THE LIE-TROTTER PRODUCT FORMULA DOES NOT HOLD FOR ARBITRARY SUMS OF GENERATORS

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Abstract. In this paper we show that in general the semigroup generated by the closure of the sum of two generators can not be represented by the Lie-Trotter product formula.

1. Introduction

In 1959, H. F. Trotter [9] proved the following theorem.

Theorem 1. *Suppose that $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are strongly continuous contraction semigroups on a Banach space E with generators $(A, D(A))$ and $(B, D(B))$, respectively. If the closure of $A + B$ generates a strongly continuous semigroup $(U(t))_{t\geq 0}$, then the generated semigroup is given by the product formula*

$$U(t)f = \lim_{n\to\infty} \left(T(\tfrac{t}{n})S(\tfrac{t}{n})\right)^n f \tag{1}$$

for all $f \in E$ and uniformly for t in compact intervals.

In 1974, more general product formulas were considered by P. R. Chernoff [1, Chap. 1, 2]. Following his approach one can prove the theorem below (cf. [2, Chap. III. 5]).

Theorem 2. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be strongly continuous semigroups on a Banach space E satisfying the stability condition*

$$\| \left(T(\tfrac{t}{n})S(\tfrac{t}{n})\right)^n \| \leq Me^{\omega t} \text{ for all } t \geq 0, n \in \mathbb{N}, \tag{2}$$

and for some constants $M \geq 1, \omega \in \mathbb{R}$. Let $(A, D(A))$ and $(B, D(B))$ be the generators of $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, respectively. Consider the sum $A + B$ defined on a subspace $D \subseteq D(A) \cap D(B)$ and assume that D and $(\lambda_0 - A - B)D$ are dense in E for some $\lambda_0 > \omega$. Then $\overline{A + B}$ generates a strongly continuous semigroup $(U(t))_{t\geq 0}$ given by

$$U(t)f = \lim_{n\to\infty} \left(T(\tfrac{t}{n})S(\tfrac{t}{n})\right)^n f$$

for all $f \in E$ and uniformly for t in compact intervals.

These results led many people to believe that if the sum of two generators is again a generator of a strongly continuous semigroup, this semigroup must be given by the so-called Lie-Trotter product formula (1).
However, this is not true in general. T.G. Kurtz and M. Pierre showed that the Lie-Trotter product formula fails for sums of m-accretive operators in the nonlinear sense [4]. In this note we construct a strongly continuous semigroup generated by the closure of the sum

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of two generators such that the condition (2) does not hold and therefore the Lie-Trotter product formula does not converge.

2. The counterexample

The following counterexample is based on a construction by G. Nickel and R. Schnaubelt, [7]. First, we construct a suitable Banach space E.
Take $0 < a_k < \frac{1}{3}$ for $k \in \mathbb{N}$ and set $b_k := 1 - a_k$ and $J_k := [0, b_k]$. Then $\frac{2}{3} < b_k < 1$. Using the functions $h_k : J_k \to \mathbb{R}$ defined by

$$h_k(x) := \begin{cases} 1 & \text{for } 0 \leq x < a_k \text{ or } 2a_k < x \leq b_k, \\ 2 & \text{for } a_k \leq x \leq 2a_k, \end{cases}$$

we introduce the measures $\mu_k := h_k \lambda$ where λ is the Lebesgue measure on J_k. Then (J_k, μ_k) is a probability space for each $k \in \mathbb{N}$. Thus the product space $(\Omega, \nu) := \prod_{k\in\mathbb{N}} (J_k, \mu_k)$ and the space

$$E := L^1(\Omega, \nu)$$

are welldefined. We denote by $\|\cdot\|$ the 1-norm on E and by χ_C the characteristic function of a set $C \subseteq \Omega$. Note that the space

$$E_0 := \mathrm{lin}\{\chi_C : C = C_1 \times \cdots \times C_n \times J_{n+1} \times \cdots, n \in \mathbb{N},\ C_k \subseteq J_k \text{ measurable}\}$$

is dense in E because the closure of E_0 contains all simple functions in E (see [10, Chap. 5]).
On E we define, for each $k \in \mathbb{N}$, a nilpotent right translation semigroup by

$$\left(e^{\tau B_k} f\right)(x_1, \cdots) := \begin{cases} f(x_1, \cdots, x_{k-1}, x_k - \tau, x_{k+1}, \cdots) & \text{if } x_k - \tau \in J_k, \\ 0 & \text{if } x_k - \tau \notin J_k, \end{cases}$$

with generator $(B_k, D(B_k))$. This semigroup has the following properties which are proved in [7, Lemma 3.1].

