SOME PROPERTIES OF THE KMS-FUNCTION

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Abstract. Let, for $j = 0,\, 1$, $H_j = H_j^* \geq -\omega_j > -\infty$, be a self-adjoint operator in the Hilbert spaces $\mathcal{H}_j$. Let $T : \mathcal{H}_1 \to \mathcal{H}_0$ be a linear operator with domain in $\mathcal{H}_1$ and range in $\mathcal{H}_0$. Let $V_j(t) = \exp(-tH_j)$, $t \geq 0$, be the strongly continuous semigroup generated by H_j, $j = 0,\, 1$. If the operators

$$(aI + H_0)\, V_0\,(t_0)\, T\,(aI + H_1)^{-1} \text{ and } (aI + H_0)^{-1}\, T\,(aI + H_1)\, V_1\,(t_0)$$

are compact, (Hilbert-Schmidt, Trace class), then so is the operator

$$\int_0^{t_0} V_0(u) T V_1\,(t_0 - u)\, du.$$

The result is applicable if $T = \mathcal{J} H_1 - H_0 \mathcal{J}$, where $\mathcal{J} : \mathcal{H}_1 \to \mathcal{H}_0$ is a bounded linear (identification) operator. In this case $\int_0^{t_0} V_0(u) T V_1(t_0 - u) du = V_0\,(t_0)\, \mathcal{J} - \mathcal{J} V_1\,(t_0)$; i.e. the difference of the semigroups. Some convergence and approximation results are presented as well. For example the operator $\int_0^{t_0} V_0(u) T V_1\,(t_0 - u)\, du$ is expressed in terms of the operator $t_0 V_0\,(t_0/2)\, T V_1\,(t_0/2)$.

1. Introduction, motivation

In [6] the authors proved a result of the following sort.

Theorem. *Let V and W be Kato-Feller potentials on $\mathbb{R}^\nu$ for which $W - V$ belongs to $L^1(\mathbb{R}^\nu)$. Then the scattering system $\left(H_0 \dot{+} V, H_0 \dot{+} W\right)$ is complete, in the sense that, for $f \in P_{ac}\left(H_0 \dot{+} W\right)$, the limits $\Omega_\pm \left(H_0 \dot{+} V, H_0 \dot{+} W\right) f := \lim_{t \to \pm\infty} e^{it\left(H_0 \dot{+} V\right)} e^{-it\left(H_0 \dot{+} W\right)} f$ exist in $L^2(\mathbb{R}^\nu)$.*

In order to prove this we write, with $H_0 = -\frac{1}{2}\Delta$ in $L^2(\mathbb{R}^\nu)$,

$$V_0\,(t_0) = \exp\left(-t_0\left(H_0 \dot{+} V\right)\right), \text{ and } V_1\,(t_0) = \exp\left(-t_0\left(H_0 \dot{+} W\right)\right).$$

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From our assumption that $W-V$ is an L^1-function it follows that the operator $D(t_0) := V_0(t_0) - V_1(t_0)$ is compact (in fact it is Hilbert-Schmidt, but we only need its compactness). In addition, $W-V$ being an L^1-function implies that the operator

$$t_0 V_0(t_0)(W-V)V_1(t_0) \quad \text{is of trace class.}$$

The latter property is even more important than the compactness of the operator $D(t_0)$. The problem left out by this approach is establishing a direct relationship between $t_0 V_0(t_0)(W-V)V_1(t_0)$ and $D(t_0)$. Such a link exists. To be specific, put $\mathcal{Q}(t_0)T = \mathbb{E}^{\text{logistic}}\left[V_0(iU_1 t_0)TV_1(-iU_1 t_0)\right]$, where U_1 is a random variable, which is logistically distributed: $\mathbb{P}^{\text{logistic}}[U_1 \in B] = \frac{\pi}{2}\int_B \frac{1}{(\cosh \pi\tau)^2} d\tau$. Then

$$\begin{aligned} &t_0 V_0(t_0)(W-V)V_1(t_0) = \mathcal{Q}(t_0)D(t_0), \quad \text{and, more significantly,} \\ &D(t_0) = \lim_{n\to\infty}\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}(t_0)^j\left(t_0 V_0(t_0)(W-V)V_1(t_0)\right). \end{aligned}$$

For this, see e.g. Proposition 4 and Theorem 11 below: the operator H_0 in §2 may be any self-adjoint operator, which is bounded below. Duhamel's formula shows that $\mathcal{D}(t_0)(W-V) = D(t_0) = V_0(t_0) - V_1(t_0)$.

2. Main results

In the following theorem we present the KMS-function, and some of its properties. Another interesting formula is the following; it is related to the KMS-condition in the theory of von Neumann-algebras: e.g. see Sunder [9, page 63 and following] and the remark prior to Theorem 2.

1. Theorem. *(a) Suppose (τ, s) belongs to the strip $\mathbb{R}\times\left(-\frac{1}{2},\frac{1}{2}\right)$, and let $T:\mathcal{H}_1\to\mathcal{H}_0$ be a linear operator with the property that the operators $V_0(t_0)T$ and $V_1(t_0)T^*$ are densely defined. Then, in form sense, the following identity is true:*

$$\begin{aligned} &t_0 V_0(-i\tau t_0)V_0\left(\left(\frac{1}{2}+s\right)t_0\right)TV_1\left(\left(\frac{1}{2}-s\right)t_0\right)V_1(i\tau t_0) \\ &= \frac{t_0}{2}\int_{-\infty}^{\infty}\frac{\cos\pi s}{\cosh\pi(\tau-\sigma)-\sin\pi s}V_0(t_0)V_0(-i\sigma t_0)TV_1(i\sigma t_0)\,d\sigma \\ &\quad + \frac{t_0}{2}\int_{-\infty}^{\infty}\frac{\cos\pi s}{\cosh\pi(\tau-\sigma)+\sin\pi s}V_0(-i\sigma t_0)TV_1(i\sigma t_0)V_1(t_0)\,d\sigma. \end{aligned} \tag{1}$$

(b) The following identity is valid as well:

$$\mathcal{D}(t_0)T = \frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\left(\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}\right)V_0(-i\tau t_0)\left\{V_0(t_0)T+TV_1(t_0)\right\}V_1(i\tau t_0)\,d\tau \tag{2}$$

$$= \frac{t_0}{\pi}\int_{-\infty}^{\infty}\log\left|\coth\left(\frac{1}{2}\pi\tau\right)\right|V_0(-i\tau t_0)\left\{V_0(t_0)T+TV_1(t_0)\right\}V_1(i\tau t_0)\,d\tau. \tag{3}$$

Proof. We outline a proof. This formula follows by virtue of the following observation. The function at the right hand side of formula (1) is harmonic on the strip $\mathbb{R}\times\left(-\frac{1}{2},\frac{1}{2}\right)$ and it possesses boundary values

$$\begin{cases} t_0V_0\left(t_0\right)V_0\left(-i\tau t_0\right)TV_1\left(i\tau t_0\right), & \text{for } s=\frac{1}{2}, \text{ and}\\ t_0V_0\left(-i\tau t_0\right)TV_1\left(i\tau t_0\right)V_1\left(t_0\right), & \text{for } s=-\frac{1}{2}.\end{cases} \tag{4}$$

The left hand side is harmonic on the same strip (in fact it is holomorphic there), and has the same boundary values. The uniqueness part on the existence of solutions to the classical Dirichlet problem on a strip, yields the formula in (1).
(b) The identity in (2) follows upon integrating the previous identity with respect to s (between $-\frac{1}{2}$ and $\frac{1}{2}$). The equality in (3) is a consequence of a very classical identity: $\dfrac{\cosh\pi\tau+1}{\cosh\pi\tau-1}=\coth^2\dfrac{1}{2}\pi\tau$.

