ON THE STRUCTURE OF THE CRITICAL SPECTRUM OF STRONGLY CONTINUOUS SEMIGROUPS

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ABSTRACT. We show certain rotational symmetries of the recently introduced critical spectrum of strongly continuous semigroups. Moreover, we study a nonautonomous differential equation with infinite delay and show how the critical spectrum can be used to characterize uniform exponential stability of the solutions.

The failure of the spectral mapping theorem

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)}, \qquad t \geq 0, \tag{SMT}$$

for a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator A on a Banach space X is the main reason for many difficulties arising in the investigation of the asymptotic behavior of these semigroups (see [4], Chapter IV and V). To overcome some of these difficulties, the *essential spectrum* $\sigma_{ess}(T(t))$ has been used by many authors and for many concrete equations. More recently, the *critical spectrum* $\sigma_{crit}(T(t))$, smaller than $\sigma_{ess}(T(t))$, has been introduced in [8] and then applied in [2] to perturbed semigroups.

In this paper, we prove symmetry properties of the critical spectrum thereby explaining certain phenomena arising in the counterexamples to (SMT) discussed by Wrobel [13]. In the final section, we study a nonautonomous differential equation with infinite delay and show, by replacing the essential spectrum by the critical spectrum, how one can obtain new criteria for uniform exponential stability.

1. The Critical Spectrum

We briefly recall the main definitions and results from [8]. To the Banach space X we associate the sequence space

$$\tilde{X} := \ell^\infty(X) = \big\{(x_n)_{n\in\mathbb{N}} \subset X : \sup_{n\in\mathbb{N}} \|x_n\| < \infty\big\}$$

endowed with the sup-norm. The strongly continuous semigroup $\mathcal{T} = (T(t))_{t\geq 0}$ on X will be extended to a (not necessarily strongly continuous) semigroup $\tilde{\mathcal{T}} = (\tilde{T}(t))_{t\geq 0}$ on $\tilde{X}$ by

$$\tilde{T}(t)(x_n)_{n\in\mathbb{N}} := (T(t)x_n)_{n\in\mathbb{N}}, \qquad (x_n)_{n\in\mathbb{N}} \in \tilde{X}.$$

This semigroup has the space of strong continuity

$$\tilde{X}_{\mathcal{T}} := \big\{(x_n)_{n\in\mathbb{N}} \in \tilde{X} : \lim_{t\downarrow 0} \sup_{n\in\mathbb{N}} \|T(t)x_n - x_n\| = 0\big\},$$

which is closed and $(\tilde{T}(t))_{t\geq 0}$-invariant. Therefore, on the quotient space

$$\hat{X} := \tilde{X}/\tilde{X}_{\mathcal{T}}$$

the quotient operators

$$\hat{T}(t)(\tilde{x} + \tilde{X}_{\mathcal{T}}) := \tilde{T}(t)\tilde{x} + \tilde{X}_{\mathcal{T}}, \qquad \tilde{x} + \tilde{X}_{\mathcal{T}} \in \hat{X},$$

are well defined and yield a semigroup $\hat{\mathcal{T}} = (\hat{T}(t))_{t\geq 0}$ of bounded operators.

1.1. Definition. *The critical spectrum of the semigroup $(T(t))_{t\geq 0}$ is defined as*

$$\sigma_{crit}(T(t)) := \sigma(\hat{T}(t)), \qquad t \geq 0,$$

while the critical growth bound is

$$\omega_{crit}(\mathcal{T}) := \omega_0(\hat{\mathcal{T}}) := \inf\Big\{w \in \mathbb{R} : \begin{array}{l} \text{there exists } M_w \text{ such that} \\ \|\hat{T}(t)\| \leq M_w e^{wt} \text{ for all } t \geq 0 \end{array}\Big\}.$$

The critical spectrum is smaller than the essential spectrum and, e.g., trivial for eventually norm continuous semigroups. However, it is large enough to ensure the following modification of (SMT) (see [8], Theorem 3.2).

