# SPECTRUM OF THE INFINITE-DIMENSIONAL LAPLACIAN

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Abstract. We prove that the spectrum of the infinite-dimensional Laplacian is the left half plane $\{\lambda \in \mathbb{C} : Re\ \lambda \leq 0\}$. As a consequence, we obtain a simple proof of the norm discontinuity of the generated semigroup.

Let $H$ be a separable, infinite dimensional, real Hilbert space and let $(e_k)$ be an orthonormal basis. We denote by $BUC(H)$ the space of all bounded, uniformly continuous functions from $H$ to $\mathbb{C}$. We shall regard $BUC(\mathbb{R}^n)$ as a subspace of $BUC(H)$ via the isometric embedding

$$J_n : BUC(\mathbb{R}^n) \to BUC(H), \quad (J_n\varphi)(x) := \varphi(x_1, \ldots, x_n),$$

for $\varphi \in BUC(\mathbb{R}^n)$, $x \in H$, and $x_k := (x, e_k)$. By $D_k$ we denote the partial derivative in the direction $e_k$. Let $\lambda_k > 0$ with $\sum_{k=1}^{\infty} \lambda_k < \infty$ be given. The infinite dimensional heat equation

$$\begin{aligned} \frac{\partial u}{\partial t}(t,x) &= \sum_{k=1}^{\infty} \lambda_k\, D_k^2\, u(t,x), \qquad t > 0,\ x \in H, \\ u(0,x) &= \varphi(x), \qquad x \in H, \end{aligned}$$

on $BUC(H)$ is solved by the strongly continuous contraction semigroup

$$P_t\varphi = \lim_{n\to\infty} P_t^n \varphi, \quad \varphi \in BUC(H).$$

The above limit exists in $BUC(H)$ uniformly in $t$ on bounded subsets of $[0,\infty)$ and, for $\varphi \in BUC(H)$, $x \in H$ and $t > 0$, we set

$$P_t^n\varphi(x) := (4\pi t)^{-\frac{n}{2}} (\lambda_1 \cdots \lambda_n)^{-\frac{1}{2}} \int_{\mathbb{R}^n} e^{-\sum_{k=1}^{n} \frac{y_k^2}{4t\lambda_k}}\, \varphi\Big(x - \sum_{k=1}^{n} y_k e_k\Big) dy. \tag{1}$$

We denote by $A$ the generator of $(P_t)$. Let $BUC^{1,1}(H)$ be the space of Fréchet differentiable $\varphi \in BUC(H)$ with a bounded Lipschitz continuous derivative. It is known that

$$D_0(H) := \{\varphi \in BUC^{1,1}(H) : D_k D_l \varphi \in BUC(H),\ \sup_{k,l\in\mathbb{N}} \|D_k D_l \varphi\|_\infty < \infty\}$$

is a $P_t$–invariant core of $A$ and

$$A\varphi = \sum_{k=1}^{\infty} \lambda_k\, D_k^2\, \varphi \qquad \text{for } \varphi \in D_0(H).$$

The reader is referred to [2] and [1] for the proofs of the above results.

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**Theorem 1.** *The spectrum of $A$ is the left half plane $\{\lambda \in \mathbb{C} : Re\ \lambda \leq 0\}$ and the spectrum of $P_t$ is the unit disc $\{\lambda \in \mathbb{C} : |\lambda| \leq 1\}$. Moreover, every $\lambda \in \sigma(A)$ is an approximate eigenvalue.*

*Proof.* Note that the restriction of $P_t$ to $BUC(\mathbb{R}^n)$ coincides with the semigroup generated by $A_n := \sum_{k=1}^n \lambda_k D_k^2$. In particular, $A_n$ is the part of $A$ in $BUC(\mathbb{R}^n)$ and, hence, $R(\lambda, A_n) = R(\lambda, A)_{|BUC(\mathbb{R}^n)}$ for $\lambda \in \rho(A) \cap \rho(A_n)$. Therefore, for these values of $\lambda$, the sequence $\|R(\lambda, A_n)\|_\infty$ is bounded. Let $V : BUC(\mathbb{R}^n) \to BUC(\mathbb{R}^n)$ be the isometry defined by

$$(V\varphi)(x) := \varphi(\sqrt{\lambda_1}x_1, \ldots, \sqrt{\lambda_n}x_n), \qquad \varphi \in BUC(\mathbb{R}^n),\ x \in \mathbb{R}^n.$$