Lemma 3. *Let E and $(e^{\tau B_k})_{\tau\geq 0}, k \in \mathbb{N}$, be defined as above. Then the following assertions hold.*

(a) *The semigroups $(e^{\tau B_k})_{\tau\geq 0}$ are strongly continuous and commute pairwise. Moreover,*

$$\|e^{\tau_k B_k}\| = \begin{cases} 2 & \text{for } 0 < \tau_k < 2a_k, \\ 1 & \text{for } 2a_k \leq \tau_k < b_k, \\ 0 & \text{for } \tau_k \geq b_k. \end{cases}$$

(b) $\| \prod_{k=1}^{n} e^{\tau_k B_k} \| = 2^n$ *for* $0 < \tau_k < 2a_k$ *and* $n \in \mathbb{N}$.

(c) *The space*

$$F := \{f \in \bigcap_{k\in\mathbb{N}} D(B_k) : \sup_{k\in\mathbb{N}} \|B_k f\| < \infty\}$$

endowed with the norm $\|f\|_F := \max\{\|f\|, \sup_{k\in\mathbb{N}} \|B_k f\|\}$ *is continuously and densely embedded in E.*

In the sequel, we need some results of the theory of nonautonomous Cauchy problems (cf. [3, Chap. 7], [6, Chap. 1, 2], [8, Chap. 1]).

Definition 4. *A nonautonomous Cauchy problem*

$$(NCP) \quad \begin{cases} \frac{d}{dt}u(t) & = B(t)u(t) \quad t \geq s, \ t,s \in \mathbb{R}, \\ u(s) & = g \in E, \end{cases}$$

for linear operators $(B(t), D(B(t)))_{t\in\mathbb{R}}$ on a Banach space E is called well-posed (on spaces $F_t, t \in \mathbb{R}$), if there are dense subspaces $F_t \subseteq D(B(t))$ of E such that for each $g \in F_s, s \in \mathbb{R}$, there exists a unique continuously differentiable function

$$t \longmapsto u(t) = u(t; s, g) \in F_t, \quad t \geq s, \ t, s \in \mathbb{R},$$

satisfying (NCP). Moreover, the solutions are required to depend continuously on initial data, i.e. if $\mathbb{R} \ni s_n \to s \in \mathbb{R}$ and $F_{s_n} \ni g_n \to g \in F_s$ in E, then $\tilde{u}(t; s_n, g_n) \to \tilde{u}(t; s, g)$ in E uniformly for t in compact intervals, where

$$\tilde{u}(t; s, g) := \begin{cases} u(t; s, g) & \text{for } t \geq s, \ t, s \in \mathbb{R}, \\ g & \text{for } t < s, \ t, s \in \mathbb{R}\ . \end{cases}$$

Now, let $I \in \{[a, b], [a, \infty), \mathbb{R}\}$. We set

$$U(t, s)g := u(t; s, g) \text{ for } g \in F_s \text{ and } \triangle_I := \{(t, s) \in I \times I : t \geq s\}.$$

A family $(U(t, s))_{(t,s)\in\triangle_I}$ of linear and bounded operators on E is called a (strongly continuous) evolution family if

(i) *$U(s, s) = Id$, $U(t, s) = U(t, r)U(r, s)$ for $t \geq r \geq s$,*
(ii) *the mapping $\triangle_I \ni (t, s) \mapsto U(t, s)$ is strongly continuous.*

An evolution family $(U(t, s))_{(t,s)\in\triangle_\mathbb{R}}$ is called exponentially bounded if

$$\| U(t, s) \| \leq Me^{\omega(t-s)} \text{ for some } M \geq 1, \omega \in \mathbb{R} \text{ and all } t \geq s \in \mathbb{R}\ .$$

One can show that the Cauchy problem (NCP) is well-posed (on $F_t, t \in \mathbb{R}$) iff there are a unique evolution family $(U(t, s))_{(t,s)\in\triangle_\mathbb{R}}$ and dense subspaces $F_t, t \in \mathbb{R}$, in E satisfying the following conditions (see [6, Prop. 2.5])

$$\begin{cases} U(t, s)F_s \subseteq F_t, \\ \text{the mapping } t \mapsto U(t, s)g, \ t \geq s, \text{ is continuously differentiable for } g \in F_s, \text{ and} \\ \frac{\partial}{\partial t}U(t, s)g = B(t)U(t, s)g\ . \end{cases} \tag{3}$$

To every exponentially bounded evolution family we can associate strongly continuous semigroups on E-valued function spaces (see [6, Chap. 1]). Here, we choose the Banach space

$$\mathcal{E} := C_0(\mathbb{R}, E) := \{f : \mathbb{R} \to E : f \text{ is continuous and } \lim_{|x|\to\infty} f(x) = 0\}$$

endowed with the supremum norm.

