GLOBAL ATTRACTORS OF NONAUTONOMOUS QUASIHOMOGENEOUS DYNAMICAL SYSTEMS

D.N.Cheban

Abstract. It is shown that nonautonomous quasihomogeneous dynamical system admits a compact global attractor. The general results obtained are applied to various differential equations both in finite dimensional spaces and in infinite dimensional spaces (such as ordinary differential equations in Banach space and some types of evolutional partial differential equations).

Introduction

N.N.Krasovskii [1,2], V.I.Zubov [3] and C.Coleman [4] showed that for homogeneous autonomous systems in a finite dimensional space the existence of a power-low asymptotics is equivalent to asymptotic stability.

A.F.Filippov [5,6] generalized this result for homogeneous differential inclusions. N.N.Ladis [7] showed that in the general case this result does not apply to periodic systems. For nonautonomous homogeneous systems (of order k=1) uniform asymptotic stability is equivalent to exponential stability (see, for example [8]). M.V.Morozov [9] obtained a similar result for periodic differential inclusions.

In the paper of author [10] connection between the uniform asymptotic stability and power-low (exponential) asymptotics of solutions of infinite dimensional homogeneous systems is studied.

The goal of the present paper is to prove that the quasihomogeneous differential system

$$x' = f(x) + F(x,t),$$

(where $f(\lambda x) = \lambda^m f(x)$ for all $\lambda > 0$ and $x \in E, |F(x,t)||x|^{-m} \to 0$ as $|x| \to +\infty$) admit a compact global attractor, if the homogeneous differential system

$$x' = f(x)$$

is uniform asymptotic stable. This problem was studied and solved within the framework of general dynamical systems with infinite dimensional phase space. The general results obtained are applied to various differential equations both in finite dimensional spaces and in infinite dimensional spaces (such as ordinary differential equations in Banach space and some types of evolutional partial differential equations).

1991 *Mathematics Subject Classification.* 34C35.
Key words and phrases. nonautonomous quasihomogeneous systems, global attractors.

§ 1. Global attractors of dynamical systems.

Let $(X.\rho)$ be a complete metric space, $\mathbb{R}(\mathbb{Z})$ is a group of real (integer) numbers, $\mathbb{S} = \mathbb{R}$ or $\mathbb{Z}, \mathbb{S}_+ = \{t \in \mathbb{S} | t \geq 0\}$ and $\mathbb{T}(\mathbb{S}_+ \subseteq \mathbb{T})$ is a subgroup of group $\mathbb{S}$.

By $(X, \mathbb{T}, \pi)$ we denote a dynamical system on the X and $xt = \pi(t,x) = \pi^t x$.

The dynamical system $(X, \mathbb{T}, \pi)$ is called [11-14] compact dissipative, if there exists a nonempty compact $K \subseteq X$ such that

$$\lim_{t \to +\infty} \rho(xt, K) = 0 \tag{1.1}$$

for all $x \in X$, moreover equality (1.1) holds uniformly with respect to (w.r.t.) $x \in X$ on each compact from X. In this case the set K is called attractor of family of all the compacts $C(X)$ in the space X .

We assume

$$J = \Omega(K) = \bigcap_{t \geq 0} \overline{\bigcup_{\tau \geq t} \pi^\tau K},$$

then [11-14] a set J does not depend of choice of attractor K and is characterized by the properties of dynamical system $(X, \mathbb{T}, \pi)$. The set J is called [15] Levinson's center of dynamical system $(X, \mathbb{T}, \pi)$.

Let us mention some facts, which will need below.

Will say that a space X has property (S), if for any compact $K \subseteq X$ there exists connected set $M \subseteq X$ such that $K \subseteq M$.

Theorem 1.1 [11-14]. *If $(X, \mathbb{T}, \pi)$ is compact dissipative dynamical system and J is its Levinson's center, then:*

1.J is invariant, i.e. $\pi^t J = J$ for all $t \in T$;

2. J is orbitally stable, i.e. for all $\varepsilon > 0$ there exists $\delta(\varepsilon) > 0$ such that $\rho(x, J) < \delta$ implies $\rho(xt, J) < \varepsilon$ for all $t \geq 0$;

3. J is attractor for a family of all compact subsets of X;

4. $J$is maximal compact invariant set of $(X, \mathbb{T}, \pi)$;

5. a set J is connected, if the space X possesses the (S)-property.

Let Y be a compact metric space and $(X, \mathbb{T}_1, \pi)((Y, \mathbb{T}_2, \sigma))$ be a dynamical system on the $X(Y)$ and $h : X \to Y$ is a homomorphism $(X, \mathbb{T}_1, \pi)$ onto $(Y, \mathbb{T}_2, \sigma)$ then the triple $< (X, \mathbb{T}_1, \pi), (Y, \mathbb{T}_2, \sigma), h >$ is called [16-19] a nonautonomous dynamical system.

Let W and Y be complete metric spaces, $(Y, \mathbb{S}, \sigma)$ is a group dynamical system on Y and $< W, \varphi, (Y, \mathbb{S}, \sigma) >$ is a skew product [20] (cocycle [21-22]) over $(Y, \mathbb{S}, \sigma)$ with the fibre W, i.e. φ is a continuous mapping $W \times Y \times \mathbb{T}$ into W, satisfying the following conditions: $\varphi(0, w, y) = w$ and $\varphi(t + \tau, w, y) = \varphi(t, \varphi(\tau, w, y), \sigma(\tau, y))$ for all $t, \tau \in \mathbb{T}, w \in W$ and $y \in Y$.

We denote by $X = W \times Y$ and define on X a dynamical system $(X, \mathbb{T}, \pi)$ by equality $\pi = (\varphi, \sigma)$ i.e. $\pi(t, (w, y)) = (\varphi(t, w, y), \sigma(t, y))$ for all $t \in \mathbb{T}$ and $(w, y) \in W \times Y$,then the triple $< (X, \mathbb{T}, \pi), ((Y, \mathbb{S}, \sigma), h >$, where $h = pr_2$, is a nonautonomous dynamical system.

For any two bounded subsets A and B from X by $\beta(A, B)$ we denote semi-deviation A to B , i.e. $\beta(A, B) = \sup\{\rho(a, B) | a \in A\}$ and $\rho(a, B) = \inf\{\rho(a, b) | b \in B\}$.

The skew product over $(Y, \mathbb{S}, \sigma)$ with the fibre W is called [13] compact dissipative, if there exists a nonempty compact $K \subseteq W$ such that

$$\lim_{t\to+\infty} \sup\{\beta(U(t,y)M, K) : y \in Y\} = 0 \tag{1.2}$$

for all $M \in C(W)$, where $U(t,y) = \varphi(t, \cdot, y)$.

Lemma 1.2. *In order to skew product $< W, \varphi, (Y, \mathbb{S}, \sigma) >$ over $(Y, \mathbb{T}, \sigma)$ with fibre W were a compact dissipative, it is necessary and sufficiently that the autonomous dynamical system $(X, \mathbb{T}, \pi)$ $\quad (X = W \times Y$ and $\pi = (\varphi, \sigma))$ will be a compact dissipative.*

By entire trajectory of semi-group dynamical system $(X, \mathbb{T}, \pi)$ (of skew product $< W, \varphi, (Y, \mathbb{S}, \sigma) >$ over $(Y, \mathbb{T}, \sigma)$ with fibre W), passing through point $x \in X$ ($(u, y) \in W \times Y$) is called continuous mapping $\gamma : \mathbb{S} \to X(\nu : \mathbb{S} \to W)$ satisfying conditions : $\gamma(0) = x(\nu(0) = w)$ and $\gamma(t+\tau) = \pi^t\gamma(\tau)$ $\quad (\gamma(t+\tau) = \varphi(t, \nu(\tau), y\tau))$ for all $t \in \mathbb{T}$ and $\tau \in \mathbb{S}$.

Theorem 1.3 [13]. *Let Y be compact, $< W, \varphi, (Y, \mathbb{S}, \sigma) >$ is compactly dissipative and K is non-empty compact, appearing in equality (1.2), then:*

1. $I_y = \Omega_y(K) \neq \emptyset$, is compact, $I_y \subseteq K$ and

$$\lim_{t\to+\infty} \beta(U(t, y^{-t})K, I_y) = 0$$

for every $y \in Y$, where

$$\Omega_y(M) = \bigcap_{t\geq 0} \overline{\bigcup_{\tau \geq t} U(\tau, y^{-\tau})M}$$

and $y^{-\tau} = \sigma(-\tau, y)$;

2. $U(t,y)I_y = I_{yt}$ for all $y \in Y$ and $t \in \mathbb{T}$;

3.

$$\lim_{t\to+\infty} \beta(U(t, y^{-t})M, I_y) = 0 \tag{1.3}$$

for all $M \in C(W)$ and $y \in Y$;

4.

$$\lim_{t\to+\infty} \sup\{\beta(U(t, y^{-t})M, I) : y \in Y\} = 0$$

whatever is $M \in C(W)$, where $I = \bigcup\{I_y : y \in Y\}$;

5. $I = pr_1 J$ and $I_y = pr_1 J_y$, where J is a center of Levinson of (X, T, π) and $J_y = J \bigcap X_y$;

6. the set I is compact ;

7. the set I is connected if the space $W \times Y$ has property (S).