1.2. Theorem. *For a strongly continuous semigroup $(T(t))_{t\geq 0}$ with generator A one has*

$$\sigma(T(t)) \setminus \{0\} = e^{t\sigma(A)} \cup \sigma_{crit}(T(t)) \setminus \{0\}, \qquad t \geq 0,$$

and therefore the spectral bound $s(A)$ and the critical growth bound $\omega_{crit}(\mathcal{T})$ determine the growth bound $\omega_0(\mathcal{T})$ by

$$\omega_0(\mathcal{T}) = \max\{s(A), \omega_{crit}(\mathcal{T})\}.$$

In the following proposition, we give a condition on the approximate point spectrum $A\sigma(A)$ or on the resolvent $R(\lambda, A)$ of the generator A in order to produce points in the critical spectrum. To that purpose, we need the following topological property of the exponential function (see [13], Lemma 4.3). For abbreviation, we put $\Gamma := \{\mu \in \mathbb{C} : |\mu| = 1\}$.

1.3. Lemma. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers satisfying $\lim_{n\to\infty} \operatorname{Re}\lambda_n = w$ and $\lim_{n\to\infty} |\operatorname{Im}\lambda_n| = \infty$. Then, for almost all $t \geq 0$, each point $\mu \in e^{wt}\Gamma$ is an accumulation point of the sequence $(e^{\lambda_n t})_{n\in\mathbb{N}}$.*

1.4. Proposition. *Let $t \geq 0$, $\mu \in \mathbb{C}$ and $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers satisfying $\lim_{n\to\infty} |\operatorname{Im}\lambda_n| = \infty$ and $\lim_{n\to\infty} e^{\lambda_n t} = \mu$. For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ we assume that either*

$$(*) \qquad (\lambda_n)_{n\in\mathbb{N}} \subset A\sigma(A)$$

or

$$(**) \qquad (\lambda_n)_{n\in\mathbb{N}} \subset \rho(A) \qquad \text{and} \qquad \lim_{n\to\infty} \|R(\lambda_n, A)\| = \infty.$$

Then we obtain

$$\mu \in \sigma_{crit}(T(t)).$$

Proof. Under both assumptions, there exists a sequence $(x_n)_{n\in\mathbb{N}} \subset D(A)$ satisfying

$$\|x_n\| = 1$$

for all $n \in \mathbb{N}$ and

$$\lim_{n\to\infty} \|\lambda_n x_n - Ax_n\| = 0.$$

Then we have

$$\begin{aligned}
\sup_{n\in\mathbb{N}} \|T(h)x_n - x_n\| &\geq \overline{\lim}_{n\to\infty} \|T(h)x_n - x_n\| \\
&\geq \overline{\lim}_{n\to\infty} (\|e^{\lambda_n h}x_n - x_n\| - \|e^{\lambda_n h}x_n - T(h)x_n\|) \\
&= \overline{\lim}_{n\to\infty} \Big(|e^{\lambda_n h} - 1| - \Big\| \int_0^h e^{\lambda_n(h-s)} T(s)(\lambda_n x_n - Ax_n)\, ds \Big\|\Big) \\
&= \overline{\lim}_{n\to\infty} |e^{\lambda_n h} - 1|
\end{aligned}$$

for all $h \geq 0$. We now put $\lim_{n\to\infty} \operatorname{Re}\lambda_n =: w$. Using Lemma 1.3, we obtain

$$\sup_{n\in\mathbb{N}} \|T(h)x_n - x_n\| \geq e^{wh} + 1$$

for almost all $h \geq 0$. Therefore, the semigroup $(T(t))_{t\geq 0}$ is not uniformly continuous on $(x_n)_{n\in\mathbb{N}}$, i.e.,

$$(x_n)_{n\in\mathbb{N}} \notin \tilde{X}_{\mathcal{T}}.$$

On the other hand,

$$\begin{aligned}
\lim_{n\to\infty} \|\mu x_n - T(t)x_n\| &= \lim_{n\to\infty} \|e^{\lambda_n t}x_n - T(t)x_n\| \\
&= \lim_{n\to\infty} \Big\| \int_0^t e^{\lambda_n(t-s)} T(s)(\lambda_n x_n - Ax_n)\, ds \Big\| \\
&= 0.
\end{aligned}$$

