^{-\frac{n}{2}} [\mathcal{F}_y \widetilde{\rho}(x, y)](\xi) \tag{1.11}$$

where

$$[\mathcal{F}_y f](\xi) = \widehat{f}(\xi) = (2\pi)^{-\frac{n}{2}} \int_{\mathbb{R}^n} e^{-i\xi \cdot y} f(y) dy.$$

Since ρ is in particular a Hilbert-Schmidt operator, its kernel $\rho(\cdot, \cdot)$ belongs to $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ and so does W. Although the reality of W can be deduced from the selfadjointness of ρ, it is necessarily negative in some region of the phase-space (see [H-al]). These properties incited many authors ([M-R], [Ma1], [Ma2] and their references) to make their analysis in $L^2(\mathbb{R}^n \times \mathbb{R}^n)$ space. Nevertheless, in statistical mechanics the unknown of the problems is always the density of particles (probability or mass) and the physically right space invokes $L^1_+(\mathbb{R}^n \times \mathbb{R}^n)$, the positive cone of the Banach lattice $L^1(\mathbb{R}^n \times \mathbb{R}^n)$.

Related to this , Husimi [Hu] showed that a Gaussian smoothing of W is indeed positive and hence suitable for a probability density. In fact, let

$$H(x, \xi, t) \equiv [W * G](x, \xi, t) = \int_{\mathbb{R}^{2n}} W(y, \eta, t) G(x - y, \xi - \eta)\, dy d\eta, \tag{1.12}$$

where $G(x, \xi) = (\pi)^{-n} \exp(-(|x|^2 + |\xi|^2))$. For $(x_0, \xi_0) \in \mathbb{R}^{2n}$, take

$$\psi_{x_0, \xi_0}(x) = (\pi)^{-n/4} e^{-\frac{|x - x_0|^2}{2}} e^{i \xi_0 x}, \tag{1.13}$$

then

$$G(x - x_0, \xi - \xi_0) = W(\psi_{x_0, \xi_0})(x, \xi) = W(\psi_{x, \xi})(x_0, \xi_0). \tag{1.14}$$

Hence

$$\begin{aligned} H(x,\xi) &= \int_{\mathbb{R}^{2n}} W(\varphi)(x_0,\xi_0)W(\psi_{x,\xi,})(x_0,\xi_0)dx_0d\xi_0 \\ &= \left(\frac{1}{4\pi}\right)^n \Big| \int_{\mathbb{R}^n} \varphi(r)\psi^*_{x,\xi}(r)dr\Big|^2 \quad \geq 0. \end{aligned} \tag{1.15}$$

All these matters including many other applications of the Husimi function can be found in [L-P]. In particular, it is proved in [E-H] that the range of the Husimi transformation of $L^2(\mathbb{R}^{2n})$ is exactly the space of the coherent state vectors of a first order pseudo–differential operator in $L^2(\mathbb{R}^{2n})$ in the sense which is defined in [E-H].

Apart from the mentioned properties of the Husimi function, in the theory of wavelets the transformation

$$\mathcal{G}_x(\varphi)(\xi) = \int_{\mathbb{R}^n} \varphi(r)\psi^*_{x,\xi}(r)\,dr$$

which appears in (1.15), is called *Gabor transform*, and its connection to uncertainty principle is reviewed in [Ch].

In spite of the fact that the pseudo-differential operator $P(x,\nabla_\xi)$ commutes with the convolution, the Husimi function does not satisfy (1.7), but it does a perturbed version of it, namely

$$\frac{\partial h}{\partial t} + \xi\cdot\nabla_x h - P(x,\nabla_\xi)h + \frac{1}{2}\nabla_x\cdot\nabla_\xi h = 0. \tag{1.16}$$