Theorem 1.4. *In conditions of theorem 1.3 the following affirmations take place;*

1. $w \in I_y (y \in Y)$ if and only if, when there exists whole trajectory $\nu : \mathbb{S} \to W$ of the skew product $< W, (Y, \mathbb{S}, \sigma) >$, satisfying the following conditions: $\nu(0) = w$ and $\nu(\mathbb{S}) = \{\nu(s) | s \in \mathbb{S}\}$ is relatively compact ;

2. $I_y (y \in Y)$ *is connected, if a space* W *possesses* $(S)-$ *property .*

Proof. To prove the first affirmation we note, that a continuous function $\nu : \mathbb{S} \to W$ is a whole trajectory of the skew product $< W, (Y, \mathbb{S}, \sigma) >$ if and only if, when $\gamma = (\nu, Id_Y)$ is a whole trajectory of semigroup dynamical system $(X, \mathbb{T}, \pi)$ $(X = W \times Y, \pi = (\varphi, \sigma))$. According to lemma 1.2 a dynamical system $(X, \mathbb{T}, \pi)$ is compactly dissipative and by theorem 1.1 the set J is compact and invariant and, consequently, the point $(w, y) = x \in J$ if and only if, through $(w, y) = x$ passe a whole trajectory $\gamma = (\nu, Id_Y)$ of dynamical system $(X, \mathbb{T}, \pi)$ which is completely included in J, i.e. $\gamma(0) = (\nu(0), y) = (w, y)$ and $\gamma(s) \in J$ for all $s \in \mathbb{S}$. In order to prove first affirmation of theorem it is sufficiently to refer to theorem 1.3 (item 5).

In order to prove the second affirmation, we note that in conditions of theorem 1.3 the set $I_y \neq \emptyset$ and is compact. Since the space W possesses the $(S)-$property, then there is a connected compact $V \supseteq I$.

According to (1.3) takes place the equality

$$\lim_{t \to +\infty} \beta(U(t, y^{-t})V, I_y) = 0.$$

We note, that $I_y \subseteq U(t, y^{-t})V$ for each $y \in Y$ and $t \in \mathbb{T}$, the mapping $U(t, y^{-y}) : W \to W$ is continuous and , consequently, the set $U(t, y^{-t})V$ is compact and connected.

Now to complete the proof of theorem 1.4 it is sufficient to refer to lemma 3.12 from [22].

The dynamical system $(X, \mathbb{T}, \pi)$ is called [11-13,23] asymptotically compact, if for every bounded positively invariant set $M \subseteq X$ there exists a nonempty compact $K \subseteq X$ such that

$$\lim_{t \to +\infty} \beta(\pi^t M, K) = 0.$$

The nonautonomous dynamical system $< (X, \mathbb{T}_1, \pi), ((Y, \mathbb{T}_2, \sigma), h >$ is called asymptotically compact if Y is compact and autonomous dynamical system $(X, \mathbb{T}_1, \pi)$ is asymptotically compact.

The nonempty set $J \subseteq X$ is called [11-13,23-24] a global attractor of dynamical system $(X, \mathbb{T}, \pi)$, if J is a maximal compact invariant set of $(X, \mathbb{T}, \pi)$ and J attracts the every bounded subset $M \subseteq X$, i.e.

$$\lim_{t \to +\infty} \beta(\pi^t M, J) = 0.$$

Theorem 1.5 [14]. *Let* Y *be compact and* (X, h, Y) *is locally trivial Banach fibering,* $|\cdot| : X \to \mathbb{R}_+$ *is the norm on* (X, h, Y) *coordinated with the metric* ρ *on* X *(i.e.* $\rho(x_1, x_2) = |x_1 - x_2|$ *for all* $x_1, x_2 \in X$ *such that* $h(x_1) = h(x_2)$ *) and* $(X, \mathbb{T}, \pi)$ *is asymptotically compact. If there are a number* $r > 0$ *and function* $V : X_r \to \mathbb{R}_+$ $(X_r = \{x \in X| \quad |x| \geq r\})$ *with the following properties:*

1. the function V *is bounded on bounded sets and for every* $c \in \mathbb{R}_+$ *the set* $\{x \in X_r | V(x) \leq c\}$ *is bounded ;*

2. $V'_\pi(x) \leq -c(|x|)$ *for all* $x \in X_r$*, where* $c : \mathbb{R}_+ \to \mathbb{R}_+$ *is positive on* $[r, +\infty)$*,* $V'_\pi(x) = \lim_{t \to +0} \sup t^{-1}[V(xt) - V(x)]$ *if* $\mathbb{T} = \mathbb{R}_+$ *and* $V'_\pi(x) = V(x1) - V(x)$ *if* $\mathbb{T} = \mathbb{Z}_+$.

Then the dynamical system $(X, \mathbb{T}, \pi)$ admits a compact global attractor.

§ 2. Homogeneous dynamical systems. Method of Lyapunov functions.

Let (X, h, Y) be a locally trivial Banach fibering, $|\cdot| : X \to \mathbb{R}_+$ is the norm on (X, h, Y) coordinated with the metric ρ on X.

The autonomous dynamical system $(X, \mathbb{R}_+, \pi)$ is called [10] homogeneous of order $m \in \mathbb{R}_+$, if for any $x \in X, t \in \mathbb{R}_+$ and $\lambda > 0$ takes place equality $\pi(t, \lambda x) = \lambda\pi(\lambda^{m-1}t, x)$.

The nonautonomous dynamical system $< (X, \mathbb{T}, \pi), ((Y, \mathbb{T}, \sigma), h >$ is called homogeneous of order $m = 1$ if the autonomous dynamical system $(X, \mathbb{T}, \pi)$ is homogeneous of order $m = 1$.

Theorem 2.1. *For an autonomous homogeneous (of order $m > 1$) dynamical system $(X, \mathbb{R}_+, \pi)$ the following assertions are equivalent:*

1.there exist positive numbers a and b such that

$$|\pi(t, x)| \leq (a|x|^{1-m} + bt)^{\frac{1}{1-m}} \tag{2.1}$$

for all $t \geq 0$ and $x \in X$;

2. for all $k > m - 1$ there exists continuous function $V : X \to \mathbb{R}_+$ with the following properties:

2.1. $V(\lambda x) = \lambda^{k-m+1}V(x)$ for all $\lambda \geq 0$ and $x \in X$;

2.2. $\alpha|x|^{k-m+1} \leq V(x) \leq \beta|x|^{k-m+1}$ for all $x \in X$, where α and β are certain positive numbers;

2.3. $V'_\pi(x) = -|x|^k$ for all $x \in X$, where $V'_\pi(x) = \frac{d}{dt}V(\pi(t, x)) \mid_{t=0}$ for $\mathbb{T} = \mathbb{R}_+$ and $V'_\pi(x) = V(\pi(1, x)) - V(x)$ for $\mathbb{T} = \mathbb{Z}_+$.

Proof. We will show that from 1. results 2.. Let a and b are positive numbers, such that inequality (2.1) takes place , then for each $k > m-1$ we define a function $V : X \to \mathbb{R}_+$ by equality

$$V(x) = \int_0^{+\infty} |\pi(t, x)|^k dt. \tag{2.2}$$

First of all we note that by equality (2.2) it is defined correctly a function $V : X \to \mathbb{R}_+$ because the integral, which figures in second member of equation (2.2) is convergent, moreover is uniformly convergent w.r.t. x on the all bounded set from X. Really, since

$$|\pi(t, x)|^k \leq (a|x|^{1-m} + bt)^{\frac{k}{1-m}}, \tag{2.3}$$

$$\int_0^{+\infty} |(a|x|^{1-m} + bt)^{\frac{k}{1-m}} dt = \frac{1}{b} \int_{a|x|^{1-m}}^{+\infty} \tau^{\frac{k}{1-m}} d\tau \tag{2.4}$$

and $\frac{k}{1-m} < -1$,then the integral (2.4) is convergent, moreover the convergence is uniform on the every bounded set from X.

We will show that a function V, defined by equality (2.2), is our unknown function. The continuity of V results from the continuity of mapping $\pi : \mathbb{T} \times X \to X$ and uniform convergence of integral (2.4) w.r.t. x on every bounded set from X .