A simple change of variables in (1) shows that $e^{tA_n} = V^{-1}e^{t\Delta_n}V$ for $t \geq 0, n \in \mathbb{N}$, where $\Delta_n$ denotes the Laplacian on $BUC(\mathbb{R}^n)$. This implies that $R(\lambda, A_n) = V^{-1}R(\lambda, \Delta_n)V$ for $\lambda \in \Sigma_\pi := \{0 \neq \lambda \in \mathbb{C} : |\arg\ \lambda| < \pi\}$, and thus

$$\|R(\lambda, \Delta_n)\|_\infty = \|R(\lambda, A_n)\|_\infty \leq C \quad \text{for} \quad \lambda \in \Sigma_\pi \cap \rho(A),\ n \in \mathbb{N}.$$

Fix $\lambda \in \Sigma_\pi$ with $Re\ \lambda < 0$. For $n \in \mathbb{N}$, the function $g_{\lambda,n}(x) := e^{\frac{\lambda}{2n}|x|^2}$ belongs to $BUC(\mathbb{R}^n)$ and $\|g_{\lambda,n}\|_\infty = 1$. Setting

$$f_{\lambda,n}(x) := (\lambda - \Delta_n)g_{\lambda,n}(x) = -\frac{\lambda^2}{n^2}|x|^2 e^{\frac{\lambda}{2n}|x|^2},$$

we compute

$$\|f_{\lambda,n}\|_\infty = \frac{2|\lambda|^2}{ne|Re\ \lambda|}.$$

So we derive

$$\|R(\lambda, A_n)\|_\infty = \|R(\lambda, \Delta_n)\|_\infty \geq \frac{\|R(\lambda, \Delta_n)f_{\lambda,n}\|_\infty}{\|f_{\lambda,n}\|_\infty} = \frac{ne|Re\ \lambda|}{2|\lambda|^2}.$$

As a result, $\lambda$ must belong to the spectrum of $A$. Since $A$ generates a contraction semigroup, the first and second assertion readily follow from standard spectral theory (cf. [4, Chap. IV]).

To prove the last assertion, we observe that $i\mathbb{R}$ is contained in the approximate point spectrum of $A$. Let $\lambda = -a^2 + ib$ with $a > 0$ and $b \in \mathbb{R}$. The first part of the proof applies to the operator $\tilde{A}$ on $BUC(\tilde{H})$ corresponding to the sequence $(\lambda_2, \lambda_3, \cdots)$, where $\tilde{H}$ is the closed subpace of $H$ spanned by the vectors $e_k$, $k \geq 2$. Thus there exist $\tilde{g}_n \in D_0(\tilde{H})$ such that $\|\tilde{g}_n\|_\infty = 1$ and $\|\tilde{A}\tilde{g}_n - ib\tilde{g}_n\|_\infty \to 0$ as $n \to \infty$. We now define

$$f_n(x) := \exp(ia\lambda_1^{-\frac{1}{2}}\, x_1)\, \tilde{g}_n(x_2, x_3, \cdots), \quad x \in H.$$

Clearly, $f_n \in D_0(H)$, $\|f_n\|_\infty = 1$, and

$$Af_n(x) = \sum_{k=1}^{\infty} \lambda_k\, D_k^2\, f_n(x) = -a^2 f_n(x) + \exp(ia\lambda_1^{-\frac{1}{2}}\, x_1)\, (\tilde{A}\tilde{g}_n)(x_2, x_3, \cdots), \qquad x \in H.$$

Hence $\lambda$ is an approximate eigenvalue of $A$. □

As a consequence of [4, Thm.II.4.18] we immediately obtain the following result from [3], see also [1], [5], [7].

**Corollary 2.** *The semigroup $(P_t)$ is not eventually norm continuous on $BUC(H)$.*

In fact we have proved slightly more. A semigroup $T(\cdot)$ with growth bound 0 is called *asymptotically norm continuous* if

$$\lim_{t\to\infty} \limsup_{h\to 0} \|T(t+h) - T(t)\| = 0.$$

In view of [6, Thm.1.9] our theorem yields

**Corollary 3.** *The semigroup $(P_t)$ is not asymptotically norm continuous on $BUC(H)$.*

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