In fact, if we denote by $T_0 \equiv -\xi\nabla_x$ and $T \equiv -\xi\nabla_x - \frac{1}{2}\nabla_x\cdot\nabla_\xi$, one can verify readly that

$$CT_0f = T_0Cf - \frac{1}{2}\nabla_x\cdot\nabla_\xi Cf = TCf \quad \forall f\in D(T_0), \tag{1.17}$$

where C is a bounded operator on $L^1(\mathbb{R}^{2n})$ defined by

$$Cf \equiv G * f \qquad \forall f \in L^1(\mathbb{R}^{2n}). \tag{1.18}$$

Consequently by acting C on (1.7), one obtains

$$\begin{aligned} \frac{\partial h}{\partial t} &= C\frac{\partial}{\partial t}w = C(T_0+P)w = TCw + PCw \\ &= (T+P)h \end{aligned}$$

which is exactly (1.16).

The equation (1.16) is called *Husimi equation* , and it can be considered once more within the context of the statistical mechanics.

The fact that $\|G\|_{L^1(\mathbb{R}^{2n})} = 1$, and

$$\int_{\mathbb{R}^n} W(x,\xi)d\xi = |\varphi(x)|^2$$

imply that

$$\int_{\mathbb{R}^{2n}} H(x,\xi)dx\, d\xi = \int_{\mathbb{R}^n} |\varphi(x)|^2 dx.$$

This together with (1.4) give us the invariants of three equations (1.2), (1.5) and (1.16), which are

$$\|\varphi(t)\|_{\mathcal{H}} = \|\rho(t)\|_{tr} = \|H(t)\|_{L^1(\mathbb{R}^{2n})} = c_0. \tag{1.19}$$

for any $t \in \mathbb{R}$, where $c_0 = \|\varphi_0\|_{\mathcal{H}} = \|\rho_0\|_{tr} = \|H(\varphi_0)(0)\|_{L^1(\mathbb{R}^{2n})}$.

As the Stone's theorem justifies the mathematical well-founded of the space $L^2(\mathbb{R}^n)$ for the Schrödinger equation, the identities (1.19) ensure also the adequacy of $\mathcal{T}(\mathcal{H})$ the space of the trace class operators for the classical Liouville equation and $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ for the Husimi equation. This property incited us to make an analysis of the Husimi equation in $L^1(\mathbb{R}^{2n})$.

The well-posedness of classical Liouville equation in $L^1(\mathbb{R}^{2n})$ is treated in [E-P] by requiring a minimal ammont of Sobolev regularity on the potential V. In the next section we will show the well-posedness of the quantum Liouville equation (1.7) in $L^1(\mathbb{R}^{2n})$. We prove that the Wigner operator $L \equiv T_0 + P = -\xi \cdot \nabla_x + P(x, \nabla_\xi)$ generates a C_0–group e^{tL} in $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and we give the exponential bound of this group with respect of the H^s norm of the potential V. For studing the Husimi equation in $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ we come up against the following difficulty that the Husimi equation (1.16) is not well-posed in any $L^p(\mathbb{R}^{2n})$. In fact, by the Fourier transform we can see that there is no real number in the resolvent set of the operator $B \equiv -\frac{1}{2}\nabla_x \cdot \nabla_\xi$ and even if we pertube it by a conservative operator $T_0 \equiv -\xi \cdot \nabla_x$ or by a bounded operator $P(x, \nabla_\xi)$ we can not remedy the ill-posdness of the Husimi operator $A \equiv T_0 + P + B$.

When an abstract Cauchy problem in a Banach space X

$$\textbf{(CP)} \qquad \begin{cases} \frac{d}{dt}u(t,\varphi) = Au(t,\varphi) & t \geq 0 \\ u(0,\varphi) = \varphi \in X, \end{cases}$$

is not well-posed, at least in its original formulation, one of the useful concept for dealing with it is a *C-existence family* which is introduced in [dL1], and, for the exponentially bounded case, in [dL2]. Roughly speaking, sometimes when A is a closed operator in X and there is no mild solution for **(CP)**, for any $\varphi \in X$ we find the following alternative. There exists a bounded operator C on X, which plays the role of a regularizing operator and for any $\varphi = C\psi \in Im(C), Y(t)\psi \equiv u(t, C\psi)$ defines a mild solution for **(CP)**. In the section 4 we define the operator C by $Cf \equiv G * f$ for all $f \in L^1(\mathbb{R}^{2n})$ and we show that this concept fits perfectly to our problem and one can define with these materials a mild and strong solution of the Husimi equation.