We now note that

$$V(\lambda x) = \int_0^{+\infty} |\pi(t,\lambda x)|^k dt = \int_0^{+\infty} \lambda^k |\pi(\lambda^{m-1}t,x)|^k dt$$
$$= \lambda^{k-m+1} \int_0^{+\infty} |\pi(\tau,x)|^k dt = \lambda^{k-m+1} V(x)$$

for all $\lambda > 0$ and $x \in X$. It is not difficult to show that the function V is positive definite. Since

$$V(x) = \int_0^{+\infty} |\pi(t,x)|^k, dt \geq \int_0^{\varepsilon} |\pi(t,x)|^k, dt = |\pi(\xi,x)|^k \tag{2.5}$$

for all $\varepsilon > 0$, where $\xi \in [0,\varepsilon]$, then passing to the limit in (2.5) as $\varepsilon \to 0$ we obtain

$$V(x) \geq |x|^k \tag{2.6}$$

for all $x \in X$. In virtue of (2.3) and (2.4) we have

$$V(x) \leq \beta |x|^{k-m+1} \tag{2.7}$$

for every $x \in X$, where

$$\beta = \frac{(m-1)a^{\frac{k-m+1}{1-m}}}{b(k-m+1)}.$$

From (2.4)-(2.7) results that the function V satisfies and condition 2.2 .

Finally, we note that

$$\frac{d}{dt} V(\pi(t,x)) = -|\pi(t,x)|^k \tag{2.8}$$

and, consequently, $V'_\pi(x) = -|x|^k$ for all $x \in X$.

We will prove now that from condition 2. results 1.. Really, we denote by $\psi(t) = V(\pi(t,x))$, then in virtue of condition 2.3 we will have

$$\psi'(t) = -|\pi(t,x)|^k \tag{2.9}$$

for all $t \geq 0$.

From condition 2.2 we have $|\pi(t,x)|^{k-m+1} \geq \frac{1}{\beta}\psi(t)$ and, consequently,

$$\psi'(t) \leq -\frac{1}{\beta^{\frac{k}{k-m+1}}} \psi(t)^{\frac{k}{k-m+1}}$$

for all $t \geq 0$. If $x \neq 0$, then $\psi(t) = V(\pi(t,x)) > 0$ for all $t \geq 0$,therefore

$$V(\pi(t,x)) \leq (V^{-\frac{m-1}{k-m+1}}(x) + \frac{m-1}{k-m+1} \frac{1}{\beta^{\frac{k}{k-m+1}}} t)^{\frac{1}{1-m}} \tag{2.10}$$

for all $x \in X$ and $t \geq 0$.

From of condition 2.2 and the inequality (2.10) results that $|\pi(t,x)| \leq (a|x|^{1-m} + bt)^{\frac{1}{1-m}}$ for all $x \in X$ and $t \geq 0$, where

$$a = (\alpha\beta)^{\frac{m-1}{k-m+1}} \quad \text{and} \quad b = (\alpha)^{\frac{m-1}{k-m+1}} (\beta)^{\frac{k}{k-m+1}} \frac{m-1}{k-m+1}.$$

The theorem is completely proved.

Corollary 2.2.. *For an autonomous homogeneous (of order $m > 1$) dynamical system $(X, \mathbb{R}_+, \pi)$ the following assertions are equivalent:*

1. the trivial motion of the dynamical system $(X, \mathbb{R}_+, \pi)$ is uniform asymptotical stable;

2. there exist positive numbers a and b such that $|\pi(t,x)| \le (a|x|^{1-m} + bt)^{\frac{1}{1-m}}$ for all $t \in \mathbb{R}_+$ and $x \in X$;

3. for every number $k > m-1$ there exists a continuous function $V : X \to \mathbb{R}_+$ which possesses properties 2.1-2.3 from the theorem 2.1..

This assertion directly follows from the theorem 2.1 and theorem 1.2 from [22].

Theorem 2.3. *Let a nonautonomous system $<(X, \mathbb{T}, \pi), ((Y, \mathbb{T}, \sigma), h>$ be homogeneous of order $m = 1$. Then the following conditions are equivalent:*

1. there exist positive numbers N and ν such that $|\pi(t,x)| \le Ne^{-\nu t}|x|$ for all $x \in X$ and $t \ge 0$;

2. for each $k > 0$ there exists a continuous function $V : X \to \mathbb{R}_+$ satisfying the following conditions :

2.1. $V(\lambda x) = \lambda^k V(x)$ for all $x \in X$ and $\lambda > 0$;

2.2. there exist a positive numbers $\alpha \ge 1$ and β so that

$$\alpha|x|^k \le V(x) \le \beta|x|^k \tag{2.11}$$

for every $x \in X$;

2.3. $V'_\pi(x) = -|x|^k$ for all $x \in X$.

Proof. Let us show that under the conditions of theorem from assumption 1. follows 2.. First of all let us show that function V, defined by equality (2.2) is the unknown, in the case when $\mathbb{T} = \mathbb{R}_+$. We note that by equality (2.2) is correctly defined a function $V : X \to \mathbb{R}_+$, such that

$$|\pi(t,x)|^k \le N^k e^{-\nu k t}|x|^k \tag{2.12}$$

for all $t \ge 0, x \in X$ and, consequently, the integral in second member of equality (2.2) is convergent uniformly w.r.t. x on every bounded set from X. In particular, the function $V$is continuous w.r.t. $x \in X$.

From (2.2),(2.6) and (2.12) follows that

$$\alpha|x|^k \le V(x) \le \beta|x|^k \tag{2.13}$$

for every $x \in X$, where $\beta = N^k(\nu k)^{-1}$ and $\alpha = 1$. From equality (2.4) we have that $V(\lambda x) = \lambda^k V(x)$ for every $x \in X$ and $\lambda > 0$, and from (2.8) implies the equality $V'_\pi(x) = -|x|^k$.

Let us show that the inverse implication takes place too . We now suppose that the conditions 2.1-2.3 of theorem hold. One denote by $\psi(t) = V(\pi(t,x))$, then the equality (2.9) takes place for all $t \ge 0$. From condition 2.2 follows that $|\pi(t,x)|^k \ge \frac{1}{\beta}\psi(t)$ and, consequently,

$$\psi'(t) \le -\frac{1}{\beta}\psi(t) \tag{2.14}$$

for all $t \ge 0$. From (2.14) we have

$$V(\pi(t,x)) \leq V(x)e^{-\frac{1}{\beta}t} \tag{2.15}$$

for any $t \geq 0$ and $x \in X$. According to (2.13) and (2.15) we have

$$|\pi(t,x)| \leq Ne^{-\nu t}|x|$$

for any $t \geq$ and $x \in X$, where $N = (\frac{\beta}{\alpha})^{\frac{1}{k}}$ and $\nu = \frac{1}{\beta k}$.

If $\mathbb{T} = \mathbb{Z}_+$, then we will define a function $V : X \to \mathbb{R}_+$ by equality

$$V(x) = \sum_{n=1}^{+\infty} |\pi(n,x)|^k. \tag{2.16}$$

The series in second member of equality (2.16) is convergent uniformly w.r.t. x on every bounded subset from X, because according to condition of theorem we have

$$|\pi(n,x)|^k \leq N^k e^{-\nu k n}|x|^k \tag{2.17}$$

for all $n \in \mathbb{Z}_+$ and $x \in X$ and, consequently, the series (2.16) is majorized by geometric series with the denominator $q = e^{-\nu k} < 1$. Thus a function V defined by equality (2.16) is continuous.

From the homogeneity of $(X,\mathbb{T},\pi)$ of order $m = 1$ and equality (2.16) follows that $V(\lambda x) = \lambda^k V(x)$ for all $\lambda > 0$ and $x \in X$. According to (2.16) and (2.17) we have

$$\alpha|x|^k \leq V(x) \leq \beta|x|^k \tag{2.18}$$

for any $x \in X$, where $\alpha = 1$ and $\beta = N^k(1-e^{-\nu k})^{-1}$. Finally, from equality (2.16) follows that $V'_\pi(x) = V(\pi(1,x)) - V(x) = -|x|^k$ for all $x \in X$.

We now will show that from conditions 2.1-2.3 of theorem (in the case, when $\mathbb{T} = \mathbb{Z}_+$) follows the estimation (2.11). Really, we denote by $\psi(n) = V(\pi(n,x))$, then from conditions 2.2-2.3 we have

$$\Delta\psi(n) = \psi(n+1) - \psi(n) = -|\pi(n,x)|^k \leq -\frac{1}{\beta}\psi(n). \tag{2.19}$$

We may assume without loss of generality that $\beta > 1$, then from (2.19) we have

$$V(\pi(n,x)) \leq \gamma^n V(x) \tag{2.20}$$

for all $x \in X$ and $n \in \mathbb{Z}_+$, where $\gamma = 1 - \beta^{-1}$. From (2.18) and (2.20) follows the inequality (2.11), if we suppose that $N = (\frac{\beta}{\alpha})^{\frac{1}{k}}$ and $\nu = -\frac{1}{k}ln(1-\beta^{-1})$. The theorem is proved.