Remark 1. We consider the space $\mathcal{H}_0\times\mathcal{H}_1$ together with $V(i\tau)=\begin{pmatrix} V_0(i\tau) & 0\\ 0 & V_1(i\tau)\end{pmatrix}$. Define the flow $\mathcal{A}_\tau$ on $\mathcal{B}\left(\mathcal{H}_0\times\mathcal{H}_1,\mathcal{H}_0\times\mathcal{H}_1\right)$ by $\mathcal{A}_\tau(T)=V\left(-i\tau t_0\right)TV\left(i\tau t_0\right)$. Define for $(f,g)\in\mathcal{H}_0\times\mathcal{H}_1$, and $T\in\mathcal{B}\left(\mathcal{H}_0\times\mathcal{H}_1,\mathcal{H}_0\times\mathcal{H}_1\right)$ the function $F_{f,g}(\tau+is)$, $\tau\in\mathbb{R}$, $-\frac{1}{2}\leq s\leq\frac{1}{2}$ by

$$F_{f,g}(\tau+is)=t_0\left\langle\mathcal{A}_\tau\left(V\left(\left(\frac{1}{2}+s\right)t_0\right)TV\left(\left(\frac{1}{2}-s\right)t_0\right)\right)\begin{pmatrix} f\\ g\end{pmatrix},\begin{pmatrix} f\\ g\end{pmatrix}\right\rangle.$$

The function $F_{f,g}$ is K(ubo)-M(artin)-S(chwinger)-admissible for the operators $V(t_0)$ and T in the sense that it is continuous on the closed strip $\left\{\tau+is:\tau\in\mathbb{R},\ -\frac{1}{2}\leq s\leq\frac{1}{2}\right\}$, and holomorphic on its interior. Moreover,

$$\begin{aligned} F_{f,g}\left(\tau+\frac{1}{2}i\right)&=t_0\left\langle V(t_0)\mathcal{A}_\tau(T)\begin{pmatrix} f\\ g\end{pmatrix},\begin{pmatrix} f\\ g\end{pmatrix}\right\rangle;\\ F_{f,g}\left(\tau-\frac{1}{2}i\right)&=t_0\left\langle \mathcal{A}_\tau(T)V(t_0)\begin{pmatrix} f\\ g\end{pmatrix},\begin{pmatrix} f\\ g\end{pmatrix}\right\rangle.\end{aligned}$$

If $T=T^*$, then $F_{f,g}\left(\tau-\frac{1}{2}i\right)=\overline{F_{f,g}\left(\tau+\frac{1}{2}i\right)}$.

Remark 2. The family of operators $t\mapsto\begin{pmatrix} V_0(t) & \mathcal{D}(t)T\\ 0 & V_1(t)\end{pmatrix}$ is a strongly continuous semigroup on the space $\mathcal{H}_0\times\mathcal{H}_1$. Implicitly, this fact is used in the proof of equality (21).

2. Theorem. *Let the hypotheses be as in Theorem 1. The following equalities are valid:*

$$\begin{aligned} &\int_0^{t_0}V_0(u)TV_1(t_0-u)du\\ &=\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}e^{i\tau t_0H_0}\left(aI+H_0\right)V_0\left(t_0\right)T\left(aI+H_1\right)^{-1}e^{-i\tau t_0H_1}d\tau\\ &\quad+\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}e^{i\tau t_0H_0}\left(aI+H_0\right)^{-1}T\left(aI+H_1\right)V_1\left(t_0\right)e^{-i\tau t_0H_1}d\tau\end{aligned}$$

$$
\begin{aligned}
&+ i \int_{-\infty}^{\infty} \frac{e^{i\tau t_0 (aI+H_0)} - I}{\sinh \pi\tau} V_0 (t_0) T (aI + H_1)^{-1} \left(e^{-i\tau t_0 (aI+H_1)} - I\right) d\tau \\
&- i \int_{-\infty}^{\infty} \left(e^{i\tau t_0 (aI+H_0)} - I\right) (aI + H_0)^{-1} T V_1 (t_0) \frac{e^{-i\tau t_0 (aI+H_1)} - I}{\sinh \pi\tau} d\tau \\
&+ \tanh \left(\frac{1}{2} t_0 (aI + H_0)\right) \left((aI + H_0)^{-1} T V_1 (t_0) - V_0 (t_0) T (aI + H_1)^{-1}\right) \\
&- \left((aI + H_0)^{-1} T V_1 (t_0) - V_0 (t_0) T (aI + H_1)^{-1}\right) \tanh \left(\frac{1}{2} t_0 (aI + H_1)\right) \qquad (5) \\
= &\frac{t_0}{2\pi} \int_{-\infty}^{\infty} \log \frac{\cosh \pi\tau + 1}{\cosh \pi\tau - 1} e^{i\tau t_0 H_0} (aI + H_0) V_0 (t_0) T (aI + H_1)^{-1} e^{-i\tau t_0 H_1} d\tau \\
&+ \frac{t_0}{2\pi} \int_{-\infty}^{\infty} \log \frac{\cosh \pi\tau + 1}{\cosh \pi\tau - 1} e^{i\tau t_0 H_0} (aI + H_0)^{-1} T (aI + H_1) V_1 (t_0) e^{-i\tau t_0 H_1} d\tau \\
&+ i \int_{-\infty}^{\infty} \frac{e^{i\tau t_0 (aI+H_0)} - I}{\sinh \pi\tau} V_0 (t_0) T (aI + H_1)^{-1} e^{-i\tau t_0 (aI+H_1)} d\tau \\
&- i \int_{-\infty}^{\infty} e^{i\tau t_0 (aI+H_0)} (aI + H_0)^{-1} T V_1 (t_0) \frac{e^{-i\tau t_0 (aI+H_1)} - I}{\sinh \pi\tau} d\tau \\
&+ \tanh \left(\frac{1}{2} t_0 (aI + H_0)\right) (aI + H_0)^{-1} T V_1 (t_0) \\
&+ V_0 (t_0) T (aI + H_1)^{-1} \tanh \left(\frac{1}{2} t_0 (aI + H_1)\right). \qquad (6)
\end{aligned}
$$

Proof. For brevity we write $\mathcal{D}(t_0) T = \int_0^{t_0} V_0(u) T V_1 (t_0 - u) \, du$, $A_0 = aI + H_0$, $A_1 = aI + H_1$. Then, from the KMS-formula (see Theorem 1 (b)) it follows that

$$
\begin{aligned}
\mathcal{D}(t_0) T &= \frac{t_0}{2\pi} \int_{-\infty}^{\infty} \log \frac{\cosh \pi\tau + 1}{\cosh \pi\tau - 1} e^{i\tau t_0 A_0} (V_0 (t_0) T + T V_1 (t_0)) e^{-i\tau t_0 A_1} d\tau \\
&= -\frac{1}{2\pi} \int_{-\infty}^{\infty} \log \frac{\cosh \pi\tau + 1}{\cosh \pi\tau - 1} e^{i\tau t_0 A_0} V_0 (t_0) T A_1^{-1} \frac{\partial}{\partial(i\tau)} \left(e^{-i\tau t_0 A_1} - I\right) d\tau \\
&\quad + \frac{1}{2\pi} \int_{-\infty}^{\infty} \log \frac{\cosh \pi\tau + 1}{\cosh \pi\tau - 1} \frac{\partial}{\partial(i\tau)} \left(e^{i\tau t_0 A_0} - I\right) A_0^{-1} T V_1 (t_0) e^{-i\tau t_0 A_1} d\tau
\end{aligned}
$$