This implies

$$(\mu x_n - T(t)x_n)_{n\in\mathbb{N}} \in c_0(X) \subset \tilde{X}_{\mathcal{T}}.$$

From this we conclude that μ is an eigenvalue of $\hat{T}(t)$, hence $\mu \in \sigma_{crit}(T(t))$. $\square$

1.5. Example. Let $X := C_0(\Omega)$ or $X := L^p(\Omega, \mu)$, where Ω is locally compact and (Ω, μ) is a σ-finite measure space, respectively. Further, let $(T_q(t))_{t\geq 0}$ be the strongly continuous semigroup generated by the multiplication operator $M_q f := qf$ for some function $q : \Omega \to \mathbb{C}$ satisfying

$$\sup_{s\in\Omega} \operatorname{Re} q(s) < \infty.$$

Then one has

$$\mu \in \sigma_{crit}(T_q(t)) \setminus \{0\}$$

if and only if there exists a sequence

$$(\lambda_n)_{n\in\mathbb{N}} \subset A\sigma(M_q)$$

of complex numbers such that $\lim_{n\to\infty} |\operatorname{Im} \lambda_n| = \infty$ and $\lim_{n\to\infty} e^{\lambda_n t} = \mu$. For the proof we refer to [8], Example 2.8 (ii).

2. Rotational Symmetry of the Critical Spectrum

Herbst [5] and Wrobel [13] noted that if (SMT) fails in the known examples, it fails dramatically in the sense that the sets $\sigma(T(t)) \setminus e^{t\sigma(A)}$ become quite large. In this section, we show how this phenomenon can be explained by the critical spectrum.

2.1. Theorem. *Let $(\lambda_n)_{n\in\mathbb{N}}$ be a sequence of complex numbers satisfying $\lim_{n\to\infty} \operatorname{Re} \lambda_n = w$ and $\lim_{n\to\infty} |\operatorname{Im} \lambda_n| = \infty$. For the generator A of a strongly continuous semigroup $(T(t))_{t\geq 0}$ we assume that either*

$$(*) \qquad (\lambda_n)_{n\in\mathbb{N}} \subset A\sigma(A)$$

or

$$(**) \qquad (\lambda_n)_{n\in\mathbb{N}} \subset \rho(A) \quad \text{and} \quad \lim_{n\to\infty} \|R(\lambda_n, A)\| = \infty.$$

Then, for almost all $t \geq 0$, we have

$$e^{wt}\Gamma \subset \sigma_{crit}(T(t)).$$

Proof. By Lemma 1.3, each point $\mu \in e^{wt}\Gamma$ is an accumulation point of the sequence $(e^{\lambda_n t})_{n\in\mathbb{N}}$ for almost all $t \geq 0$. Proposition 1.4 implies that each point $\mu \in e^{wt}\Gamma$ belongs to $\sigma_{crit}(T(t))$ for almost all $t \geq 0$. $\square$

Using the following abscissa of uniform boundedness of the resolvent

$$s_0^\infty(A) := \inf \left\{ w \in \mathbb{R} : \begin{array}{l} \text{there exist } s_w \text{ and } C_w \text{ such that } r + is \in \rho(A) \\ \text{and } \|R(r+is, A)\| \leq C_w \text{ for } r \geq w \text{ and } |s| \geq s_w \end{array} \right\}$$

introduced by Blake (see [1]), we obtain that the corresponding circles are contained in $\sigma_{crit}(T(t))$ for almost all $t \geq 0$. (For other abscissas and their significance for the asymptotic behavior of semigroups see [12].)

2.2. Corollary. *We have*

$$e^{ts_0^\infty(A)}\Gamma \subset \sigma_{crit}(T(t))$$

for almost all $t \geq 0$.

In particular, we obtain the estimate

$$s_0^\infty(A) \leq \omega_{crit}(\mathcal{T}).$$

In case X is a Hilbert space, it has been shown by Blake that

$$s_0^\infty(A) = \omega_{crit}(\mathcal{T})$$

(see [1], Theorem 4.4).