Corollary 2.4. *Let a nonautonomous dynamical system* $< (X,\mathbb{T},\pi),((Y,\mathbb{T},\sigma),h >$ *be homogeneous of order* $m = 1$*. Then the following conditions are equivalent:*

1. the zero section of fibering (X,h,Y) *is uniform asymptotic stable, i.e.*

1.a. for all $\varepsilon > 0$ *there exists* $\delta(\varepsilon) > 0$ *such that* $|\pi(t,x)| < \varepsilon$ *for all* $t \geq 0$ *and* $|x| < \delta$*;*

1.b. $\lim_{t\to+\infty} |\pi(t,x)| = 0$ *for all* $x \in X$, *moreover this equality takes place uniformly w.r.t.* x *on every bounded set from* X*;*

2. there exist positive numbers N *and* ν *such that* $|\pi(t,x)| \le Ne^{-\nu t}|x|$ *for all* $t \ge 0$ *and* $x \in X$*;*

3. for all $k > 0$ *there exists a continuous function* $V : X \to \mathbb{R}_+$ *satisfying the conditions 2.1-2.3 of theorem 2.3 .*

This affirmation follows from the theorem 2.3 and theorem 1.1 from [10].

Remark 2.5. *In case, when space* E *is finite dimensional the theorem 2.1 and theorem 2.2 (for autonomous system with* $\mathbb{T} = \mathbb{R}_+$ *) generalizes and precises some results of V.I.Zubov (see, for example [3], theorems 36 and 37).*

§ 3. Differentiable homogeneous systems.

Let H be a Hilbert space with a scalar product $< \cdot, \cdot >$ and norm $|\cdot| = \sqrt{< \cdot, \cdot >}$. We denote by $C(E)(C^1(E))$ the space of all continuous (differentiable continuous) functions defined on the space E wit values in some Banach space .

The function $f : E \times \Omega \to F$ is called homogeneous of order k w.r.t. variable $u \in E$ if the equality $f(\lambda x, \omega) = \lambda^k f(x,\omega)$ takes place for all $\lambda > 0, x \in E$ and $\omega \in \Omega$.

Lemma 3.1. *The following assertions take place.*

1. Let Ω *be a compact ,* $f \in C(E \times \Omega), f(0,\omega) = 0$ *for all* $\omega \in \Omega, f(\lambda u,\omega) = \lambda^m f(u,\omega)$ *(for every* $\lambda > 0$*and* $(u,\omega) \in E \times \Omega$*)). Then there exists* $M > 0$ *such that* $|f(u,\omega)| \le M|u|^m$ *for all* $(u,\omega) \in E \times \Omega$.

2. If the function $f \in C^1(E \times \Omega)$ *is homogeneous (of order* $m \ge 1$*) then the function* $D_u f(\cdot,\omega) : E \to L(E,F) \quad (\omega \in \Omega)$ *will be homogeneous (of order* $m-1$*), where* $L(E,F)$ *is a space of all linear continuous operators* $A : E \to F$*;*

3.The function $f \in C^1(E)$ *is homogeneous (of order* m*) if and only if,* f *satisfies the equation* $Df(x)x = mf(x)$ *for all* $x \in E$*, where* $Df(x)$ *is a derivative of Frechet of function* $f \in C^1(E)$ *in the point* x;

Proof. First of all we note that in conditions of Lemma 3.1 there exists $\delta_0 > 0$ such that $|f(u,\omega)| \le 1$ for all $|u| \le \delta$ and $\omega \in \Omega$. If we suppose that it is not so, then there exist $\delta_n \to 0, |u_n| < \delta_n$ and $\omega_n \in \Omega$ such that $|f(u_n,\omega_n)| > 1$. Since Ω is compact we may assume that the sequence $\{\omega_n\}$ is convergent. Let $\omega_n \to \omega_0$, then according to continuity of f one have $|f(0,\omega_0)| \ge 1$. On the other hand we have $f(0,\omega_0) = 0$. The obtained contradiction prove the required affirmation.

Thus there exists $\delta > 0$ such that $|f(u,\omega)| \le 1$ for all $|u| \le \delta$ and $\omega \in \Omega$ and, consequently, we have

$$|f(u,\omega)| \le |f(\frac{|u|}{\delta}\frac{u\delta}{|u|},\omega)| = \frac{|u|^m}{\delta^m}|f(\frac{\delta u}{|u|},\omega)| \le \frac{1}{\delta^m}|u|^m.$$

Let $f \in C^1(E)$ and f is homogeneous (of order m), then

$$f(x+h) - f(x) = Df(x)h + r(x,h) \tag{3.1}$$

and $|r(x,h)| \to 0$ as $|h| \to 0$. We now will replace x by λx in equality (3.1), then we obtain

$$f(\lambda x + h) - f(\lambda x) = Df(\lambda x) + r(\lambda x, h)$$

and, consequently,

$$f(x+u) - f(x) = \lambda^{-m+1} Df(\lambda x)h + \lambda^{-m} r(\lambda x, \lambda u) \tag{3.2}$$

for all $\lambda > 0$, where $u = \lambda^{-1}h$. From (3.2) follows, that $\lambda^{-m+1}Df(\lambda x) = Df(x)$, i.e. $Df(\lambda x) = \lambda^{m-1}Df(x)$ for any $\lambda > 0$.

Finally, let us prove the third affirmation of Lemma. Let $f \in C^1(E)$ be a homogeneous function (of order m). Having differentiated the identity $f(\lambda x) = \lambda^m f(x)$ w.r.t. λ and taking $\lambda = 1$ one obtain the identity $Df(x)x = mx$. Let now the identity $Df(x)x = mx$ takes place. We denote by $\varphi(\lambda) = \lambda^{-m}f(\lambda x)(\lambda > 0)$. Thus $Df(\lambda x)\lambda x = mf(\lambda x)$, then

$$\varphi'(\lambda) = -m\lambda^{-m+1}f(\lambda x) + \lambda^{-m}Df(\lambda x)x$$

$$= -m\lambda^{-m-1}f(\lambda x) + \lambda^{-m-1}mf(\lambda x) = 0$$

for all $\lambda > 0$ and, consequently, $\varphi(\lambda) = const$ for all $\lambda > 0$. Thus $\varphi(1) = f(x)$, when $\varphi(\lambda) = f(x)$ for any $\lambda > 0$, i.e. $f(\lambda x) = \lambda^m f(x)$ for every $\lambda > 0$ and for all $x \in E$. The lemma is proved.

A function $f \in C(E)$ is said to be regular, if for a differential equation

$$x' = f(x) \tag{3.3}$$

are fulfilled the conditions of existence, uniqueness on $\mathbb{R}_+$ and continuous dependence on initial data, i.e. by equation (3.3) is generated a semigroup dynamical system $(E, \mathbb{R}_+, \pi)$, where $\pi(t,x)$ is a solution of equation (3.3) with initial condition $\pi(0,x) = x$.

Theorem 3.2. *Let $f \in C'(E)$ be regular and homogeneous (of order $m > 1$), then the following conditions are equivalent:*

1. the zero solution of equation (3.3) is uniform asymptotic stable;

2. for all sufficiently large k there exists a continuously differentiable function $V : E \to \mathbb{R}_+$ satisfying the following conditions:

2.1. $V(\lambda x) = \lambda^{k-m+1}V(x)$ for all $\lambda \geq 0$ and $x \in E$;

2.2. there exist positive numbers α and β such that $\alpha|x|^{k-m+1} \leq V(x) \leq \beta|x|^{k-m+1}$ for all $x \in E$;

2.3. $V'_\pi(x) = DV(x)f(x) = -|x|^k$ for any $x \in E$, where $DV(x)$ is a derivative of Frechet of function V in the point x.

Proof. We suppose that a zero solution of equation (3.3) is uniform asymptotic stable. Denote by $(E, \mathbb{R}_+, \pi)$ a semigroup dynamical system, generated by equation (3.3). Thus, $f \in C^1(E)$ is homogeneous (of order $m > 1$), then according to theorem 3.4 [3] the equality $\pi(t, \lambda x) = \lambda\pi(\lambda^{m-1}t, x)$ takes place for all $x \in E, \lambda \geq 0$ and $t \in \mathbb{R}_+$, i.e. a dynamical system $(E, \mathbb{R}_+, \pi)$ is homogeneous (of order $m > 1$). According to the theorem 2.1 and corollary 2.2 by equality (2.2) is defined a continuous function $V : E \to \mathbb{R}_+$, satisfying the condition 2.1-2.3. of theorem 3.2. Let us show that in the conditions of theorem 3.2 function V will be continuously differentiable and that the equality $V'_\pi(x) = DV(x)x$ takes place. To this end we will formally differentiate the equality (2.2) w.r.t. x, then we will have

$$DV(x)u = \int_0^{+\infty} k|\pi(t,x)|^{k-2} Re < D_x\pi(t,x)u, \pi(t,x) > dt. \tag{3.4}$$