Before introducing the notion of the C-existence family we should be certain that the operator A is closable and define an adequate close extension of A. We prove actually that the minimal and maximal extensions of A coincide. This will be the topic of the section 3.

2. Well-posedness of the Wigner equation in $L^1(\mathbb{R}^{2n})$

Now, we are going to analyse the well-posedness of the Wigner equation (1.7). With this end in view, we need to introduce the Sobolev space $H^s(\mathbb{R}^n) = \{f \in \mathcal{S}' \mid \|f\|_{H^s} < \infty\}$, where

$$\|f\|_{H^s} = \left(\int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}}.$$

We recall that for any $f \in H^s(\mathbb{R}^n)$, and $s > \frac{n}{2}$

$$\|\widehat{f}\|_1 \leq c\, \|f\|_{H^s}, \tag{2.1}$$

where $c = \int_{\mathbb{R}^n} (1 + |\xi|^2)^{-s} d\xi$.

Theorem 2.1. *If for some $s > \frac{n}{2}$, the potential V lies in $H^s(\mathbb{R}^n)$, then*

(a) *the oprator $P = P(x, \nabla_\xi)$ is bounded in $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ and we have*

$$\|P(x, \nabla_\xi) w\|_1 \leq \delta \|w\|_1 \tag{2.2}$$

where $\delta = 2(2\pi)^{-n/2} C \|V\|_{H^s}$;

(b) *the Wigner operator $L = T_0 + P$ generates a C_0–group e^{tL} on $L^1(\mathbb{R}^n_x \times \mathbb{R}^n_\xi)$ which satisfies*

$$\|e^{tL} f\|_1 \leq e^{\delta\, |t|} \|f\|_1, \qquad \forall \quad t \in \mathbb{R}. \tag{2.3}$$

Proof. (a) The estimation (2.2) follows from

$$\begin{aligned}\|P(x,\nabla_\xi)w\|_1 &= \|k \underset{\xi}{*} w\|_1 \\ &\le \sup_x \Big(\int_{\mathbb{R}^n} |k(x,\xi)|d\xi \Big) \|w\|_1 \\ &\le \delta\, \|w\|_1\end{aligned}$$

for, $k(x,\xi) = \frac{1}{i}(2\pi)^{-n/2}\left[2^n e^{2i\xi x}\widehat{V}(2\xi) - (-2)^n e^{-2i\xi x}\widehat{V}(2\xi)^*\right]$.

(b) Taking $V \equiv 0$ in (1.7), the advection equation (1.10) is thus governed by a unitary C_0–group $e^{tT_0}f(x,\xi) = f(x-t\xi,\xi)$. By virtue of the part (a) the operator P being bounded, (2.2) follows from perturbation theorem of the generator of a strongly continuous semigroup by a bounded operator (see [Paz]).

□

Corollary 2.2. *If for some $s > \frac{n}{2}$ the initial datum φ_0 of the Schrödinger equation (1.2) is such that the function $x \to \rho_0(x,y) = \varphi_0(x+\frac{y}{2})\varphi_0^*(x-\frac{y}{2})$ belongs to $L^1(\mathbb{R}^n)$ and the functions $y \to \widetilde{\rho}_0(x,y)$ and V both lie in $H^s(\mathbb{R}^n)$, then the Wigner function $W \in L^1(\mathbb{R}^{2n})$.*

Proof. Take $t = 0$ in (1.11) and

$$w_0(x,\xi) = W(x,\xi,0) = (2\pi)^{-\frac{n}{2}}[\mathcal{F}_y\widetilde{\rho}_0(x,y)](\xi). \tag{2.4}$$