The following result characterizes the well-posedness of the Cauchy problem (NCP) by certain properties of the domain $D(\mathcal{G})$ (cf. [5, Prop. 2.9], [6, Theorem 2.9]).

Proposition 5. *Let E be a Banach space and $(B(t), D(B(t)))_{t\in\mathbb{R}}$ a family of linear operators on E. Then the following assertions are equivalent.*

(i) *The nonautonomous Cauchy problem (NCP) is well-posed.*

(ii) *There exist a unique semigroup $(\mathcal{U}(t))_{t\geq 0}$ defined by*

$$(\mathcal{U}(t)f)(s) := U(s, s-t)f(s-t)$$

for $f \in \mathcal{E}, s \in \mathbb{R}$ and $t \geq 0$ with generator $(\mathcal{G}, D(\mathcal{G}))$ and an invariant core $\mathcal{D} \subseteq \{f \in C^1(\mathbb{R}, E) : f, f' \in C_0(\mathbb{R}, E)\} \cap D(\mathcal{G})$ of $\mathcal{G}$ such that

$$\mathcal{G}f = B(\cdot)f - f'$$

for $f \in \mathcal{D}$.

Now, we can state our main result.

Theorem 6. *There exist strongly continuous semigroups $(\mathcal{T}(t))_{t\geq 0}$ and $(\mathcal{S}(t))_{t\geq 0}$ on $\mathcal{E}$ with generators $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$, respectively, having the following properties.*

(i) *There exists a core $\mathcal{D}$ for $\mathcal{A}+\mathcal{B}$ such that $\overline{(\mathcal{A}+\mathcal{B}, \mathcal{D})}$ generates a strongly continuous semigroup $(\mathcal{U}(t))_{t\geq 0}$ on $\mathcal{E}$.*

(ii) *The semigroup $(\mathcal{U}(t))_{t\geq 0}$ is not given by the Trotter product formula, i.e., there exists a function $f \in \mathcal{E}$ and $t \geq 0$ such that*

$$\mathcal{U}(t)f \neq \lim_{n\to\infty} \left(\mathcal{S}(\tfrac{t}{n})\mathcal{T}(\tfrac{t}{n})\right)^n f \ .$$

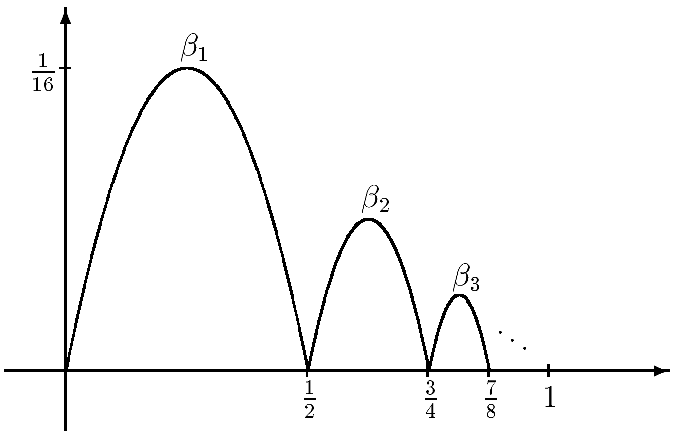
Proof. Let $(\mathcal{T}(t))_{t\geq 0}$ be the right translation semigroup on $\mathcal{E}$ with generator $(\mathcal{A}, D(\mathcal{A}))$ given by

$$\begin{aligned} \mathcal{A}f &= -f' \\ \text{with domain } D(\mathcal{A}) &= \{f \in C^1(\mathbb{R}, E) : f, f' \in \mathcal{E}\} \ . \end{aligned}$$

Let $k \in \mathbb{N}$. Set $t_k := 1 - 2^{-k+1}$ and define functions $\beta_k \in C_c(\mathbb{R})$ by

$$\beta_k(s) := \begin{cases} -(s-t_k)(s-t_{k+1}) & \text{for } s \in [t_k, t_{k+1}], \\ 0 & \text{for } s < t_k \text{ or } s > t_{k+1}, \end{cases}$$

and as shown below.