We now will show that the integral figuring in the second hand of formula (3.4), is uniformly convergent w.r.t. x and u on the every ball from E and, consequently, the equality (3.4) really defines a derivative of function V. We note that an operator-function $U(t,x) = D_x\pi(t,x)$ satisfies the operational equation

$$X' = \mathcal{A}(t,x)X, \tag{3.5}$$

where $\mathcal{A}(t,x) = D_x f(\pi(t,x))$, and initial condition $U(0,x) = I$ $\quad$ ($I-$is identity operator in E). According to Lemma 3.1 the function $\mathcal{A}(t,x) = D_x f(\pi(t,x))$ is homogeneous w.r.t. $\pi(t,x)$ of order $m-1$ and there exists a number $M > 0$ such that

$$||\mathcal{A}(t,x)|| \leq M|\pi(t,x)|^{m-1} \tag{3.6}$$

for all $x \in E$ and $t \geq 0$. Since the zero solution of equation (3.3) is uniform asymptotic stable, then in virtue of corollary 2.2 there exist positive numbers a and b such that

$$|\pi(t,x)| \leq (a|x|^{1-m} + bt)^{\frac{1}{1-m}} \tag{3.7}$$

for any $x \in E$ and $t \geq 0$ and, consequently, from (3.6)-(3.7) one have

$$||\mathcal{A}(t,x)|| \leq M(a|x|^{1-m} + bt)^{-1}. \tag{3.8}$$

Since

$$||U(t,x)|| \leq e^{\int_0^t ||\mathcal{A}(\tau,x)||, d\tau}, \tag{3.9}$$

then from (3.8) and (3.9) we have

$$||U(t,x)|| \leq (a|x|^{1-m} + bt)^{\frac{M}{b}} (a^{-1}|x|^{m-1})^{\frac{M}{b}} \tag{3.10}$$

for every $t \geq 0$ and $x \in E$. In virtue of inequality (3.10) we have

$$\begin{aligned} |\pi(t,x)|^{k-2} |Re < D_x\pi(t,x)u, \pi(t,x) > | &\leq |\pi(t,x)|^{k-1} ||D_x\pi(t,x)|| |u| \\ &\leq (a|x|^{1-m} + bt)^{-\frac{k-1}{m-1} + \frac{M}{b}} a^{-\frac{M}{b}} |x|^{(m-1)\frac{m}{b}} |u| \end{aligned} \tag{3.11}$$

for all $t \geq 0$ and $x, u \in E$. From estimation (3.11) follows that for sufficiently large k the integral in second member of equality (3.4) is uniformly convergent w.r.t. x and u on the each ball from E and, consequently, the function V ,defined by equality (2.2), in conditions of theorem 3.2 is continuously differentiable.

We now note, that

$$V'_\pi(x) = \frac{d}{dt}V(\pi(t,x))\Big|_{t=0} = DV(\pi(t,x))f(\pi(t,x))\Big|_{t=0} = DV(x)f(x)$$

for all $x \in E$. On the other hand, according to theorem 2.1 $V'_\pi(x) = -|x|^k$.

For finishing the proof of the theorem it is sufficient to notice that in conformity with theorem 2.1 takes place the inverse statement, i.e. conditions 2.1-2.3 of theorem 3.2 imply 1.. The theorem is completely proved.

Now let $f \in C^1(E \times F)$ and $\Phi \in C^1(F)$. Consider the differential system

$$\begin{cases} u' = f(u,\omega) \\ \omega' = \Phi(\omega) \end{cases} \qquad (\omega \in \Omega), \tag{3.12}$$

where Ω is a certain differentiable compact submanifold in F, f and Φ are regular functions, i.e. for the differential system (3.12) end equation

$$\omega' = \Phi(\omega) \tag{3.13}$$

are fulfilled the conditions of existence, uniqueness and continuous dependence on initial data on the $\mathbb{R}_+$ for (3.12) and on the $\mathbb{R}$ for (3.13). We denote by $(\Omega, \mathbb{R}, \sigma)$ a group dynamical system, generated by equation (3.13), then the system (3.12) may be written in the form of nonautonomous equation

$$u' = f(u, \omega t), \quad (\omega \in \Omega) \tag{3.14}$$

where $\omega t = \sigma(t,\omega)$. Let $\varphi(t,u,\omega)$ be a solution of equation (3.14) satisfying the initial condition $\varphi(0,u,\omega) = u$, then it is not difficult to see that the triple $< E, \varphi, (\Omega, \mathbb{R}, \sigma) >$ is a skew product (cocycle) over $(\Omega, \mathbb{R}, \sigma)$ with the fibre E. It is easy to check that if a function f is homogeneous of order $m = 1$ w.r.t. $u \in E$, then takes place the equality $\varphi(t, \lambda u, \omega) = \lambda \varphi(t,u,\omega)$, for all $t \geq 0, \lambda > 0, u \in E$ and $\omega \in \Omega$.

Theorem 3.3. *Let a function $f \in C^1(E \times F)$and $\Phi \in C^1(F)$ are regular, Ω is compact and the function f is homogeneous (of order $m = 1$) w.r.t. variable $u \in E$, then the following conditions are equivalent:*

1. the trivial solution of equation (3.14) is uniform exponential stable, i.e. there exist positive numbers N and ν such that

$$|\varphi(t,u,\omega)| \leq Ne^{-\nu t}|u| \tag{3.15}$$

for all $u \in E, t \geq 0$ and $\omega \in \Omega$;

2. for all number $k > 1$ there exists a continuously differentiable function $V : E \times \Omega \to \mathbb{R}_+$ satisfying the following conditions:

2.1. $V(\lambda u, \omega) = \lambda^k V(u,\omega)$ for all $u \in \Omega$ and $\lambda > 0$;

2.2. there exist positive numbers α and β such that $\alpha|u|^k \leq V(u,\omega) \leq \beta|u|^k$ for all $u \in E$ and $\omega \in \Omega$;

2.3.

$$V'_\pi(u,\omega) = \frac{d}{dt}V(\varphi(t,u,\omega),\omega t)\Big|_{t=0} = D_uV(u,\omega)f(u,\omega) + D_\omega V(u,\omega)\Phi(\omega) = -|u|^k$$

for all $u \in E$ and $\omega \in \Omega$, where $D_u V(D_\omega V)$ is a partial Frechet's derivative of function V w.r.t. variable $u \in E(\omega \in \Omega)$.

Proof. Consider a nonautonomous dynamical system $< (X, \mathbb{R}_+, \pi), (\Omega, \mathbb{R}, \sigma), h >$ generated by differential system (3.12), where $X = E \times \Omega, h = pr_2$ and $\pi = (\varphi, \sigma)$. The triple (X, h, Ω) is the trivial vectorial boundle with the fibre E, moreover the norm of element $x = (u, \omega) \in E \times \Omega$ is defined by equality $|x| = |u|$, where $|u|$ is a norm of element u in the space E. Then in conditions of theorem the zero section of the nonautonomous dynamical system constructed above will satisfy conditions of theorem 2.3 .Thus to finish the proof of theorem 3.3 it is sufficient to show that function $V : E \times \Omega \to \mathbb{R}_+$ from the theorem 2.3 in conditions of theorem 3.3 will be continuously differentiable and satisfies the condition

$$V'_\pi(u, \omega) = D_u V(u, \omega) f(u) + D_\omega V(u, \omega)\Phi(\omega)$$

for all $(u, \omega) \in E \times \Omega$.

Let us show that a function $V : E \times \Omega \to \mathbb{R}_+$, defined by equality (2.2), i.e.

$$V(u, \omega) = \int_0^{+\infty} |\varphi(t, u, \omega)|^k dt \tag{3.16}$$

for all $(u, \omega) \in E \times \Omega$ (such that $|\pi(t, x)| = |(\varphi(t, u, \omega), \omega t)| = |\varphi(t, u, \omega)|$) is continuously differentiable. To this end we will formally differentiate the equality (3.16) w.r.t. variable $u \in E$, thus we obtain

$$D_u V(u, \omega)v = \int_0^{+\infty} k|\varphi(t, u, \omega)|^{k-2} Re < D_u\varphi(t, u, \omega)v, \varphi(t, u, \omega) > dt. \tag{3.17}$$

Let us show that integral in the second hand of (3.17) is uniformly convergent w.r.t. $\omega \in \Omega$ and u, v on every bounded set from E and since by formula (3.17) is really defined a derivative of Frechet of function V w.r.t. variable $u \in E$. First of all we note that operator-function $U(t, u, \omega) = D_u\varphi(t, u, \omega)$ satisfies the operational equation

$$X' = \mathcal{B}(t, u, \omega)X, \tag{3.18}$$

where $\mathcal{B}(t, u, \omega) = D_u f(\varphi(t, u, \omega), \omega t)$, and initial condition $U(0, u, \omega) = I \quad (I-$ is the identity operator in E). According to lemma 3.1 function $\mathcal{B}(t, u, \omega) = D_u f(\varphi(t, u, \omega), \omega t)$ is homogeneous w.r.t. $\varphi(t, u, \omega)$ of order $m = 1$ and there exists a number $M > 0$ such that

$$||\mathcal{B}(t, u, \omega)|| \leq M \tag{3.19}$$

for all $(u, \omega) \in E \times \Omega$ and $t \geq 0$. From inequality (3.19) follows that

$$||U(t, u, \omega)|| \leq e^{\int_0^t ||\mathcal{B}(\tau, u, \omega)|| d\tau} \leq e^{Mt}$$

for all $t \geq 0$ and $(u, \omega) \in E \times \Omega$ and, consequently,

$$|\varphi(t,u,\omega)|^{k-2}|Re < D_u\varphi(t,u,\omega)v, \varphi(t,u,\omega) > |$$
$$\leq |\varphi(t,u,\omega)|^{k-1}\|D_u\varphi(t,u,\omega)\||v| \leq M|v|N^{k-1}e^{-\nu(k-1)t}|u| \quad (3.20)$$

for any $t \geq 0, u, v \in E$ and $\omega \in \Omega$. From inequality (3.20) follows that for $k > 1$ the integral in the second hand of (3.16) is convergent, moreover uniformly convergent w.r.t. $\omega \in \Omega$ and u, v on the every bounded subset from E. Thus, the function $V(u,\omega)$, defined by formula (3.16) is continuously differentiable w.r.t. variable $u \in E$ and by equality (3.17) is defined its derivative of Frechet w.r.t. variable $u \in E$.