Since

$$\iint_{\mathbb{R}^{2n}} |w_0(x,\xi)|dxd\xi \le (2\pi)^{\frac{n}{2}} \int_{\mathbb{R}^n} \|\widetilde{\rho}_0(x,\cdot)\|_{H^s}dx < \infty,$$

one can choose w_0 as an initial datum of (1.7). So that Theorem 2.1 concludes, since e^{tL} produces a unique solution of (1.7) for the initial datum w_0. □

3. Extensions of the Husimi operator

In this section we will prove that the Husimi operator A is closable and the minimal and maximal closed extensions of A coincide. Let us define the regularized transport operator T on $\mathcal{S}(\mathbb{R}^{2n})$ by

$$[Tf](x,\xi) \equiv [T_0 f + Bf](x,\xi) = -\xi\cdot\nabla_x f(x,\xi) - \frac{1}{2}\nabla_x\cdot\nabla_\xi f(x,\xi).$$

and the Husimi operator $A = T + P$.

Theorem 3.1. *The operator A is closable in $L^1(\mathbb{R}^{2n})$.*

Proof. Since A is a bounded perturbation of T, it suffices to show that T is closable in $L^1(\mathbb{R}^{2n})$. Let $\{\varphi_k\}_{k\mathbb{N}}$ be a sequence of functions in $\mathcal{S}(\mathbb{R}^{2n})$, converging to 0 in $L^1(\mathbb{R}^{2n})$, such that $T\varphi_k$ converges to a function f in $L^1(\mathbb{R}^{2n})$. If we denote by $T^* = \xi \cdot \nabla_x - \frac{1}{2}\nabla_x \cdot \nabla_\xi$ the formal adjoint of T and by $< \cdot, \cdot >$ the pairing between $L^1(\mathbb{R}^{2n})$ and $L^\infty(\mathbb{R}^{2n})$, then we have $< T\varphi_k, \psi > = < \varphi_k, T^*\psi >$, for all $\psi \in \mathcal{S}(\mathbb{R}^{2n})$ and $k \in \mathbb{N}$. Letting $k \to \infty$ gives us $< f, \psi > = < 0, T^*\psi > = 0$, for all $\psi \in \mathcal{S}(\mathbb{R}^{2n})$, so by the density of $\mathcal{S}(\mathbb{R}^{2n})$ in $L^1(\mathbb{R}^{2n})$ we get $f = 0$, as desired. □

The operator A acting on $\mathcal{S}(\mathbb{R}^{2n})$ has a closure in $L^1(\mathbb{R}^{2n})$. Let us denote by A_m the smallest closed extension of A in $L^1(\mathbb{R}^{2n})$, which is called the *minimal extension of* A and can be defined also in the following manner (see [Sch]): $f \in L^1(\mathbb{R}^{2n})$ is in $D(A_m)$ and $A_m f = g$, if there exists a sequence $\{\varphi_k\}$ of functions in $\mathcal{S}(\mathbb{R}^{2n})$, such that $\varphi_k \to f$ and $A\varphi_k \to g$ in $L^1(\mathbb{R}^{2n})$.

The operator A has another closed extension A_M on $L^1(\mathbb{R}^{2n})$ which is called the *maximal extension of* A and is defined as follows (see also [Sch]): we say that $f \in D(A_M)$ and $A_M f = g$, if f and g belong to $L^1(\mathbb{R}^{2n})$ and $< f, A^*\varphi > = < g, \varphi >$ for all $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$. Clearly A_M is an extension of A_m ($A_m \subset A_M$).

Theorem 3.2. *The minimal and the maximal extensions of the Husimi operator A coincide.*

Proof. Since A is a bounded perturbation of T, it suffices to show that $D(T_M) \subseteq D(T_m)$ (see [Sch]).