For each $k \in \mathbb{N}$, we set

$$a_k := \frac{1}{2} \int_{\mathbb{R}} \beta_k(s) ds = \frac{2^{-3k}}{12} \ .$$

Now, corresponding to the sequence $(a_k)_{k\in\mathbb{N}}$ we can define the spaces E and F and the operators $(B_k, D(B_k))$ as above.
In the next step, we introduce multiplication operators by

$$(B(s), D(B(s))) := \begin{cases} (\beta_k(s)B_k, & D(B_k)) \quad \text{for } s \in]t_k, t_{k+1}], \ k \in \mathbb{N}, \\ (0, & E) \qquad\quad \text{for } s \leq 0 \text{ or } s \geq 1. \end{cases}$$

By Lemma 3, the operators $B(s)$ generate commuting, uniformly bounded, strongly continuous semigroups $(e^{\tau B(s)})_{\tau\geq 0}$ and $D(B(s))$ contains the dense subspace F for all $s \in \mathbb{R}$. Moreover, we obtain

$$\|B(t)f - B(s)f\| \leq 2^{-k}|t-s| \cdot \|f\|_F \quad \text{and} \quad \|B(s)f\| \leq |\beta_k(s)| \cdot \|f\|_F$$

for $t_k < s, t < t_{k+1}$ and $f \in F$. Hence, $B(\cdot)$ is a generator of an vector-valued multiplication semigroup $(\mathcal{S}(t))_{t\geq 0}$ on $\mathcal{E}$ (cf. [2, Chap. III. 4]). For $(B(\cdot), D(B(\cdot)))$ we write $(\mathcal{B}, D(\mathcal{B}))$.

To prove (ii), we calculate the Trotter products

$$\begin{aligned}\left[\left(\mathcal{S}(\tfrac{t}{n})\mathcal{T}(\tfrac{t}{n})\right)^n f\right](s+t) = \exp\left(\tfrac{t}{n}B(s+t)\right)\exp\left(\tfrac{t}{n}B(s+t-\tfrac{t}{n})\right) \\ \cdots \exp\left(\tfrac{t}{n}B(s+\tfrac{t}{n})\right) f(s)\end{aligned}$$

for $t \geq 0, \ s \in \mathbb{R}$ and $f \in \mathcal{E}$.
For $n = 2^l, l \in \mathbb{N}, s = 0$ and $t = 1$ we obtain by the commutativity of the semigroups

$$\begin{aligned}\left[\left(\mathcal{S}(\tfrac{1}{2^l})\mathcal{T}(\tfrac{1}{2^l})\right)^{2^l} f\right](1) &= \exp\left(\frac{1}{2^l}B(1)\right)\prod_{k=1}^{l}\exp\left(\frac{1}{2^l}\sum_{r=2^l(1-2^{-k+1})+1}^{2^l(1-2^{-k})}\beta_k(\tfrac{r}{2^l})B_k\right) f(0) \\ &=: \prod_{k=1}^{l}\exp\left(b_k^{(l)}B_k\right) f(0) \ .\end{aligned}$$

Observe that $b_l^{(l)} = 0$ and $b_k^{(l)} > 0$ for $k = 1, ..., l-1$ and $l \geq 2$. Let us now calculate $b_k^{(l)}, 1 \leq k < l$. We shift the functions β_k to $\tilde{\beta}_k$ defined by

$$\tilde{\beta}_k(s) := -s(s - 2^{-k}) \ .$$

Hence, for $1 \le k < l$, we have

$$b_k^{(l)} = \frac{1}{2^l}\sum_{r=1}^{2^{-k+l}} -\frac{r}{2^l}\left(\frac{r}{2^l}-2^{-k}\right) = \frac{1}{2^{2l}}\left[-\frac{1}{2^l}\left[\frac{2^{-k+l}(2^{-k+l}+1)(2^{-k+1+l}+1)}{6}\right]\right.$$
$$\left. + 2^{-k}\left[\frac{2^{-k+l}(2^{-k+l}+1)}{2}\right]\right]$$
$$= \frac{2^{-k}(2^{-k+l}+1)\left[-(2^{-k+1+l}+1)+3\cdot 2^{-k+l}\right]}{6\cdot 2^{2l}}$$
$$= \frac{2^{-k}(2^{-k+l}+1)\left(2^{-k+l}-1\right)}{6\cdot 2^{2l}}$$
$$= \frac{2^{-k}(2^{-2k+2l}-1)}{6\cdot 2^{2l}}$$
$$= \frac{2^{-3k}}{6} - \frac{2^{-k}}{6\cdot 2^{2l}}$$
$$< \frac{2^{-3k}}{6} = 2a_k \ .$$

Since all $b_k^{(l)} < 2a_k$ for $k = 1, ..., l-1$, Lemma 3 implies the instability of the Lie-Trotter products, indeed, we can choose a function $f_l \in \mathcal{E}$ with $\| f_l \| = 1$ such that

$$\left\| \left[\left(\mathcal{S}(\tfrac{1}{2^l})\mathcal{T}(\tfrac{1}{2^l})\right)^{2^l} f_l\right](1)\right\| \ge 2^{l-1}-1 \quad \text{for } l \in \mathbb{N}\ .$$

So the principle of uniform boundedness implies that the Lie-Trotter product formula does not converge.