(integration by parts)

$$
\begin{aligned}
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial}{\partial(i\tau)} \left(\log \frac{\cosh \pi\tau + 1}{\cosh \pi\tau - 1} e^{i\tau t_0 A_0}\right) V_0 (t_0) T A_1^{-1} \left(e^{-i\tau t_0 A_1} - I\right) d\tau \\
&\quad - \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(e^{i\tau t_0 A_0} - I\right) A_0^{-1} T V_1 (t_0) \frac{\partial}{\partial(i\tau)} \left(\log \frac{\cosh \pi\tau + 1}{\cosh \pi\tau - 1} e^{-i\tau t_0 A_1}\right) d\tau \\
&= \frac{t_0}{2\pi} \int_{-\infty}^{\infty} \log \frac{\cosh \pi\tau + 1}{\cosh \pi\tau - 1} e^{i\tau t_0 A_0} A_0 V_0 (t_0) T A_1^{-1} \left(e^{-i\tau t_0 A_1} - I\right) d\tau \\
&\quad + i \int_{-\infty}^{\infty} \frac{1}{\sinh \pi\tau} e^{i\tau t_0 A_0} V_0 (t_0) T A_1^{-1} \left(e^{-i\tau t_0 A_1} - I\right) d\tau
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}\left(e^{i\tau t_0A_0}-I\right)A_0^{-1}TA_1V_1\left(t_0\right)e^{-i\tau t_0A_1}d\tau\\
&-i\int_{-\infty}^{\infty}\frac{1}{\sinh\pi\tau}\left(e^{i\tau t_0A_0}-I\right)A_0^{-1}TV_1\left(t_0\right)e^{-i\tau t_0A_1}d\tau\\
=&\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}e^{i\tau t_0A_0}A_0V_0\left(t_0\right)TA_1^{-1}e^{-i\tau t_0A_1}d\tau\\
&-\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}e^{i\tau t_0A_0}d\tau\,A_0V_0\left(t_0\right)TA_1^{-1}\\
&+i\int_{-\infty}^{\infty}\frac{1}{\sinh\pi\tau}\left(e^{i\tau t_0A_0}-I\right)V_0\left(t_0\right)TA_1^{-1}\left(e^{-i\tau t_0A_1}-I\right)d\tau\\
&+V_0\left(t_0\right)TA_1^{-1}\,i\int_{-\infty}^{\infty}\frac{1}{\sinh\pi\tau}\left(e^{-i\tau t_0A_1}-I\right)d\tau\\
&+\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}e^{i\tau t_0A_0}A_0^{-1}TA_1V_1\left(t_0\right)e^{-i\tau t_0A_1}d\tau\\
&-A_0^{-1}TA_1V_1\left(t_0\right)\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}e^{-i\tau t_0A_1}d\tau\\
&-i\int_{-\infty}^{\infty}\frac{1}{\sinh\pi\tau}\left(e^{i\tau t_0A_0}-I\right)A_0^{-1}TV_1\left(t_0\right)\left(e^{-i\tau t_0A_1}-I\right)d\tau\\
&-i\int_{-\infty}^{\infty}\frac{1}{\sinh\pi\tau}\left(e^{i\tau t_0A_0}-I\right)d\tau A_0^{-1}TV_1\left(t_0\right).
\end{aligned}\tag{7}
$$

From (7) together with the equalities:

$$
\begin{aligned}
-i\int_{-\infty}^{\infty}\frac{e^{i\tau t_0A_0}-I}{\sinh\pi\tau}d\tau&=A_0\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}e^{i\tau t_0A_0}d\tau=\tanh\left(\frac{1}{2}t_0A_0\right);\\
i\int_{-\infty}^{\infty}\frac{e^{-i\tau t_0A_1}-I}{\sinh\pi\tau}d\tau&=A_1\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}e^{-i\tau t_0A_1}d\tau=\tanh\left(\frac{1}{2}t_0A_1\right),
\end{aligned}\tag{8}
$$

we obtain

$$
\begin{aligned}
&\mathcal{D}\left(t_0\right)T\\
&=\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\log\frac{\cosh\pi\tau+1}{\cosh\pi\tau-1}e^{i\tau t_0A_0}\left(A_0V_0\left(t_0\right)TA_1^{-1}+A_0^{-1}TA_1V_1\left(t_0\right)\right)e^{-i\tau t_0A_1}d\tau\\
&\quad+i\int_{-\infty}^{\infty}\frac{1}{\sinh\pi\tau}\left(e^{i\tau t_0A_0}-I\right)\left(V_0\left(t_0\right)TA_1^{-1}-A_0^{-1}TV_1\left(t_0\right)\right)\left(e^{-i\tau t_0A_1}-I\right)d\tau\\
&\quad+\tanh\left(\frac{1}{2}t_0\left(aI+H_0\right)\right)\left(A_0^{-1}TV_1\left(t_0\right)-V_0\left(t_0\right)TA_1^{-1}\right)\\
&\quad-\left(A_0^{-1}TV_1\left(t_0\right)-V_0\left(t_0\right)TA_1^{-1}\right)\tanh\left(\frac{1}{2}t_0\left(aI+H_1\right)\right).
\end{aligned}\tag{9}
$$

This proves equality (5) in Theorem 1. Another appeal to (8) yields equality (6). In the sequel we write $\mathcal{D}\left(t_0\right)T=\int_0^{t_0}V_0(u)TV_1\left(t_0-u\right)du$.

3. Corollary. *If the operators*

$$(aI + H_0)\, V_0\,(t_0)\, T\,(aI + H_1)^{-1} \quad \textit{and} \quad (aI + H_0)^{-1}\, T\,(aI + H_1)\, V_1\,(t_0)$$

are compact (Hilbert-Schmidt, trace class), then so is the operator $\mathcal{D}\,(t_0)\, T$. Moreover,

$$\begin{aligned}
&\left\| \int_0^{t_0} V_0(u) T V_1\,(t_0 - u)\, du \right\| \\
&\le t_0 \left(\left\| (aI + H_0)\, V_0\,(t_0)\, T\,(aI + H_1)^{-1} \right\| + \left\| (aI + H_0)^{-1}\, T\,(aI + H_1)\, V_1\,(t_0) \right\| \right) \\
&\quad + \left\| (aI + H_0)^{-1}\, T V_1\,(t_0) \right\| + \left\| V_0\,(t_0)\, T\,(aI + H_1)^{-1} \right\|. \qquad (10)
\end{aligned}$$

Here $\|\cdot\|$ stands respectively for the usual operator norm (in the compact operator case), Hilbert-Schmidt norm (in the Hilbert-Schmidt case), or the trace norm (in the trace class situation).

Proof. This corollary is obtained from Theorem 2 (equality (6)) via the following observations:

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \log \frac{\cosh \pi\tau + 1}{\cosh \pi\tau - 1} d\tau = 2 \int_{-\infty}^{\infty} \frac{\tau}{\sinh \pi\tau} d\tau = 1,$$

in conjunction with the operator equalities

$$e^{i\tau t_0 A_0} - I = i\tau t_0 A_0 \int_0^1 e^{is\tau t_0 A_0} ds, \text{ and } I - e^{-i\tau t_0 A_1} = i\tau t_0 A_1 \int_0^1 e^{-is\tau t_0 A_1} ds,$$

and the inequalities:

$$\left\| \tanh\left(\frac{1}{2} t_0 A_0 \right) \right\| \le 1 \text{ and } \left\| \tanh\left(\frac{1}{2} t_0 A_1 \right) \right\| \le 1.$$

The result is applicable if $T = \mathcal{J} H_1 - H_0 \mathcal{J}$, where $\mathcal{J} : \mathcal{H}_1 \to \mathcal{H}_0$ is a bounded linear (identification) operator. In this case $\int_0^{t_0} V_0(u) T V_1(t_0 - u) du = V_0\,(t_0)\,\mathcal{J} - \mathcal{J} V_1\,(t_0)$; i.e. the difference of the semigroups. Among other things, in the sequel identity (11) below is used substantially.

4. Proposition. *In the presence of the hypotheses of Theorem 1, the following identities are true:*

$$\begin{aligned}
&t_0 V_0(t_0/2) T V_1(t_0/2) = \mathbb{E}^{\text{logistic}} \left[V_0\,(i t_0 U_1) \int_0^{t_0} V_0(u) T V_1(t_0 - u) du\, V_1\,(-i t_0 U_1) \right] \\
&= \frac{\pi}{2} \int_{-\infty}^{\infty} \frac{1}{(\cosh \pi\tau)^2} V_0(i\tau t_0) \int_0^{t_0} V_0(u) T V_1(t_0 - u) du V_1(-i\tau t_0) d\tau. \qquad (11)
\end{aligned}$$

Here T is a linear operator from $\mathcal{H}_1$ to $\mathcal{H}_0$, and U_1 is a *logistically distributed* random variable:

$$\mathbb{P}^{\text{logistic}}\left[U_1 \in B\right] = \frac{\pi}{2} \int_B \frac{1}{(\cosh \pi\tau)^2} d\tau = \pi^2 \int_0^{\infty} \frac{\sinh \pi\tau}{(\cosh \pi\tau)^3} \int_{-\tau}^{\tau} 1_B(\sigma) d\sigma\, d\tau.$$

Remark. A not necessarily rigorous proof of Proposition 4 will be given in Proposition 10. The method used here is based in double Stieltjes operator integrals: [1, 2, 3, 4]. Some information on this topic can be found in Yafaev [14], page 225–228 as well.