For $h \in D(T_M)$, there exists $f \in L^1(\mathbb{R}^{2n})$ such that $< h, T^*\varphi > = < f, \varphi >$, for all $\varphi \in \mathcal{S}(\mathbb{R}^{2n})$ and we have $T_M h = f$. For having $h \in D(T_m)$ and $T_m h = f$ we have juste to find a sequence $\{\varphi_k\}_{k\in\mathbb{N}}$ in $\mathcal{S}(\mathbb{R}^{2n})$ such that $\varphi_k \to h$ and $T\varphi_k \to f$ in $L^1(\mathbb{R}^{2n})$. For

$$G_k(x,\xi) = \left(\frac{k^2}{\pi}\right)^n \exp\left(-k^2(|x|^2 + |\xi|^2)\right),$$

let us define the operator $C_k \in \mathcal{L}(L^1(\mathbb{R}^{2n}))$ by $C_k f \equiv G_k * f = < f, \psi_k >$, for all $f \in L^1(\mathbb{R}^{2n})$ and $\varphi_k = C_k h$. Clearly $\varphi_k \to h$ in $L^1(\mathbb{R}^{2n})$. As in (1.17) a simple calculation shows that

$$C_k T_0 g = T_0 C_k g - \frac{1}{2k^2}\nabla_x \cdot \nabla_\xi C_k g \qquad \forall g \in D(T_0),$$

Since C_k commutes with $\nabla_x \cdot \nabla_\xi$, thus

$$C_k f = T C_k h - \frac{1}{2k^2}\nabla_x \cdot \nabla_\xi C_k h$$

So, for proving that $T\varphi_k = TC_k h \to f$, it suffices to show that

$$\lim_{k\to\infty} \|\frac{1}{2k^2}\nabla_x \cdot \nabla_\xi C_k h\|_1 = 0 \tag{3.1}$$

First we show this property for a smooth function $h_0 \in C_0^\infty(\mathbb{R}^{2n})$. This comes from

$$\begin{aligned}\|\frac{1}{2k^2}\nabla_x \cdot \nabla_\xi C_k h_0\|_1 &= \|\frac{1}{2k^2}[\nabla_x \cdot \nabla_\xi h_0] * G_k\|_1 \\ &\leq \frac{1}{2k^2}\|\nabla_x \cdot \nabla_\xi h_0\|_1 \|G_k\|_1 \\ &= \frac{1}{2k^2}\|\nabla_x \cdot \nabla_\xi h_0\|_1 \longrightarrow 0 \quad \text{as} \quad k \to \infty.\end{aligned}$$

Then for an arbitary $h \in L^1(\mathbb{R}^{2n})$, we choose a $h_0 \in C_0^\infty(\mathbb{R}^{2n})$ such that $\|h - h_0\|_1$ be small enough and we write

$$\begin{aligned}\|\frac{1}{2k^2}\nabla_x \cdot \nabla_\xi C_k h\|_1 &\leq \|\frac{1}{2k^2}[\nabla_x \cdot \nabla_\xi G_k] * (h - h_0)\|_1 + \frac{1}{2k^2}\|\nabla_x \cdot \nabla_\xi C_k h_0\|_1 \\ &\leq \frac{1}{2k^2}\|\nabla_x \cdot \nabla_\xi G_k\|_1 \|h - h_0\|_1 + \frac{1}{2k^2}\|\nabla_x \cdot \nabla_\xi C_k h_0\|_1.\end{aligned}$$

Now since $\frac{1}{2k^2}\|\nabla_x \cdot \nabla_\xi G_k\|_1 = \frac{1}{2}\|\nabla_x \cdot \nabla_\xi G\|_1$, we can use the $\frac{\varepsilon}{2}$ argument for concluding (3.1). $\square$

4. A C-existence family for the Husimi equation

Throughout this section we will write $D(A)$ for the domain of a linear operator A defined on a Banach space $(X, \|\cdot\|)$. We will denote by $[D(A)]$ the normed vector space with the graph norm $\|f\|_{[D(A)]} \equiv \|Af\| + \|f\|$. We designate $\mathcal{L}(Y)$ for the space of linear bounded operators from the space Y into itself and $C \in \mathcal{L}(X)$. Basic material on mild and strong C-existence families may be found in [dL2], along with more extensive references. For understanding these notions we will compare them with the notions of the mild and strong solutions for **(CP)**.