On the other hand, one can show that the nonautonomous Cauchy problem

$$\begin{cases} \frac{d}{dt}u(t) &=& B(t)u(t), \quad t \ge s,\ t,s \in \mathbb{R}, \\ u(s) &=& g \in E, \end{cases}$$

on E is wellposed on F in the sense of Definition 4. To that purpose, we define on E the bounded operators

$$U(t,s) := \exp\left(\int_s^t B(\tau)d\tau\right)$$
$$= \exp\left(B_m \int_s^{t_{m+1}} \beta_m(\tau)d\tau\right)\cdot \prod_{k=m+1}^{n-1} \exp\left(B_k\int_{t_k}^{t_{k+1}}\beta_k(\tau)d\tau\right)\cdot \exp\left(B_n\int_{t_n}^{t}\beta_n(\tau)d\tau\right),$$

for $0 \le s \le t < 1$ with $s \in [t_m, t_{m+1}[$, $t \in [t_n, t_{n+1}[$ for some $m, n \in \mathbb{N}$. It is obvious that $(U(t,s))_{\triangle_{[0,\delta]}}$ with $\delta < 1$ is an evolution family satisfying the conditions (3). In addition, we have

$$\| U(t,s) \| \le 4 \quad \text{for all } 0 \le s \le t < 1\ , \tag{4}$$

and

$$\lim_{(t,s)\to(1,s_0)} U(t,s)f =: U(1,s_0)f$$

exists for all functions $f \in E_0$ and $0 \le s_0 \le 1$. Since the subspace E_0 is dense in E, we can extend the operator $U(1,s_0)$ to a bounded operator on E denoted by the same symbol. It is easy to see that $(U(t,s))_{\triangle_{[0,1]}}$ is an evolution family (see Definition 4). Finally, we extend $(U(t,s))_{\triangle_{[0,1]}}$ to an evolution family with parameter set $\triangle_{\mathbb{R}}$ by setting

$$U(t,s) := \begin{cases} Id, & t \ge s \ge 1 \text{ or } s \le t \le 0, \\ U(\min\{t,1\}, \max\{s,0\}), & \text{otherwise} . \end{cases}$$

Now, for $f \in F$ (cf. Lemma 3(c)) we have

$$\frac{\partial}{\partial t} U(t,s)f = B(t)U(t,s)f = U(t,s)B(t)f, \quad 0 \le s \le t < 1 .$$

Using $\|B(t)f\| \longrightarrow 0$ for $t \nearrow 1$ and the estimate (4), we obtain

$$U(t,s)B(t)f \longrightarrow 0 \quad \text{for } t \nearrow 1 .$$

Moreover, we have

$$\begin{aligned} \left\| \frac{U(1,s)f - U(1-h,s)f}{h} \right\| &= \| \lim_{\epsilon \searrow 0} \frac{1}{h} \left(U(1-\epsilon,s)f - U(1-h,s)f \right) \| \\ &= \| \frac{1}{h} \int_{1-h}^{1} B(\tau)U(\tau,s)f d\tau \| \\ &\le \sup_{1-h\le\tau\le 1} \|B(\tau)U(\tau,s)f\| \xrightarrow{h\searrow 0} 0 \end{aligned}$$

for $f \in F$ and $0 \le s < 1-h$. Therefore, the nonautonomous Cauchy problem is wellposed on F.

Hence, by Proposition 5, there exists a strongly continuous semigroup $(\mathcal{U}(t))_{t\ge 0}$ with generator $(\mathcal{G}, D(\mathcal{G}))$ and an invariant core $\mathcal{D}$ such that

$$\mathcal{G}f = (\mathcal{A}+\mathcal{B})f \quad \text{for } f \in \mathcal{D} .$$

Thus $\overline{(\mathcal{A}+\mathcal{B}, \mathcal{D})}$ is the generator of $(\mathcal{U}(t))_{t\ge 0}$ and the property (i) of the theorem holds. $\square$

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