Put $\psi(\sigma) = \dfrac{\pi}{2} \dfrac{1}{(\cosh \pi\sigma)^2}$. Another relevant formula is the following one.

5. Proposition. *Under the assumptions of Theorem 1, the following identity is true:*

$$\begin{aligned}
& t_0 V_0\left(t_0/2\right) T V_1\left(t_0/2\right)-\mathcal{D}\left(t_0\right) T \\
& =\int_0^1 \int_{-\infty}^{\infty} \exp \left(-i s \sigma t_0\left(a I+H_0\right)\right)\left(a I+H_0\right)^{-1} T V_1\left(t_0\right) \\
& \quad \frac{\partial}{\partial \sigma}\left(\sigma \psi(\sigma) \frac{\exp \left(i s \sigma t_0\left(a I+H_1\right)\right)-I}{s}\right) d \sigma\, d s \\
& \quad -\int_0^1 \int_{-\infty}^{\infty} \frac{\partial}{\partial \sigma}\left(\sigma \psi(\sigma) \frac{I-\exp \left(-i s \sigma t_0\left(a I+H_0\right)\right)}{s}\right) \\
& \quad V_0\left(t_0\right) T\left(a I+H_1\right)^{-1} \exp \left(i s \sigma t_0\left(a I+H_1\right)\right) d \sigma\, d s \\
& \quad +\left(I-\frac{1}{2}\left(a I+H_0\right)\left(\sinh \left(\frac{1}{2}\left(a I+H_0\right)\right)\right)^{-1}\right)\left(a I+H_0\right)^{-1} T V_1\left(t_0\right) \\
& \quad -V_0\left(t_0\right) T\left(a I+H_1\right)^{-1}\left(\frac{1}{2}\left(a I+H_1\right)\left(\sinh \left(\frac{1}{2}\left(a I+H_1\right)\right)\right)^{-1}-I\right) .
\end{aligned} \tag{12}$$

Proof. From equality (24) (with $n=1$) in Proposition 10 below we obtain

$$\begin{aligned}
& \mathcal{D}\left(t_0\right) T-t_0 V_0\left(t_0/2\right) T V_1\left(t_0/2\right) \\
& =i t_0 \mathbb{E}^{\text{logistic}}\left[U_1 \int_0^1 V_0\left(i s U_1 t_0\right)\left(T V_1(t_0)-V_0(t_0) T\right) V_1\left(-i s U_1 t_0\right) d s\right] \\
& =\int_0^1 \int_{-\infty}^{\infty} \sigma \psi(\sigma) i t_0 \exp \left(-i s \sigma t_0\left(a I+H_0\right)\right) T V_1\left(t_0\right) \\
& \qquad \exp \left(i s \sigma t_0\left(a I+H_1\right)\right) d \sigma\, d s \\
& \qquad -\int_0^1 \int_{-\infty}^{\infty} \sigma \psi(\sigma) i t_0 \exp \left(-i s \sigma t_0\left(a I+H_0\right)\right) V_0\left(t_0\right) T \\
& \qquad \exp \left(i s \sigma t_0\left(a I+H_1\right)\right) d \sigma\, d s \\
& =i t_0 \int_0^1 \int_{-\infty}^{\infty} \sigma \psi(\sigma) \exp \left(-i s \sigma t_0\left(a I+H_0\right)\right) T V_1\left(t_0\right) \\
& \qquad \left(\exp \left(i s \sigma t_0\left(a I+H_1\right)\right)-I\right) d \sigma\, d s \\
& \qquad -i t_0 \int_0^1 \int_{-\infty}^{\infty} \sigma \psi(\sigma)\left(\exp \left(-i s \sigma t_0\left(a I+H_0\right)\right)-I\right) V_0\left(t_0\right) T \\
& \qquad \exp \left(i s \sigma t_0\left(a I+H_1\right)\right) d \sigma\, d s \\
& \quad +i t_0 \int_0^1 \int_{-\infty}^{\infty} \sigma \psi(\sigma) \exp \left(-i s \sigma t_0\left(a I+H_0\right)\right) d \sigma\, d s T V_1\left(t_0\right) \\
& \quad -i t_0 V_0\left(t_0\right) T \int_0^1 \int_{-\infty}^{\infty} \sigma \psi(\sigma) \exp \left(i s \sigma t_0\left(a I+H_1\right)\right) d \sigma\, d s \\
& =\int_0^1 \int_{-\infty}^{\infty} \sigma \psi(\sigma) \frac{\partial}{\partial \sigma}\left(\frac{-\exp \left(-i s \sigma t_0\left(a I+H_0\right)\right)}{s}\right) \\
& \quad \left(a I+H_0\right)^{-1} T V_1\left(t_0\right)\left(\exp \left(i s \sigma t_0\left(a I+H_1\right)\right)-I\right) d \sigma\, d s
\end{aligned}$$

$$
\begin{aligned}
&+\int_0^1\int_{-\infty}^{\infty}\sigma\psi(\sigma)\left(\exp\left(-is\sigma t_0\left(aI+H_0\right)\right)-I\right)V_0\left(t_0\right)T\left(aI+H_1\right)^{-1}\\
&\quad\frac{\partial}{\partial\sigma}\left(\frac{-\exp\left(is\sigma t_0\left(aI+H_1\right)\right)}{s}\right)d\sigma\,ds\\
&+it_0\int_0^1\int_{-\infty}^{\infty}\sigma\psi(\sigma)\exp\left(-is\sigma t_0\left(aI+H_0\right)\right)d\sigma\,dsTV_1\left(t_0\right)\\
&-it_0V_0\left(t_0\right)T\int_0^1\int_{-\infty}^{\infty}\sigma\psi(\sigma)\exp\left(is\sigma t_0\left(aI+H_1\right)\right)d\sigma\,ds
\end{aligned}
$$

(integration by parts)

$$
\begin{aligned}
&=\int_0^1\int_{-\infty}^{\infty}\exp\left(-is\sigma t_0\left(aI+H_0\right)\right)\left(aI+H_0\right)^{-1}TV_1\left(t_0\right)\\
&\quad\frac{\partial}{\partial\sigma}\left(\sigma\psi(\sigma)\frac{\exp\left(is\sigma t_0\left(aI+H_1\right)\right)-I}{s}\right)d\sigma\,ds\\
&-\int_0^1\int_{-\infty}^{\infty}\frac{\partial}{\partial\sigma}\left(\sigma\psi(\sigma)\frac{I-\exp\left(-is\sigma t_0\left(aI+H_0\right)\right)}{s}\right)\\
&\quad V_0\left(t_0\right)T\left(aI+H_1\right)^{-1}\exp\left(is\sigma t_0\left(aI+H_1\right)\right)d\sigma\,ds\\
&+it_0\int_0^1\int_{-\infty}^{\infty}\sigma\psi(\sigma)\exp\left(-is\sigma t_0\left(aI+H_0\right)\right)d\sigma\,dsTV_1\left(t_0\right)\\
&-it_0V_0\left(t_0\right)T\int_0^1\int_{-\infty}^{\infty}\sigma\psi(\sigma)\exp\left(is\sigma t_0\left(aI+H_1\right)\right)d\sigma\,ds
\end{aligned}
$$