Having formally differentiated the equality (3.16) w.r.t. variable $\omega \in F$ we will obtain

$$D_\omega V(u,\omega)w = \int_0^{+\infty} k|\varphi(t,u,\omega)|^{k-2} Re < D_\omega\varphi(t,u,\omega)w, \varphi(t,u,\omega) > dt \quad (3.21)$$

for all $(u,\omega) \in E \times \Omega$. Let us show that integral in the second member of (3.21) is uniformly convergent w.r.t. $\omega \in \Omega$ and $(u,w) \in E \times F$ on every bounded subset from $E \times F$. First of all we note that from (3.14) we have

$$\varphi(t,u,\omega) = u + \int_0^t f(\varphi(\tau,u,\omega),\omega\tau)d\tau$$

for all $t \geq 0$ and $(u,\omega) \in E \times F$ and, consequently,

$$D_\omega\varphi(t,u,\omega) = \int_0^t D_u f(\varphi(\tau,u,\omega),\omega\tau)D_\omega\varphi(\tau,u,\omega)$$
$$+ D_\omega f(\varphi(\tau,u,\omega),\omega\tau)D\Phi(\omega\tau)d\tau. \quad (3.22)$$

We denote by $\mathcal{V}(t,u,\omega)$ the operator-function $D_\omega\varphi(t,u,\omega)$, then from (3.22) follows that $\mathcal{V}(t,u,\omega)$ satisfies the operational equation

$$Y' = \mathcal{B}(t,u,\omega)Y + \mathcal{F}(t,u,\omega), \quad (3.23)$$

where $\mathcal{F}(t,u,\omega) = D_\omega f(\varphi(t,u,\omega),\omega t)D\Phi(\omega t)$, and initial condition

$$Y(0) = \mathcal{O} \quad (3.24)$$

($\mathcal{O}$ is a null operator, acting from F into F). From equality (3.23) and condition (3.24) follows that

$$\mathcal{V}(t,u,\omega) = \int_0^t U(t,\tau,u,\omega)\mathcal{F}(\tau,u,\omega)d\tau \quad (3.25)$$

for all $t \geq 0$ and $(u,\omega) \in E \times \Omega$, where $U(t,\tau,u,\omega)$ is a solution of operational equation (3.18), satisfying the initial condition $U(\tau,\tau,u,\omega) = U(\tau,u,\omega)$, therefore takes place the estimate

$$\|U(t,\tau,u,\omega)\| \le e^{\int_\tau^t \|\mathcal{B}(\tau,u,\omega)\|d\tau} \le e^{M(t-\tau)} \quad (3.26)$$

for any $t \ge \tau (t, \tau \in \mathbb{R}_+)$ and $(u,\omega) \in E \times \Omega$.

Thus, a function $f(u,\omega)$ is homogeneous w.r.t. u (of order $m = 1$), then $D_\omega f(u,\omega)$ will be the same and, in virtue of Lemma 3.1 there exists a number $M > 0$ such that

$$\|D_\omega f(u,\omega)\| \le M_1|u|$$

for all $u \in E$ and $\omega \in \Omega$. Thus,

$$\|\mathcal{F}(t,u,\omega)\| = \|D_\omega f(\varphi(t,u,\omega),\omega t) D\Phi(\omega t)| \le L|\varphi(t,u,\omega)| \quad (3.27)$$

for any $t \ge 0$ and $(u,\omega) \in E \times \Omega$, where $L = M_1 \max\{\|D\Phi(\omega)\| : \omega \in \Omega\}$. From (3.15),(3.25), (3.26) and (3.27) we have

$$\begin{aligned}
|\varphi(t,u,\omega)|^{k-2}|Re < D_\omega\varphi(t,u,\omega)w > | &\le |\varphi(t,u,\omega)|^{k-1}\|D_\omega\varphi(t,u,\omega)\||w| \\
&\le |\varphi(t,u,\omega)|^{k-1}|w| \int_0^t \|U(t,\tau,u,\omega)\| \cdot \|\mathcal{F}(\tau,u,\omega)\|d\tau \\
&\le |\varphi(t,u,\omega)|^{k-1}|w| \int_0^t e^{M(t-\tau)} L|\varphi(\tau,u,\omega)|d\tau \\
&\le N^{k-1}e^{-\nu(k-1)t}|u|^{k-1}|w| \int_0^t e^{M(t-\tau)} LNe^{-\nu\tau}|u|d\tau \\
&= N^k L e^{-\nu(k-1)t} \frac{(e^{Mt} - e^{-\nu t})}{M+\nu} |u|^k|w|
\end{aligned}$$

for every $t \ge 0, u \in E$ and $\omega \in \Omega$. From this follows, that for sufficiently large k the integral (3.21) is uniformly convergent w.r.t. $\omega \in \Omega$ and $(u,w) \in E \times F$ on the every bounded subset from $E \times F$. Therefore the function $V(u,\omega)$ defined by formula (3.16) is continuously differentiable w.r.t. variable $\omega \in \Omega$ and by formula (3.21) is defined its Frechet derivative w.r.t. variable $\omega \in \Omega$. So, the function $V : E \times \Omega \to \mathbb{R}_+$ is continuously differentiable w.r.t. variable $u \in E$ and variable $\omega \in \Omega$ both and, consequently, it is continuously differentiable in sense of Frechet.

Let us note that

$$\begin{aligned}
\frac{d}{dt}V((\varphi(t,u,\omega),\omega t) &= D_u V(\varphi(t,u,\omega),\omega t) D_u\varphi(t,u,\omega) + D_\omega V(\varphi(t,u,\omega),\omega t)\frac{d}{dt}\omega t \\
&= D_u V(\varphi(t,u,\omega),\omega t) f(\varphi(t,u,\omega),\omega t) + D_\omega V(\varphi(t,u,\omega),\omega t)\Phi(\omega t)
\end{aligned}$$

for any $t \ge$ and $(u,\omega) \in E \times \Omega$ and, consequently,

$$V'_\pi(u,\omega) = \frac{d}{dt}V(\varphi(t,u,\omega),\omega t)\Big|_{t=0} = D_u V(u,\omega) f(u,\omega) + D_\omega V(u,\omega)\Phi(u).$$

Thus, in conditions of theorem 3.3 from 1. follows 2..

For finishing the proof of the theorem it is sufficient to remark that according to theorem 2.3 from conditions 2.1-2.3 of theorem 2.3 follows 1.. The theorem is completely proved.

Remark 3.4. *For finite dimensional systems the theorem 3.2 generalizes and precises the theorem 38 from [3].*

§ 4. Global attractors of quasihomogeneous systems.

Let us consider a differential equation

$$x' = f(x), \tag{4.1}$$

where $f \in C(E)$ is regular and homogeneous of order $m \geq 1$. Along with equation (4.1) consider also perturbed equation

$$x' = f(x) + F(t, x) \tag{4.2}$$

with the regular perturbation. Let us remember (see, for example [3,7,25]), that equation (4.2) is called quasihomogeneous if

$$\lim \sup_{|x| \to +\infty} \frac{|F(t, x)|}{|x|^m} = c, \tag{4.3}$$

where c is a certain nonnegative sufficiently small constant. Moreover, the limiting relation (4.3) takes place uniformly w.r.t. $t \in \mathbb{R}$.

Along with equation (4.2) consider the family of equations

$$x' = f(x) + G(t, x), \tag{4.4}$$

where $G \in H(f) = \overline{\{F_\tau : \tau \in \mathbb{R}\}}$ and the bar denotes closure in $C(\mathbb{R}\times E, E)$($C(\mathbb{R} \times E, E)$ is equipped by topology of convergence on every compact from $\mathbb{R}\times E$). Let us denote by $(Y, \mathbb{R}, \sigma)$ a dynamical system of translations on $Y = H(F)$, by $\varphi(t, u, f + G)$ a solution of equation (4.4), satisfying the condition $\varphi(0, u, f + G) = u$ and $< (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h >$ (i.e. $h = pr_2$ and $\pi = (\varphi, \sigma)$) a nonautonomous dynamical system generated by equation (4.4).