Definition 4.1. (a) A *strong solution* of **(CP)** is $u(t,\varphi)$ such that

$$t \mapsto u(t,\varphi) \in C([0,\infty), [D(A)]) \cap C^1([0,\infty), X),$$

satisfying **(CP)**.

(b) A *mild solution* of **(CP)** is $u(t,\varphi)$ such that $t \mapsto u(t,\varphi) \in C([0,\infty), X)$, and for all $t \geq 0$, $\int_0^t u(s,\varphi)ds \in D(A)$, with

$$A\left(\int_0^t u(s,\varphi)ds\right) = u(t,\varphi) - \varphi \quad (t \geq 0), u(0,\varphi) = \varphi.$$

Definition 4.2. (a) A *mild C-existence family* for A is a strongly continuous family of operators $\{Y(t)\}_{t\geq 0} \subseteq \mathcal{L}(X)$ such that for any $\varphi \in X,\ t \geq 0$, $\int_0^t Y(s)\varphi ds \in D(A)$, with

$$A\left(\int_0^t Y(s)\varphi \, ds\right) = Y(t)\varphi - C\varphi.$$

(b) The mild C-existence family $\{Y(t)\}_{t\geq 0}$ is a *strong C-existence family for A*, or simply *C-existence family for A* if $\{Y(t)\ |_{[D(A)]}\}_{t\geq 0}$ is contained in $\mathcal{L}([D(A)])$ and is strongly continuous, with

$$\left(\int_0^t AY(s)\varphi \, ds\right) = Y(t)\varphi - C\varphi,$$

for all $\varphi \in D(A)$.

Here are some consequences of having a C-existence family.

Lemma 4.3 [dL2, Theorem 2.6 and Proposition 2.9]**.**

(a) If there exists a mild C-existence family, $\{Y(t)\}_{t\geq 0}$ for A, then **(CP)** *has a mild solution for any $\varphi = C\psi \in Im(C), u(t, C\psi) \equiv Y(t)\psi$, and the sequence of the solutions $u(t, C\psi_n)$ converges to zero, uniformly on compact subsets of $[0, \infty)$, whenever $\psi_n \to 0$.*

(b) If there exists a strong C-existence family, $\{Y(t)\}_{t\geq 0}$ for A, then **(CP)** *has a strong solution for any $\varphi = C\psi \in C(D(A)), u(t, C\psi) \equiv Y(t)\psi$, and the both $u(t, C\psi_n)$ and $Au(t, C\psi_n)$ converge to zero, uniformly on compact subsets of $[0, \infty)$, whenever ψ_n and $A\psi_n$ both converge to zero.*

(c) Suppose A is closed and there exists $\delta \in \mathbb{R}$ such that A has no eigenvalues in (δ, ∞). Then all mild solutions of **(CP)** *are unique.*

In the sequel we construct a C-existence family for the Husimi operator A. In Theorem 2.1 we have construct the C_0-group e^{tL} generated by the Wigner operator L on $L^1(\mathbb{R}^{2n})$. Now, let us consider the family $\{Y(t)\}_{t\in\mathbb{R}}$ defined by $Y(t) \equiv Ce^{tL}$, where the operator C is given by (1.18). Since C is a bounded operator in $L^1(\mathbb{R}^{2n})$ and

$$L\int_0^t e^{sL}\phi \, ds = e^{tL}\phi - \phi \quad (\forall t \in \mathbb{R}), \quad \phi \in L^1(\mathbb{R}^{2n})$$

by acting C, we obtain

$$
\begin{aligned}
Y(t)\phi - C\phi &= CL\int_0^t e^{sL}\phi\, ds \\
&= AC\int_0^t e^{sL}\phi\, ds \\
&= A\int_0^t Y(s)\phi\, ds.
\end{aligned}
$$

This together with the facts that $\psi(t) = \int_0^t e^{sL}\phi\, ds \in D(L)$ for any $\phi \in X$ and $CL\psi = AC\psi$ infer that $C\psi = \int_0^t Y(s)\phi\, ds = \int_0^t Ce^{sL}\phi\, ds \in D(A)$. Consequently the family $\{Y(t)\}_{t\in\mathbb{R}}$ is a mild C–existence family and from the strong continuity of e^{tL} it follows that it is even a strong C–existence family.