In what follows the sequence $U_1,\ldots,U_n,\ldots$ denotes a sequence of independent identically distributed random variables, each distributed according to the *logistic* law. The operator $\mathcal{Q}(t_0)(T)$ is defined by

$$
\mathcal{Q}\left(t_0\right)T=\mathbb{E}^{\text{logistic}}\left[e^{it_0U_1H_0}Te^{-it_0U_1H_1}\right].
$$

From Proposition 4 it follows that

$$
t_0V_0(t_0/2)TV_1(t_0/2)=\mathcal{Q}\left(t_0\right)\left[\mathcal{D}\left(t_0\right)T\right].
$$

6. Theorem. *The following identity is true:*

$$
\begin{aligned}
&\left(I-\mathcal{Q}\left(t_0\right)\right)^{n+1}\left[\int_0^a V_0(s)TV_1(a-s)ds\right]\\
&=it_0\sum_{j=0}^{n}\binom{n}{j}(-1)^j\mathbb{E}^{\text{logistic}}\Big[\exp\left(it_0\left(U_1+\cdots+U_j\right)H_0\right)\\
&\quad U_{n+1}\int_0^1\exp\left(i\sigma t_0U_{n+1}H_0\right)\left(TV_1(a)-V_0(a)T\right)\exp\left(-it_0U_{n+1}H_1\right)d\sigma\\
&\quad\exp\left(-it_0\left(U_1+\cdots+U_j\right)H_1\right)\Big].
\end{aligned}\tag{13}
$$

Proof. For $n=0$ the above equality coincides with (24) in Proposition 10. For general n employ induction.

7. Corollary. *Let all operators of the form $V_0(t)TV_1(t)$, $t>0$, be compact (Hilbert-Schmidt, trace class). Suppose that, for $t>0$,*

$$\lim_{\substack{t_2>t_1>0,\\ t_2\downarrow 0}}\left\|V_0(t)T\int_{t_1}^{t_2}e^{-as}V_1(s)ds\right\|=\lim_{\substack{t_2>t_1>0,\\ t_2\downarrow 0}}\left\|\int_{t_1}^{t_2}e^{-as}V_0(s)dsTV_1(t)\right\|=0.$$

Then the operators $\mathcal{D}(t)T$, $t>0$, are compact.

A similar statement is valid in the Hilbert-Schmidt (trace class) situation: the Hilbert-Schmidt (trace class) norm replaces the operator norm.

8. Corollary. *Suppose that the operator*

$$\begin{aligned}S(t_0)&:=it_0\int_{-\infty}^{\infty}\tau\psi(\tau)\int_0^1 e^{ist_0\tau H_0}\left(TV_1\left(t_0\right)-V_0\left(t_0\right)T\right)e^{-ist_0\tau H_1}ds\,d\tau\\&=it_0\mathbb{E}^{\text{logistic}}\left[\int_0^{U_1}e^{ist_0H_0}\left(TV_1\left(t_0\right)-V_0\left(t_0\right)T\right)e^{-ist_0H_1}ds\right]\\&=it_0\frac{1}{2}\int_{-\infty}^{\infty}\left(\operatorname{sgn}(\tau)-\tanh\pi\tau\right)e^{it_0\tau H_0}\left(TV_1\left(t_0\right)-V_0\left(t_0\right)T\right)e^{-it_0\tau H_1}d\tau\end{aligned}$$

is Hilbert-Schmidt. Then, in Hilbert-Schmidt norm, the difference

$$\int_0^{t_0}V_0(s)TV_1\left(t_0-s\right)ds-\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}\left(t_0\right)^j\left(t_0V_0\left(\frac{t_0}{2}\right)TV_1\left(\frac{t_0}{2}\right)\right)\tag{14}$$

converges to zero if n tends to ∞.

Proof. From Theorem 6 and Proposition 4 we get

$$\begin{aligned}&\int_0^{t_0}V_0(s)TV_1\left(t_0-s\right)ds-\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}\left(t_0\right)^j\left[t_0V_0\left(t_0/2\right)TV_1\left(t_0/2\right)\right]\\&=\left(I-\mathcal{Q}\left(t_0\right)\right)^nS\left(t_0\right).\end{aligned}\tag{15}$$

An argument in [13] yields the conclusion in Corollary 8.

9. Theorem. *Suppose that the operators*

$$\left(aI+H_0\right)^{-1}TV_1\left(t_0\right)+V_0\left(t_0\right)T\left(aI+H_1\right)^{-1}\quad\text{and}\tag{16}$$

$$\left(aI+H_0\right)^{-1}T\left(aI+H_1\right)V_1\left(t_0\right)-\left(aI+H_0\right)V_0\left(t_0\right)T\left(aI+H_1\right)^{-1}\tag{17}$$

are compact (Hilbert-Schmidt, trace class). Then the operator $\int_0^{t_0}V_0(u)TV_1\left(t_0-u\right)du$ is compact (Hilbert-Schmidt, trace class) if and only if $V_0\left(\frac{1}{2}t_0\right)TV_1\left(\frac{1}{2}t_0\right)$ is so. Moreover, if (16) and (17) are satisfied, then the Riemann approximation

$$\frac{t_0}{n}\sum_{j=0}^{n-1}V_0\left(\frac{2j+1}{2n}t_0\right)TV_1\left(\frac{2n-2j-1}{2n}t_0\right)$$

converges with respect to the operator norm (in the compact case), in the Hibert-Schmidt norm (in the Hilbert-Schmidt case), and in the trace norm (in the trace situation).

Remark. From the proof it will follow that (16) and (17) may be relaxed to the following assumption. For $\tau \in \mathbb{R}$ the operator

$$\begin{aligned} & i\int_0^\tau e^{i\sigma H_0}\left(TV_1\left(t_0\right)-V_0\left(t_0\right)T\right)e^{-i\sigma H_1}d\sigma \quad \text{(integration by parts)} \\ & = e^{i\tau H_0}\left(\left(aI+H_0\right)^{-1}TV_1\left(t_0\right)+V_0\left(t_0\right)T\left(aI+H_1\right)^{-1}\right)e^{-i\tau H_1} \\ & \quad -\left(\left(aI+H_0\right)^{-1}TV_1\left(t_0\right)+V_0\left(t_0\right)T\left(aI+H_1\right)^{-1}\right) \\ & \quad + i\int_0^\tau e^{i\sigma H_0}\left(\left(aI+H_0\right)^{-1}T\left(aI+H_1\right)V_1\left(t_0\right)-\left(aI+H_0\right)V_0\left(t_0\right)T\left(aI+H_1\right)^{-1}\right) \\ & \qquad\qquad e^{-i\sigma H_1}d\sigma \end{aligned} \tag{18}$$

is compact (Hilbert-Schmidt, trace class), and converges to 0 in the operator norm (or in Hilbert-Schmidt norm, or in trace class norm, as the case may be), if τ tends to 0.

For the proof of Theorem 9 a number of identities are required. They are collected in the following proposition.

10. Proposition. *Let the hypotheses be as in Theorem 9. Put*

$$\mathcal{D}\left(t_0\right)T = \int_0^{t_0} V_0(u)TV_1\left(t_0-u\right)du; \tag{19}$$

$$\mathcal{D}_n\left(t_0\right)T = \frac{t_0}{n}\sum_{j=0}^{n-1} V_0\left(\frac{2j+1}{2n}t_0\right)TV_1\left(\frac{2n-2j-1}{2n}t_0\right). \tag{20}$$

The following equalities are valid:

$$\sum_{j=0}^{n-1} V_0\left(\frac{j}{n}t_0\right)\mathcal{D}\left(\frac{1}{n}t_0\right)TV_1\left(\frac{n-j-1}{n}t_0\right) = \mathcal{D}\left(t_0\right)T; \tag{21}$$

$$t_0V_0\left(\frac{1}{2}t_0\right)TV_1\left(\frac{1}{2}t_0\right) = \int_{-\infty}^{\infty}\psi(\tau)e^{i\tau t_0 H_0}\mathcal{D}\left(t_0\right)Te^{-i\tau t_0 H_1}d\tau; \tag{22}$$

$$\mathcal{D}_n\left(t_0\right)T = n\int_{-\infty}^{\infty}\psi(n\tau)e^{i\tau t_0 H_0}\mathcal{D}\left(t_0\right)Te^{-i\tau t_0 H_1}d\tau; \tag{23}$$

$$e^{i\tau H_0}\mathcal{D}\left(t_0\right)Te^{-i\tau H_1}-\mathcal{D}\left(t_0\right)T = i\int_0^\tau e^{i\sigma H_0}\left(TV_1\left(t_0\right)-V_0\left(t_0\right)T\right)e^{-i\sigma H_1}d\sigma. \tag{24}$$