Theorem 4.1. *Let $f \in C^1(E)$ be regular, homogeneous of order $m \geq 1$ and a zero solution of equation (4.1) is uniform asymptotic stable. If $f + F \in C(\mathbb{R} \times E, E)$ is regular, equation (4.2) is quasihomogeneous and nonautonomous dynamical system $< (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h >$ generated by equation (4.2) is asymptotically compact, then the following assertions take place:*

1. a set $I_G = \{u \in E | \sup\{|\varphi(t, u, f+G)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact and connected for each $G \in H(F)$;

2. $\varphi(t, I_G, f + G) = I_{\sigma(t,f+G)}$ for all $t \in \mathbb{R}_+$ and $G \in H(F)$;

3. a set $I = \bigcup\{I_G | G \in H(F)\}$ is compact and connected;

4. the equalities

$$\lim_{t\to+\infty} \beta(\varphi(t, M, f + G_{-t}), I_G) = 0$$

and

$$\lim_{t\to+\infty} \beta(\varphi(t, M, f + G), I) = 0$$

take place for all $G \in H(F)$ and bounded subset $M \subseteq E$.

Proof. Let $< (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h >$ be a nonautonomous dynamical system, generated by equation (4.2). Thus, $f \in C^1(E)$ and a zero solution of equation (4.1) is uniform asymptotic stable, then according to theorem 3.2 by equality (2.2) is defined a continuously differentiable function $V : E \to \mathbb{R}_+$, satisfying conditions 2.1-2.3 of theorem 3.2 . Let us define a function $\mathcal{V} : X \to \mathbb{R}_+$ in the following way: $\mathcal{V}(x) = V(u)$ for all $x = (u, G) \in E \times H(F)$. Note that

$$\begin{aligned} \mathcal{V}'_\pi(x) &= \frac{d}{dt} V(\varphi(t,u,f+G))\Big|_{t=0} = DV(\varphi(t,u,f+G))\frac{d}{dt}\varphi(t,u,f+G)\Big|_{t=0} \\ &= DV(u)(f(u)+G(0,u)) = DV(u)f(u) + DV(u)G(0,u) \\ &= -|u|^k + DV(u)G(0,u). \end{aligned} \tag{4.5}$$

In virtue of condition 2.1 of theorem 3.2 the function V is homogeneous of order $k-m+1$, then according to lemma 3.1 its Frechet derivative $DV(u)$ will be homogeneous of order $k-m$ and, consequently, there exists a number $L > 0$ such that

$$\|DV(u)\| \le L|u|^{k-m} \tag{4.6}$$

for all $u \in E$. From equality (4.3) follows, that for all $\varepsilon > 0$ there exists $r = r(\varepsilon) > 0$ such that

$$|G(t,u)| \le (c+\varepsilon)|u|^m \tag{4.7}$$

for all $|u| \ge r, t \in \mathbb{R}$ and $G \in H(F)$. From (4.6)-(4.7) we have

$$|DV(u)G(0,u)| \le L(c+\varepsilon)|u|^k \tag{4.8}$$

and, consequently, from (4.5),(4.8) we will obtain

$$\mathcal{V}'_\pi(x) \le |u|^k(-1 + L(c+\varepsilon)) = -\gamma|x|^k$$

for each $x \in X_r$, where $\gamma = 1 - l(c+\varepsilon) > 0$ since C and ε are a sufficiently small positive numbers.

To finish the proof of theorem it is sufficient to refer on theorems 1.3-1.5 .

Theorem 4.2. *Let $f \in C^1(E), \Phi \in C^1(F)$ and $f + F \in C^1(E \times F, E)$ be regular, $\Omega \subseteq F$ be a compact invariant set of dynamical system (3.13) , the function f be homogeneous (of order $m > 1$) and a zero solution of equation (3.3) be uniform asymptotical stable. If*

$$|F(u,\omega)| \le c|u|^m$$

for all $|u| \ge r$ and $\omega \in \Omega$,where $r$and c are certain positive numbers, and moreover c is sufficiently small and dynamical system

$$\begin{cases} u' = f(u) + F(u,\omega) \\ \omega' = \Phi(\omega) \end{cases} \qquad (\omega \in \Omega)$$

is asymptotically compact, then the following assertions take place:

1. a set $I_\omega = \{u \in E | \sup\{|\varphi(t,u,\omega)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact and connected for each $\omega \in \Omega$,where $\varphi(t,u,\omega)$ is a solution of equation $u' = f(u) + F(u,\omega t)$ satisfying the initial condition $\varphi(0,u,\omega) = u$;

2. $\varphi(t, I_\omega, \omega) = I_{\sigma(t,\omega)}$ *for all* $t \in \mathbb{R}_+$ *and* $\omega \in \Omega$*;*
3. a set $I = \bigcup\{I_\omega | \omega \in \Omega\}$ *is compact and connected;*
4. the equalities

$$\lim_{t\to+\infty} \beta(\varphi(t, M, \omega_{-t}), I_\omega) = 0 \tag{4.9}$$

and

$$\lim_{t\to+\infty} \beta(\varphi(t, M, \omega), I) = 0 \tag{4.10}$$

take place for all $\omega \in \Omega$ *and bounded subset* $M \subseteq E$.

The proof of formulated assertion is carried out using the same scheme as in theorem 4.1, therefore its proof may be we omitted.

Theorem 4.3. *Let* $f \in C^1(E \times F, E), \Phi \in C^1(F)$ *and* $f + F \in C^1(E \times F, E)$ *be regular,* $\Omega \subseteq F$ *is a compact invariant set of dynamical system (3.13), function* f *is homogeneous (of order* $m = 1$*) w.r.t. variable* $u \in E$ *and a zero solution of equation (3.14) is uniform asymptotic stable.*
If

$$|F(u, \omega)| \leq c|u| \tag{4.11}$$

for all $|u| \geq r$ *and* $\omega \in \Omega$, *where* r *and* c *are certain positive numbers, and moreover* c *is sufficiently small and dynamical system*

$$\begin{cases} u' = f(u, \omega) + F(u, \omega) \\ \omega' = \Phi(\omega) \end{cases} \qquad (\omega \in \Omega) \tag{4.12}$$

is asymptotically compact. Then the following assertions take place:
1. a set $I_\omega = \{u \in E | \sup\{|\varphi(t, u, \omega)| : t \in \mathbb{R}\} < +\infty\}$ *is not empty, compact and connected for each* $\omega \in \Omega$*,where* $\varphi(t, u, \omega)$ *is a solution of equation*

$$u' = f(u, \omega t) + F(u, \omega t) \tag{4.13}$$

satisfying the initial condition $\varphi(0, u, \omega) = u$*;*
2. $\varphi(t, I_\omega, \omega) = I_{\sigma(t,\omega)}$ *for all* $t \in \mathbb{R}_+$ *and* $\omega \in \Omega$*;*
3. a set $I = \bigcup\{I_\omega | \omega \in \Omega\}$ *is compact and connected;*
4. the equalities (4.9) and (4.10) take place for every bounded subset $M \subseteq E$ *and* $\omega \in \Omega$.

Proof. Let $< (X, \mathbb{R}_+, \pi), (Y, \mathbb{R}, \sigma), h >$ be a nonautonomous dynamical system, generated by equation (4.13) (or by differential system (4.12)). Since $f \in C^1(E \times F, E)$ and a zero solution of equation (3.14) is uniform asymptotic stable, then according to theorem 3.3 by equality

$$V(u, \omega) = \int_0^{+\infty} |\varphi(t, u, \omega)|^k dt$$

is defined a continuously differentiable function $V : X = E \times \Omega \to \mathbb{R}_+$, satisfying conditions 2.1-2.3 of theorem 2.3 . Let us remark that

$$
\begin{aligned}
V'_\pi(u,\omega) &= \frac{d}{dt}V((\varphi(t,u,\omega),\omega t)\Big|_{t=0} \\
&= \{D_uV(\varphi(t,u,\omega),\omega t)\frac{d}{dt}\varphi(t,u,\omega) + D_\omega V(\varphi(t,u,\omega),\omega t)\frac{d}{dt}\omega t\}\Big|_{t=0} \\
&= D_uV(u,\omega)[f(u,\omega)+F(u,\omega)] + D_\omega V(u,\omega)\Phi(\omega) \\
&= -|u|^k + D_uV(u,\omega)F(u,\omega).
\end{aligned}
\tag{4.14}
$$

Since function $V(u,\omega)$ is homogeneous of order $k>1$ w.r.t. variable $u\in E$, then in virtue of Lemma 3.1 there exists a number $L>0$ such that

$$\|D_uV(u,\omega)\| \le L|u|^{k-1} \tag{4.15}$$

for all $u\in E$ and $\omega\in\Omega$. From (4.11) and (4.15) follows, that

$$|D_uV(u,\omega)F(u,\omega)| \le cL|u|^k \tag{4.16}$$

for each $|u|\ge r$ and $\omega\in\Omega$ and, consequently, according to (4.14), (4.16) we have

$$V'_\pi(u,\omega) \le -|u|^k + cL|u|^k = -\gamma|u|^k$$

for all $\omega\in\Omega$ and $|u|\ge r$, where $\gamma = 1-cL>0$ since a number c is sufficiently small.