From these results we can infer that

Theorem 4.4. *Under the assumptions of Corollary 2.2, the Husimi function $H(x,\xi,t)$ given in (1.12) is a strong solution of the equation (1.16).*

Proof. The estimation (2.3) implies that $\|Y(t)\| \le e^{\delta|t|}$, i.e. all mild solutions of **(CP)** are exponentially bounded. Hence there is no eigenvalues in (δ,∞). Since A is closed the assertion (c) of Lemma 4.3 yields the uniqueness of the solution $Y(t)w_0$ starting with the initial datum $w_0(x,\xi)$ given by (2.4) which lies in $L^1(\mathbb{R}^{2n})$. This produces exactly the Husimi function. □

References

[A-L] R. Alicki and K. Lendi, *Quantum Dynamical Semigroups*, Lecture Notes in Phys., vol. 286, Springer–Verlag, Berlin, New York, 1987.

[Ch] C.K. Chui, *An Intruduction to Wavelets*, Acad. Press, San Diego, New York, 1992.

[Da] E.B. Davies, *Quantum Theory of Open Systems*, Acad. Press, London, New York, 1976.

[E-H] H. Emamirad and R. Holtz, *Vecteurs d'état cohérent et image de la transformée de Husimi*, C. R. Acad. Sc. Paris (Série I) **324** (1997), 1295–1300.

[E-P] H. Emamirad and V. Protopopescu, *On the Liouville Equation in L^1 Spaces*, Letters Appl. Math. **9** (1996), 49–53.

[dL1] R. deLaubenfels, *Existence and uniqueness families for the abstract Cauchy problem*, London Math. Soc **84** (1991), 310–338.

[dL2] R. deLaubenfels, *Existence families, functionnal calculi and evolution equations*, Lect. Notes Math., vol. 1570, Springer–Verlag, 1994.

[H-al] M. Hillery, R.F. O'Connell, M.O. Scully, E.P. Wigner, *Distribution function in physics; fundamentals*, Phys. Rep. **106** (1984), 121–167.

[Hu] K. Husimi, *Some formal properties of the density matrix*, Proc. Phys. Math. Soc. Japan **22** (1940), 264–314.

[L-P] P.L. Lions and T. Paul, *Sur les mesures de Wigner*, Rev. Mat. Iberoam **9** (1993), 553–618.

[Ma1] P.A. Markowich, *On the equivalence of the Schrödinger and the quantum Liouville equation*, Math. Meth. Appl. Sci. **11** (1989), 459–469.

[Ma2] ———, *Kinetic Models for Semiconductors*, Nonequilibrium Problems in Many–Particles Systems (C. Cercignani and M. Pulvirenti, eds.), Lecture Notes in Math., vol. 1551, Springer–Verlag, Berlin, New York, 1993.

[M-R] P.A. Markowich and C.A. Ringhofer, *An analysis of quantum Liouville equation*, ZAMM **69** (1989), 121–127.

[Paz] A. Pazy, *Semigroups of Linear Operators and Application to Partial Differential Equations*, Springer Verlag, New York, Berlin, 1983.

[Sch] M. Schechter, *Spectra of partial differential Operators*, North–Holland Publishing Company, Amsterdam, London, 1971.

[Wi] E.P. Wigner, *On the quantum correction for thermodynamique equilibrium*, Phys. Rev. **40** (1932), no. 2, 749–759.

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