2.3. Corollary. *Let X be a Hilbert space. Then*

$$e^{t\omega_{crit}(\mathcal{T})}\Gamma \subset \sigma_{crit}(T(t))$$

for almost all $t \geq 0$.

For multiplication semigroups, it follows from the description of the critical spectrum in Example 1.5 that the condition in Theorem 2.1 always holds for points in the critical spectrum.

2.4. Corollary. *Let $(T_q(t))_{t\geq 0}$ be a multiplication semigroup as in Example 1.5. Then $\sigma_{crit}(T_q(t))$ consists of full circles for almost all $t \geq 0$, i.e.*

$$\sigma_{crit}(T_q(t)) \cdot \Gamma = \sigma_{crit}(T_q(t))$$

for almost all $t \geq 0$.

Finally, we consider positive semigroups on Banach lattices (see [7]) and obtain a certain dichotomic behavior.

2.5. Corollary. *Let $(T(t))_{t\geq 0}$ be a positive semigroup with generator A on a Banach lattice X and assume that the peripheral spectrum*

$$\sigma_+(A) := \{\lambda \in \sigma(A) : \operatorname{Re}\lambda = s(A)\}$$

is cyclic. If $\omega_{crit}(\mathcal{T}) < s(A)$, then we have $\sigma_+(A) = \{s(A)\}$, i.e.,

$$\operatorname{Re}\lambda < s(A)$$

for all $\lambda \in \sigma(A) \setminus \{s(A)\}$.

Proof. If $s(A) + ir \in \sigma(A)$ for some $r \in \mathbb{R}$, then the cyclicity of $\sigma_+(A)$ implies $s(A) + inr \in \sigma(A)$ for all $n \in \mathbb{N}$. By the assumption $\omega_{crit}(\mathcal{T}) < s(A)$ and by condition $(*)$ in Theorem 2.1, we conclude $r = 0$. □

Semigroups satisfying the condition $\omega_{crit}(\mathcal{T}) < s(A)$ are called *asymptotically norm continuous*. This class of semigroups has been studied intensively in the papers by Martínez and Mazón [6], Blake [1] and Thieme [11].

The opposite situation is considered in the following result proved by W. Arendt and O. El Mennaoui (oral communication).

2.6. Theorem (Arendt-El Mennaoui). *Let $(T(t))_{t\geq 0}$ be a positive semigroup with generator A on a Banach lattice X. If $s(A) < \omega_{crit}(\mathcal{T})$, then*

$$[e^{ts(A)}, e^{t\omega_{crit}(\mathcal{T})}] \subset \sigma_{crit}(T(t))$$

for all $t \geq 0$.

Proof. We first assume that $s(A) < 0 \leq \omega_{crit}(\mathcal{T})$. By positivity this implies exponential stability, i.e., there exist constants $M, \varepsilon > 0$ such that

$$\|T(t)x\| \leq Me^{-\varepsilon t}\|Ax\|$$

for all $x \in D(A)$ (see [4], Proposition VI.1.14 or [9], Theorem 1.4.1). Assume now that

$$1 \notin \sigma_{crit}(T(t))$$

for some $t \geq 0$, which, by Theorem 1.2, implies that

$$1 \notin \sigma(T(t)).$$

Then

$$R(1, T(t))x = \sum_{n=0}^{\infty} T(nt)x \geq 0$$

for all $0 \leq x \in D(A)$. The density of $D(A)$ then implies

$$R(1, T(t)) \geq 0.$$

Therefore, we use [10], Exercise V.5.(e) to conclude that

$$r(T(t)) < 1,$$

hence

$$\omega_0(\mathcal{T}) < 0.$$

By Theorem 1.2, we must have

$$\omega_{crit}(\mathcal{T}) < 0,$$

which is a contradiction. Thus, we conclude that

$$1 \in \sigma_{crit}(T(t))$$

for all $t \geq 0$. By rescaling, we obtain

$$[e^{ts(A)}, e^{t\omega_{crit}(\mathcal{T})}] \subset \sigma_{crit}(T(t))$$

for all $t \geq 0$. □

A comparison of these results with the known examples of semigroups having $s(A) < \omega_0(\mathcal{T})$ leads us to the following open problem.