For finishing the proof of the theorem it is sufficiently to refer on theorem 1.3-1.5 . The theorem is proved.

Remark 4.4. *1. In case, when E is finite dimensional, the dissipativity of equation (4.2) is established in [3] (theorem 39 and corollary 1).*

2. If E is a finite dimensional Hilbertian space, then the nonautonomous dynamical system, generated by equation (4.2) will be asymptotically compact, if the function $F\in C(\mathbb{R}\times E,E)$ is bounded and uniformly continuous w.r.t. variable $t\in\mathbb{R}$ uniformly w.r.t. x on the every compact from E.

3. If Ω is compact, then the nonautonomous dynamical system generated by equation (4.13) is asymptotically compact.

4. The conditions of asymptotical compactness of nonautonomous dynamical system generated by quasilinear equation in Banach space are given in [14].

§ 5. Quasilinear parabolic equations.

Let $A: D(A)\to E$ be a linear closed operator with dense domain. An operator A is called [26] sectorial if for some $\varphi\in(0,\frac{\pi}{2})$, some $M\ge 1$, and some real a, the sector

$$S_{a,\varphi} = \{\lambda : \varphi \le |arg(\lambda-a)| \le \pi, \lambda\ne a\}$$

lies in the resolvent set $\rho(A)$ and $\|(I\lambda - A)^{-1}\| \le M|\lambda-a|^{-1}$ for all $\lambda\in S_{a,\varphi}$. If A is a sectorial operator, then there exists an $a_1>0$ such that $Re\sigma(A+a_1I)>0$ $(\sigma(A)=\mathbb{C}\setminus\rho(A))$. Let $A_1 = A+a_1I$. For $0<\alpha<1$, one defines the operator [26]

$$A_1^{-\alpha} = \frac{sin\pi\alpha}{\pi}\int_0^{+\infty}\lambda^{-\alpha}(\lambda I + A_1)^{-1}d\lambda,$$

which is linear, bounded, and one-to-one. Set $E^\alpha = D(A_1^\alpha)$, and let us equip the space E^α with the scalar product $< x,y >_\alpha = < A_1^\alpha x, A_1^\alpha y > (x,y \in E^\alpha), E^0 = E, X^1 = D(A)$ and $|\cdot|_\alpha = \sqrt{<\cdot,\cdot>_\alpha}$. Then E^α is a Hilbert space with the scalar product $< x,y >_\alpha$, and is densely continuously embedded in E. If the operator A admits a compact resolvent, then the embedding $E^\alpha \to E^\beta$ is compact for $\alpha > \beta \geq 0$ [26].

Consider the differential system

$$\begin{cases} u' + Au = F(u,\omega) \\ \omega' = \Phi(\omega) \end{cases} \qquad (\omega \in \Omega), \tag{5.1}$$

where $F \in C^1(E^\alpha \times F, E)$ and $\Phi \in C^1(F)$.

Theorem 5.1. *Let $F \in C^1(E^\alpha \times F, E)$ and $\Phi \in C^1(F)$ be regular, Ω be a compact invariant set of dynamical system (3.13). If a trivial solution of equation*

$$u' + Au = 0 \tag{5.2}$$

is uniform asymptotical stable and there exists a sufficiently small constant $c \geq 0$ such that

$$|F(u,\omega)| \leq c|u| \tag{5.3}$$

for all $|u| \geq r$ and $\omega \in \Omega$, where r is a sufficiently large positive number. Then the following assertions take place:

1. a set $I_\omega = \{u \in E^\alpha | \sup\{|\varphi(t,u,\omega)| : t \in \mathbb{R}\} < +\infty\}$ is not empty, compact in E and connected for each $\omega \in \Omega$,where $\varphi(t,u,\omega)$ is a solution of equation $u' + Au = F(u,\omega t)$ satisfying the initial condition $\varphi(0,u,\omega) = u$;

2. $\varphi(t, I_\omega, \omega) = I_{\sigma(t,\omega)}$ for all $t \in \mathbb{R}_+$ and $\omega \in \Omega$;

3. a set $I = \bigcup\{I_\omega | \omega \in \Omega\}$ is compact in E and connected;

4. the equalities (4.9) and (4.10) take place w.r.t. semi-deviation in the space E for every bounded subset $M \subseteq E^\alpha$ and $\omega \in \Omega$.

This assertion follows from theorem 4.3 and compactness of embedding $i : E^\alpha \to E$.

Remark 5.2. *We note that for quasilinear parabolic equations the affirmation analogous to theorem 4.1 takes place.*

References

[1] N.N.Krasovskii, *Problems in the Theory of the Stability of Motion [in Russian]*, Fizmatgiz, Moscow, 1959.

[2] N.N.Krasovskii, *On first approximation stability*, Prikl.Mat.Mekh [J.Appl.Math.Mekh.].1955, 19, No 5, p.516-530.

[3] V.I.Zubov, *Stability of Motion [in Russian]*, Vysshaya Shkola, Moscow, 1973.

[4] C.Coleman, *Growth and decay estimates near non-elementary points*, Canad. J. Math. 1970, v.22, No6, p.1156-1167.

[5] A.F.Filippov, *Differential Equations with Discontinuous Right-Hand Side*, Differentsial'nye Uravneniya [Differential Equations]. 1979, v.15, No6, p.1018-1027.

[6] Filippov A.F., *Differential Equations with Discontinuous Right-Hand Side [in Russian].*, Nauka, Moscow, 1985.

[7] N.N.Ladis, *The asymptotic of solutions to quasihomogeneous differential systems.*, Differentsial'nye Uravneniya [Differential Equations] .1973, v.9, No12, p.2257-2260.

[8] A.Halanay, *Teoria Calitativa a ecuatiilor differentiale.*, Bucuresti, 1963.

[9] M.V.Morozov, *Some properties of periodic differential inclusions*, Izv.Vyssh.Uchebn.Zaved. Mat.[Iz.VUZ] . 1993, No4(371), p.95-100.

[10] D.N.Cheban, *The asymptotics of solutions of infinite dimensional homogeneous dynamical systems*, Matematicheskie zametki [Mathematical notes]. 1998, v.63, No1, p.115-126.

[11] Hale J., *Asymptotic behavior of dissipative systems.*, Mathematical surveys and Monographs, 25, American Math. Soc., Providence, R.I., 1988.

[12] Cheban D.N., Fakeeh D.S., *Global attractors of the dynamical systems without uniqueness.*, "Sigma", Kishinev, 1994.

[13] Cheban D.N., *Global attractors of the infinite-dimensional nonautonomous dynamical systems, I.*, Izvestiya AN RM. Matematika, No. 3 (25) 1997 p.42-55.

[14] Cheban D.N., *Global attractors of the infinite-dimensional nonautonomous dynamical systems, II.*, Izvestiya AN RM. Matematika, No. 2 (27) 1998 p.25-28.

[15] V.V.Zhikov, *On stability and instability of Levinson's center*, Differentsial'nye Uravneniya [Differential Equations] .1972, v.8, No12, p.2167-2070.

[16] D.N.Cheban, *On the stability of Levinson's center of nonautonomous dissipative dynamical systems*, Differentsial'nye Uravneniya [Differential Equations] .1984, v.20, No11, p.2016-2018.

[17] Cheban D.N., *Nonautonomous dissipative dynamical systems.*, Soviet Math. Dokl., v.33, N1, 1986, p.207-210.

[18] Cheban D.N., *Nonautonomous dissipative dynamical systems. Thesis of doctor of physical-mathematical sciences.*, Minsk, 1991.

[19] I.U.Bronshtein, *Nonautonomous Dynamical Systems [in Russian]*, Shtiintsa, Kishinev, 1984.

[20] G.R.Sell, *Lecture on Topological Dynamics and Differential equations*, Van-Nostrand-Rein hold, Prinston, N.Y., 1971.

[21] P.E.Kloeden and B.Schmalfuss paper Lyapunov functions and attractors under variable time-step discretization., Discrete and Continuous Dynamical Systems. 1996, v.2, No2, p.163-172.

[22] H.Crauel, F.Flandoli, *Attractors for random dynamical systems*, Probability Theory and Related Fields . 1994, 100, p.365-393.

[23] Ladyzhenskaya O.A., *On finding minimal global attractors for Navier-Stokes and other partial differential equations.*, Uspekhi matematicheskikh nauk, 1987, vol.42, issue 6 (258), p.25-60.

[24] A.V.Babin and M.I.Vishik, *Attractors of Evolution Equations*, Nort-Holland, Amsterdam, London, New York, Tokyo, 1992.

[25] M.A.Krasnosel'skii and P.P.Zabreiko, *Geometric methods of nonlinear analysis*, Nauka, Moscow, 1975.

[26] D.Henry, *Geometric Theory of Semilinear Parabolic Equations*, Springer-Verlag, Berlin-Heidelberg-New York, 1981.

D.N.Cheban
State University of Moldova
Faculty of Mathematics and Informatics
60, A.Mateevich str.
Chişinău, MD-2009
MOLDOVA
E-mail address: cheban@usm.md