Proof of Proposition 10. A proof of (21) reads as follows (see Remark 2 prior to Theorem 2 as well):

$$\sum_{j=0}^{n-1} V_0\left(\frac{j}{n}t_0\right)\mathcal{D}\left(\frac{1}{n}t_0\right)V_1\left(\frac{n-j-1}{n}t_0\right)T$$

$$
\begin{aligned}
&= \sum_{j=0}^{n-1} \int_0^{t_0/n} V_0\left(\frac{j}{n}t_0 + s\right) T V_1\left(\frac{n-j}{n}t_0 - s\right) ds \\
&= \sum_{j=0}^{n-1} \int_{jt_0/n}^{(j+1)t_0/n} V_0\left(s\right) T V_1\left(t_0 - s\right) ds = \mathcal{D}\left(t_0\right) T.
\end{aligned}
$$

An informal proof of (22) based on double Stieltjes operator integrals reads as follows:

$$
\begin{aligned}
&\frac{\pi}{2}\int_{-\infty}^{\infty} \frac{1}{(\cosh \pi\tau)^2} V_0(i\tau t_0) \int_0^{t_0} V_0(u) T V_1(t_0 - u) du V_1(-i\tau t_0) d\tau \\
&= \int\!\!\int \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{\exp\left(-i\tau t_0(\xi-\eta)\right)}{(\cosh \pi\tau)^2} d\tau \frac{\exp(-t_0\eta) - \exp(-t_0\xi)}{\xi - \eta} E_0(d\xi) T E_1(d\eta) \\
&= \int\!\!\int \frac{\frac{1}{2}t_0(\xi-\eta)}{\sinh\left(\frac{1}{2}(\xi-\eta)t_0\right)} \frac{\sinh\left(\frac{1}{2}t_0(\xi-\eta)\right)}{\frac{1}{2}(\xi-\eta)} \exp\left(-\frac{1}{2}t_0(\xi+\eta)\right) E_0(d\xi) T E_1(d\eta) \\
&= t_0 \int\!\!\int \exp\left(-\frac{1}{2}t_0(\xi+\eta)\right) E_0(d\xi) T E_1(d\eta) = t_0 V_0(t_0/2) T V_1(t_0/2).
\end{aligned}
$$

A proof based on complex function theory is to found in [13] and in [11]. A proof of (23) proceeds as follows:

$$
\mathcal{D}_n\left(t_0\right) T = \frac{t_0}{n} \sum_{j=0}^{n-1} V_0\left(\frac{j}{n}t_0\right) V_0\left(\frac{1}{2n}t_0\right) T V_1\left(\frac{1}{2n}t_0\right) V_1\left(\frac{n-j-1}{n}t_0\right)
$$

(employ (22) with t_0/n replacing t_0)

$$
\begin{aligned}
&= \sum_{j=0}^{n-1} V_0\left(\frac{j}{n}t_0\right) \int_{-\infty}^{\infty} \psi(\tau) e^{i\tau(t_0/n)H_0} \mathcal{D}\left(\frac{1}{n}t_0\right) T e^{-i\tau(t_0/n)H_1} d\tau V_1\left(\frac{n-j-1}{n}t_0\right) \\
&= n \int_{-\infty}^{\infty} \psi(n\tau) e^{i\tau t_0 H_0} \mathcal{D}\left(t_0\right) T e^{-i\tau t_0 H_1} d\tau.
\end{aligned}
$$

(In the ultimate equality we employed equality (21).) This shows equality (23). In order to prove (24) we proceed as follows. The reasoning can be made rigorous upon replacing T by $\left(aI + H_0\right)^{-1} T \left(aI + H_1\right)^{-1}$. We write

$$
\begin{aligned}
&e^{i\tau H_0} \mathcal{D}\left(t_0\right) T e^{-i\tau H_1} - \mathcal{D}\left(t_0\right) T \\
&= i \int_0^{\tau} e^{i\sigma H_0} \mathcal{D}\left(t_0\right) \left(H_0 T - T H_1\right) e^{-i\sigma H_1} d\sigma \\
&= i \int_0^{\tau} e^{i\sigma H_0} \int_0^{t_0} V_0(u) \left(H_0 T - T H_1\right) V_1\left(t_0 - u\right) du\, e^{-i\sigma H_1} d\sigma \\
&= i \int_0^{\tau} e^{i\sigma H_0} \int_0^{t_0} \left\{ -\frac{\partial}{\partial u}\left(V_0(u) T V_1\left(t_0 - u\right)\right) \right\} du e^{-i\sigma H_1} d\sigma \\
&= i \int_0^{\tau} e^{i\sigma H_0} \left(T V_1\left(t_0\right) - V_0\left(t_0\right) T\right) e^{-i\sigma H_1} d\sigma.
\end{aligned}
$$

Proof of Theorem 9. The assertion in Theorem 9 follows from the identity:

$$
\begin{aligned}
&\mathcal{D}_n(t_0)T - \mathcal{D}(t_0)T \qquad (25)\\
&= i\,n\int_{-\infty}^{\infty}\psi(n\tau)\int_0^{\tau t_0} e^{i\sigma H_0}\left(TV_1(t_0) - V_0(t_0)T\right)e^{-i\sigma H_1}d\sigma\,d\tau \quad \text{(see formula (18))}\\
&= n\int_{-\infty}^{\infty}\psi(n\tau)\Big\{e^{i\tau t_0 H_0}\left((aI+H_0)^{-1}TV_1(t_0) + V_0(t_0)T(aI+H_1)^{-1}\right)e^{-i\tau t_0 H_1}\\
&\quad - \left((aI+H_0)^{-1}TV_1(t_0) + V_0(t_0)T(aI+H_1)^{-1}\right)\Big\}\,d\tau\\
&\quad + in\int_{-\infty}^{\infty}\psi(n\tau)\int_0^{\tau t_0} e^{i\sigma H_0}\\
&\quad\left((aI+H_0)^{-1}T(aI+H_1)V_1(t_0) - (aI+H_0)V_0(t_0)T(aI+H_1)^{-1}\right)e^{-i\sigma H_1}d\sigma\,d\tau.
\end{aligned}
$$

Our assumptions imply that the operator-valued function

$$\tau \mapsto e^{i\tau t_0 H_0}\left((aI+H_0)^{-1}TV_1(t_0) + V_0(t_0)T(aI+H_1)^{-1}\right)e^{-i\tau t_0 H_1}$$

is continuous for the operator norm, and that the same is true for the function

$$
\begin{aligned}
\tau \mapsto \int_0^{\tau t_0} e^{i\sigma H_0}&\Big((aI+H_0)^{-1}T(aI+H_1)V_1(t_0)\\
&- (aI+H_0)V_0(t_0)T(aI+H_1)^{-1}\Big)e^{-i\sigma H_1}d\sigma.
\end{aligned}
$$

Consequently equality (25) yields $\lim_{n\to\infty}\mathcal{D}_n(t_0)T = \mathcal{D}(t_0)T$ for the operator norm. Another appeal to (25) for $n = 1$ yields the compactness of $\mathcal{D}(t_0)T$. This follows because $\mathcal{D}_1(t_0)T = t_0V_0(t_0/2)TV_1(t_0/2)$. A similar argument works in the Hilbert-Schmidt and trace class context. The equality in (25) is a consequence of the equalities in Proposition 10:

$$
\begin{aligned}
&\mathcal{D}_n(t_0)T - \mathcal{D}(t_0)T \quad \text{(see (23))}\\
&= n\int_{-\infty}^{\infty}\psi(n\tau)e^{i\tau t_0 H_0}\mathcal{D}(t_0)Te^{-i\tau t_0 H_1}d\tau - \mathcal{D}(t_0)T\\
&= n\int_{-\infty}^{\infty}\psi(n\tau)\left(e^{i\tau t_0 H_0}\mathcal{D}(t_0)Te^{-i\tau t_0 H_1} - \mathcal{D}(t_0)T\right)d\tau \text{ (see (24))}\\
&= i\,n\int_{-\infty}^{\infty}\psi(n\tau)\int_0^{\tau t_0} e^{i\sigma H_0}\left(TV_1(t_0) - V_0(t_0)T\right)e^{-i\sigma H_1}d\sigma\,d\tau. \qquad (26)
\end{aligned}
$$

The equality in (25) then follows from (18) and (26).