2.7. Problem. *Do there exist strongly continuous semigroups $(T(t))_{t\geq 0}$ such that the critical spectrum $\sigma_{crit}(T(t))$ does not contain full circles for almost all $t \geq 0$?*

3. An Application

Let X be a Banach space. On the Banach space $C_0(\mathbb{R}_+, X)$, we consider the operator

$$Gf := -f'(\cdot) + A(\cdot)f(\cdot)$$

with domain

$$D(G) := \left\{ f \in C_0(\mathbb{R}_+, X) : \begin{array}{l} f \text{ is differentiable, the function } -f'(\cdot) + A(\cdot)f(\cdot) \\ \text{belongs to } C_0(\mathbb{R}_+, X), \text{ and } f'(0) = -Bf(0) - Lf \end{array} \right\}.$$

Here, $(A(s))_{s\geq 0}$ is a bounded strongly continuous family of bounded operators on X, B is the generator of a strongly continuous semigroup on the same Banach space X, and L is a bounded operator from $C_0(\mathbb{R}_+, X)$ into X. The operators $(A(s))_{s\geq 0}$ generate an evolution family $\mathcal{U} = (U(s,r))_{s\geq r\geq 0}$ on X solving the evolution equation

$$\frac{d}{dt}x(t) = A(t)x(t), \qquad t \geq 0.$$

We assume the asymptotic behavior of $(U(s,r))_{s\geq r\geq 0}$ and in particular the growth bound

$$\omega_0(\mathcal{U}) := \inf \left\{ w \in \mathbb{R} : \begin{array}{l} \text{there exists } M_w \text{ such that} \\ \|U(s,r)\| \leq M_w e^{w(s-r)} \text{ for } s \geq r \end{array} \right\}$$

to be known. We are now interested in solutions satsifying the boundary condition given by B and L at $s = 0$. Note that, without this boundary condition, the operator G would be the generator of the evolution semigroup corresponding to $(U(s,r))_{s\geq r\geq 0}$ (see [3] or [4], Section VI.9). We start by verifying that G is the generator of a strongly continuous semigroup.

3.1. Lemma. *The operator G generates a strongly continuous semigroup $(T(t))_{t\geq 0}$ on $C_0(\mathbb{R}_+, X)$.*

Proof. In case $A(\cdot) = 0$, this can be found, e.g., in [4], Theorem VI.6.1. In the general case, the assertion follows from the bounded perturbation theorem for semigroup generators (see [4], Theorem III.1.3). ☐

Next, we prove that the semigroup $(T(t))_{t\geq 0}$ satisfies a generalization of the translation property (see [7], B-IV).

3.2. Lemma. *Let $(U(s,r))_{s\geq r\geq 0}$ be the evolution family generated by $(A(s))_{s\geq 0}$. Then we have*

$$(T(t)f)(s) = \begin{cases} U(s,0)(T(t-s)f)(0) & \textit{for } s \leq t, \\ U(s,s-t)f(s-t) & \textit{for } t \leq s \end{cases}$$

for each $f \in C_0(\mathbb{R}_+, X)$.

Proof. It suffices to take $f \in D(G)$. By definition of $(T(t))_{t\geq 0}$, we have

$$\frac{d}{dr}(T(t-s+r)f)(r) = A(r)(T(t-s+r)f)(r).$$

This implies

$$(T(t-s+r)f)(r) = \begin{cases} U(r,0)(T(t-s)f)(0) & \text{for } s \leq t, \\ U(r,s-t)f(s-t) & \text{for } t \leq s, \end{cases}$$

hence

$$(T(t)f)(s) = \begin{cases} U(s,0)(T(t-s)f)(0) & \text{for } s \leq t, \\ U(s,s-t)f(s-t) & \text{for } t \leq s. \end{cases}$$

☐

3.3. Lemma. *Let $(V(t))_{t\geq 0}$ be the strongly continuous semigroup generated by $A(0)+B$. Then we have*

$$(T(t)f)(0) = V(t)f(0) + \int_0^t V(t-\sigma)LT(\sigma)f\,d\sigma$$

for each $f \in C_0(\mathbb{R}_+, X)$ and $t \geq 0$.