In the proof of Theorem 12 the following lemma will be used. The symbol $\widehat{\psi}'(\xi)$ denotes the first derivative of $\widehat{\psi}$ with respect to ξ. A similar convention is employed for the higher derivatives of $\widehat{\psi}$.

11. Lemma. *Put* $\psi(\sigma) = \dfrac{\pi}{2}\dfrac{1}{(\cosh \pi\sigma)^2}$. *Then its Fourier transform* $\widehat{\psi}$ *is given by* $\widehat{\psi}(\xi) = \dfrac{\frac{1}{2}\xi}{\sinh \frac{1}{2}\xi}$. *In addition set* $C_n(\xi) = \xi \left(1 - \left(\dfrac{\partial}{\partial \xi}\right)^2\right)\left(\left(1-\widehat{\psi}(\xi)\right)^{n+1}\dfrac{1}{\xi}\right)$. *Then the following assertions are valid.*

(a) *The derivative of* $C_n(\xi)$ *is given by*

$$\begin{aligned} C_n'(\xi) = & -(n+1)\left(1-\widehat{\psi}(\xi)\right)^n \widehat{\psi}'(\xi) \\ & + \frac{4}{\xi^3}(n+1)\left(1-\widehat{\psi}(\xi)\right)^n \widehat{\psi}'(\xi) + \frac{4}{\xi^3}\left(1-\widehat{\psi}(\xi)\right)^{n+1} \\ & - 3(n+1)n\left(1-\widehat{\psi}(\xi)\right)^{n-1} \widehat{\psi}'(\xi)\widehat{\psi}''(\xi) \\ & + (n+1)\left(1-\widehat{\psi}(\xi)\right)^n \widehat{\psi}'''(\xi) \\ & + (n+1)n(n-1)\left(1-\widehat{\psi}(\xi)\right)^{n-2} \widehat{\psi}'(\xi)^3 \\ & + \frac{1}{\xi^2}(n+1)n\left(1-\widehat{\psi}(\xi)\right)^{n-1} \widehat{\psi}'(\xi)^2. \end{aligned}$$

(b) *Moreover, there exists a constant* C *such that, for* $n \geq 2$,

$$|C_n'(\xi)| \leq C\frac{n}{2}\left(1-\widehat{\psi}(\xi)\right)^{\frac{1}{2}n-1} |\psi'(\xi)| + \frac{4}{\xi^3}\left(1-\widehat{\psi}(\xi)\right)^{n+1}.$$

(c) *Consequently, for* $\tau > 0$, *and* $n \geq 2$

$$\int_0^\infty |C_n'(\xi)| \frac{1}{\tau\xi+1} d\xi \leq \left(C + \frac{2}{3}\right)\int_0^\infty \left(1-\widehat{\psi}(\xi)\right)^{\frac{1}{2}n} \frac{\tau}{(\tau\xi+1)^2} d\xi.$$

(d) *For* $\tau > 0$ *the following inequalities are valid:*

$$\left|\int_0^\infty C_n(\xi)\frac{\sin \tau\xi}{\xi}d\xi\right| \leq \int_0^\infty |C_n'(\xi)| \left|\int_{\tau\xi}^\infty \frac{\sin\eta}{\eta}d\eta\right| d\xi \leq \left(\frac{\pi}{2}+2\right)\int_0^\infty |C_n'(\xi)| \frac{1}{1+\tau\xi}d\xi.$$

Proof. (a) The expression for $C_n'(\xi)$ is obtained after a tedious, but straight-forward calculation.
(b) We note that the functions $\widehat{\psi}$ and its first, second and third derivative obey the following inequalities:

$$\begin{aligned} & 0 \leq -\widehat{\psi}'(\xi) \leq \frac{1}{2}\widehat{\psi}(\xi), \quad -\frac{1}{12}\widehat{\psi}(\xi) \leq \widehat{\psi}''(\xi) \leq \frac{1}{4}\widehat{\psi}(\xi), \quad \text{and,} \\ & -\frac{7}{20}\left(-\widehat{\psi}'(\xi)\right) \leq -\widehat{\psi}'''(\xi) \leq \frac{1}{4}\left(-\widehat{\psi}'(\xi)\right). \end{aligned}$$

So the constants $C_1 = \dfrac{1}{2}$, $C_2 = \dfrac{1}{4}$, and $C_3 = \dfrac{7}{20}$ respectively satisfy

$$\left|\widehat{\psi}'(\xi)\right| \leq C_1\widehat{\psi}(\xi), \quad \left|\widehat{\psi}''(\xi)\right| \leq C_2\widehat{\psi}(\xi), \quad \left|\widehat{\psi}'''(\xi)\right| \leq C_3\left|\widehat{\psi}'(\xi)\right|.$$

The existence of these constants is guaranteed by the following arguments. Put $f(\xi) = -\frac{1}{\xi} + \frac{1}{2}\coth\frac{1}{2}\xi$. Then $0 \le f(\xi) \le \min\left(\frac{1}{2}, \frac{1}{12}\xi\right)$, and $-\frac{4}{15} \le \frac{6}{\xi^2} - \frac{1}{2f(\xi)\xi} \le 0$. We also have

$$\begin{aligned}
-\widehat{\psi}'(\xi) &= \widehat{\psi}(\xi) f(\xi),\\
\widehat{\psi}''(\xi) &= \widehat{\psi}(\xi)\left(2f(\xi)^2 + 2\frac{f(\xi)}{\xi} - \frac{1}{4}\right) = \widehat{\psi}(\xi)\left\{\frac{1}{4} - \frac{\sinh\xi - \xi}{\xi\left(\cosh\xi - 1\right)}\right\}, \quad \text{and, finally,}\\
\widehat{\psi}'''(\xi) &= \widehat{\psi}'(\xi)\left\{6f(\xi)^2 + 12\frac{f(\xi)}{\xi} + \frac{6}{\xi^2} - \frac{1}{2\xi f(\xi)} - \frac{5}{4}\right\}\\
&= \widehat{\psi}'(\xi)\left\{\frac{1}{4} - \frac{\frac{1}{2}\left(\cosh 2\xi + 1\right) + 4\cosh\xi - 5 - 3\xi\sinh\xi}{\frac{1}{2}\xi\sinh 2\xi - \xi\sinh\xi - \cosh 2\xi + 4\cosh\xi - 3}\right\}.
\end{aligned}$$

Moreover we have

$$\begin{aligned}
&0 \le \frac{\sinh\xi - \xi}{\xi\left(\cosh\xi - 1\right)} \le \frac{1}{3}, \quad \text{and}\\
&0 \le \frac{\frac{1}{2}\left(\cosh 2\xi + 1\right) + 4\cosh\xi - 5 - 3\xi\sinh\xi}{\frac{1}{2}\xi\sinh 2\xi - \xi\sinh\xi - \cosh 2\xi + 4\cosh\xi - 3} \le \frac{3}{5}.
\end{aligned}$$

Since, in addition,

$$\begin{aligned}
&0 \le 1 - \psi(\xi) \le \min\left(\frac{1}{24}\xi^2, 1\right), \quad \text{and for } 0 < y < 1,\\
&(n+1)\left(1-y\right)^{\frac{1}{2}n} y \le 2, \quad (n+1)\left(1-y\right)^{\frac{1}{2}n-1} y \le 6, \quad \text{and}\\
&(n+1)\left(n-1\right)\left(1-y\right)^{\frac{1}{2}n-1} y^2 \le 16,
\end{aligned}$$

the inequality in (b) follows from (a).

(c) This inequality follows from (b) in conjunction with integration by parts. The constant C may be chosen to be

$$C = 3 + \frac{1}{3}\frac{1}{\sqrt{6}} + 12C_2 + 3C_3 + 32C_1^2 + \frac{1}{2}C_1.$$

The choice $C_1 = \frac{1}{2}$, $C_2 = \frac{1}{4}$, and $C_3 = \frac{7}{20}$ yields $C < 15.5$.

(d) The first inequality in (d) is obtained upon integration by parts. The second one is a consequence of

$$\left|\int_\rho^\infty \frac{\sin\eta}{\eta} d\eta\right| \le \left(\frac{\pi}{2} + 2\right)\frac{1}{1+\rho}, \quad \rho > 0,$$

which in turn can also be proved upon employing integration by parts.

12. Theorem. *Suppose that the operator $TV_1(t_0) - V_0(t_0)T$ is bounded (Hilbert-Schmidt, trace class). Then, in operator norm (Hilbert-Schmidt norm, trace class norm) sense, the difference*

$$\int_0^{t_0} V_0(s)TV_1(t_0-s)\,ds - \sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}(t_0)^j\left(t_0V_0\left(\frac{t_0}{2}\right)TV_1\left(\frac{t_0}{2}\right)\right)$$

converges to zero if n tends to ∞.

Proof. Put $\chi(\tau) = \frac{1}{2}(\operatorname{sgn}(\tau) - \tanh\pi\tau)$. Then its Fourier transform is given by

$$\widehat{\chi}(\xi) = \frac{1}{\xi}\left(1 - \widehat{\psi}(\xi)\right). \tag{27}$$

First we notice (see Corollary 8):

$$\begin{aligned}&\int_{-\infty}^{\infty}\tau\psi(\tau)\int_0^1 e^{ist_0\tau H_0}\left(TV_1(t_0) - V_0(t_0)T\right)e^{-ist_0\tau H_1}ds\,d\tau\\ &= \int_{-\infty}^{\infty}\psi(\tau)\int_0^{\tau} e^{ist_0H_0}\left(TV_1(t_0) - V_0(t_0)T\right)e^{-ist_0H_1}ds\,d\tau\end{aligned}$$

(apply Fubini's theorem)

$$= \frac{1}{2}\int_{-\infty}^{\infty}(\operatorname{sgn}(\tau) - \tanh\pi\tau)\,e^{it_0\tau H_0}\left(TV_1(t_0) - V_0(t_0)T\right)e^{-it_0\tau H_1}d\tau. \tag{28}$$

Put $F(\tau) = \exp(it_0\tau H_0)\left(TV_1(t_0) - V_0(t_0)T\right)\exp(-it_0\tau H_1)$. We also write $m_\psi(B) = \int_B\psi(\tau)d\tau = \mathbb{P}^{\text{logistic}}[U_1 \in B]$, if B is a Borel subset of $\mathbb{R}$. With δ we denote the Dirac measure at 0, and with $f * \mu(\tau) = \int f(\tau-\sigma)d\mu(\sigma)$ the convolution product of the function f and the Borel measure μ. From (27) and (28) in conjunction with (15) we infer

$$\begin{aligned}&\int_0^{t_0} V_0(s)TV_1(t_0-s)\,ds - \sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}(t_0)^j\left[t_0V_0(t_0/2)TV_1(t_0/2)\right]\\
&= it_0\left(I - \mathcal{Q}(t_0)\right)^n\mathbb{E}^{\text{logistic}}\left[\int_0^{U_1} e^{ist_0H_0}\left(TV_1(t_0) - V_0(t_0)T\right)e^{-ist_0H_1}ds\right]\\
&= it_0\sum_{j=0}^{n}\binom{n}{j}(-1)^j\\
&\qquad \mathbb{E}^{\text{logistic}}\left[\exp\left(it_0(U_1+\cdots+U_j)H_0\right)\int_0^{U_{n+1}}F(\tau)d\tau\,\exp\left(-it_0(U_1+\cdots+U_j)H_1\right)\right]\\
&= it_0\int_{-\infty}^{\infty}\left[\chi * (\delta - m_\psi)^{*n}\right](\tau)F(\tau)d\tau\\
&= \frac{it_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\chi * \widehat{(\delta - m_\psi)^{*n}}\right](\xi)e^{i\tau\xi}d\xi\,F(\tau)d\tau\end{aligned}$$

$$= \frac{it_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\widehat{\psi}(\xi)\right)^{n+1}\frac{1}{\xi}e^{i\tau\xi}d\xi\, F(\tau)d\tau$$
$$= -\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\widehat{\psi}(\xi)\right)^{n+1}\frac{1}{\xi}\sin(\tau\xi)d\xi\, F(\tau)d\tau$$
$$= -\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\widehat{\psi}(\xi)\right)^{n+1}\frac{1}{\xi}\left(1-\left(\frac{\partial}{\partial\xi}\right)^2\right)\frac{\sin(\tau\xi)}{1+\tau^2}F(\tau)d\xi\, d\tau$$

(integration by parts)

$$= -\frac{t_0}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left[\xi\left(1-\left(\frac{\partial}{\partial\xi}\right)^2\right)\left(\left(1-\widehat{\psi}(\xi)\right)^{n+1}\frac{1}{\xi}\right)\right]\frac{\sin(\tau\xi)}{\xi}d\xi\frac{1}{1+\tau^2}F(\tau)d\tau$$
$$= -\frac{t_0}{\pi}\int_0^{\infty}\int_0^{\infty}C_n(\xi)\frac{\sin(\tau\xi)}{\xi}d\xi\frac{1}{1+\tau^2}\left(F(\tau)-F(-\tau)\right)d\tau$$
$$= -\frac{t_0}{\pi}\int_0^{\infty}\left(\int_0^{\infty}C_n(\xi)\frac{\sin(\tau\xi)}{\xi}d\xi\right)\frac{1}{1+\tau^2}\left(F(\tau)-F(-\tau)\right)d\tau$$

(integration by parts)

$$= -\frac{t_0}{\pi}\int_0^{\infty}\left(\int_0^{\infty}C_n'(\xi)\int_{\tau\xi}^{\infty}\frac{\sin\eta}{\eta}d\eta\, d\xi\right)\frac{1}{1+\tau^2}\left(F(\tau)-F(-\tau)\right)d\tau, \tag{29}$$

where, as in Lemma 11, $C_n(\xi) = \xi\left(1-\left(\frac{\partial}{\partial\xi}\right)^2\right)\left(\left(1-\widehat{\psi}(\xi)\right)^{n+1}\frac{1}{\xi}\right)$. Then by Lemma 11

$$\left\|\int_0^{t_0}V_0(s)TV_1(t_0-s)\,ds-\sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}(t_0)^j\left[t_0V_0(t_0/2)\,TV_1(t_0/2)\right]\right\|$$
$$\leq \frac{t_0}{\pi}\int_0^{\infty}\left|\int_0^{\infty}C_n'(\xi)\int_{\tau\xi}^{\infty}\frac{\sin\eta}{\eta}d\eta\, d\xi\right|\frac{1}{1+\tau^2}\left\|(F(\tau)-F(-\tau))\right\|d\tau$$
$$\leq \frac{t_0}{\pi}C_0\int_0^{\infty}\int_0^{\infty}\left(1-\widehat{\psi}(\xi)\right)^{\frac{1}{2}n}\frac{\tau}{(1+\tau\xi)^2}d\xi\frac{1}{1+\tau^2}\left\|(F(\tau)-F(-\tau))\right\|d\tau, \tag{30}$$

where $C_0 = \left(\frac{\pi}{2}+2\right)\left(C+\frac{2}{3}\right) < 83.2$. By Lebesgue's dominated convergence theorem, the final expression in (30) tends to zero, as n tends to ∞. This proves Theorem 12.

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