Proof. As above, it suffices to prove the assertion for $f \in D(G)$. In this case, we obtain

$$\frac{d}{dt}(T(t)f)(0) = (A(0)+B)(T(t)f)(0) + LT(t)f.$$

Hence, $u(t) := (T(t)f)(0)$ is the solution of an inhomogenous Cauchy problem for the operator $A(0) + B$. Therefore, we obtain the solution as

$$(T(t)f)(0) = V(t)f(0) + \int_0^t V(t-\sigma)LT(\sigma)f \, d\sigma$$

for $t \geq 0$. □

Putting these facts together, we obtain the following integral equation for the semigroup $(T(t))_{t\geq 0}$.

3.4. Lemma. *The semigroup $(T(t))_{t\geq 0}$ satisfies the relation*

$$(T(t)f)(s) = \begin{cases} U(s,0)(V(t-s)f(0) + \int_0^{t-s} V(t-s-\sigma)LT(\sigma)f \, d\sigma) & \text{for } s \leq t, \\ U(s,s-t)f(s-t) & \text{for } t \leq s \end{cases}$$

for each $f \in C_0(\mathbb{R}_+, X)$.

We are now well prepared for the main result of this section.

3.5. Theorem. *If the semigroup $(V(t))_{t\geq 0}$ generated by $A(0)+B$ is immediately norm continuous, then*

$$\omega_{crit}(\mathcal{T}) = \omega_0(\mathcal{U}).$$

Proof. For each $f \in C_0(\mathbb{R}_+, X)$, we have

$$(T(t)f)(s) = \begin{cases} U(s,0)(V(t-s)f(0) + \int_0^{t-s} V(t-s-\sigma)LT(\sigma)f \, d\sigma) & \text{for } s \leq t, \\ U(s,s-t)f(s-t) & \text{for } t \leq s. \end{cases}$$

We now put

$$(R(t)f)(s) := \begin{cases} U(s,0)(V(t-s)f(0) + \int_0^{t-s} V(t-s-\sigma)LT(\sigma)f \, d\sigma) & \text{for } s \leq t-1, \\ (t-s)U(s,0)V(t-s)f(0) & \text{for } t-1 \leq s \leq t, \\ 0 & \text{for } t \leq s. \end{cases}$$

Since $(V(t))_{t\geq 0}$ is immediately norm continuous, $(R(t))_{t\geq 0}$ is a norm continuous family of bounded operators on $C_0(\mathbb{R}_+, X)$. Moreover, for each $w > \omega_0(\mathcal{U})$ we can find a constant M such that

$$\|T(t) - R(t)\| \leq Me^{wt}.$$

Therefore, we obtain

$$\omega_{crit}(\mathcal{T}) \leq \omega_0(\mathcal{U}).$$

On the other hand, since

$$(T(t)f)(s) = U(s,s-t)f(s-t)$$

for $t \leq s$, we must have

$$\omega_{crit}(\mathcal{T}) = \omega_0(\mathcal{U}).$$

$\square$

Using Theorem 1.2, we can draw the following conclusion.

3.6. Corollary. *If the semigroup $(V(t))_{t\geq 0}$ generated by $A(0)+B$ is immediately norm continuous, then*

$$\omega_0(\mathcal{T}) = \max\{s(G), \omega_0(\mathcal{U})\}.$$

In particular, if $(U(s,r))_{s\geq r\geq 0}$ is uniformly exponentially stable and $s(G) < 0$, then $(T(t))_{t\geq 0}$ is uniformly exponentially stable.

We remark that $\sigma(G)$ and therefore $s(G)$ can be described by an operator-valued *characteristic equation* analogous to the scalar case in [4], Example V.3.9. This idea will be pursued in a subsequent paper.

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