Existence Families and Differential Operators

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Abstract. Let X, Y be Banach spaces. We introduce an existence family of bounded linear operators from Y to X, as a new tool for handling the abstract Cauchy problem in X: $u'(t) = Au(t)$ $(t \geq 0)$, $u(0) = x$. It turns out to be a more useful tool than the traditional operator families. We also take a look at a class of variable coefficient elliptic differential operators in $L^p(\Omega)$ $(1 < p < \infty)$, and reveal their relationship to our new operator system.

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1. Introduction

Let's look at the abstract Cauchy problem

$$\begin{cases} u'(t) = Au(t), & t \geq 0, \\ u(0) = x, \end{cases} \tag{ACP}$$

where A is a linear operator in a Banach space X.

Over the past decade, the study of some generalizations of the classical strongly continuous semigroups (cf. [2, 6, 8, 12]), such as integrated semigroups, regularized semigroups and existence families of bounded linear operators on X, has attracted much attention because these new types of operator families equip (ACP) with nice operator systems for controlling over both wellposed and illposed problems (see, e.g., [1, 3-5, 9-11, 13-15] and the references cited in). In this paper, we shall introduce a more general operator system, E_0-existence family $\{E(t)\}_{t\geq 0}$ for A, of bounded linear operators mapping a Banach space Y (might be different from X) into X, as a new tool for dealing with (ACP). This idea is somewhat different from the traditional one (tying (ACP) with a linear operator family of X to itself, which originates from the pioneer works of Hille and Yosida in 1948). This new system turns out to be more useful (in some respects) than the known ones of X to itself. As one will see, it produces a larger set of initial data for which the solutions exist and gives a better estimate of the solutions. As an application, we consider a class of elliptic differential operators in $L^p(\Omega)$ $(1 < p < \infty)$ with variable coefficients and Dirichlet (or Neumann) boundary condition. We will show that these operators have E_0-existence families of $L^2(\Omega)$ to $L^p(\Omega)$ (if $p \geq 2$) for suitable E_0 (Theorem 4.1).

In this paper, Section 2 is a presentation of the definitions and basic properties of the new existence family as well as (its companion) uniqueness family, including their connections with (ACP). In Section 3, we prove an interpolation theorem for existence families, which will be needed in the proof of Theorem 4.1. The final Section 4 focuses on the issue of when differential operators, as well as the matrix operators with them as entries, have E_0-existence families.

Throughout this paper, all operators are linear. X, Y, Z will be Banach spaces. We denote by $\mathbf{L}(Y,\ X)$ the space of all bounded linear operators from Y into X; $\mathbf{L}(X,\ X)$ will be abbreviated to $\mathbf{L}(X)$. We will write $\mathcal{D}(A)$ for the domain of the operator A in X, $\mathcal{R}(A)$ for the range of A, $\rho(A)$ for the resolvent set of A, and A^* for the adjoint of A. $[\mathcal{D}(A)]$ will mean the normed space $\mathcal{D}(A)$ with the graph norm

$$\|x\|_{[\mathcal{D}(A)]} := \|x\| + \|Ax\|, \qquad x \in \mathcal{D}(A).$$

When $C \in \mathbf{L}(Y,\ X)$, $[\mathcal{R}(C)]$ will denote the Banach space $\mathcal{R}(C)$ with the norm

$$\|x\|_{[\mathcal{R}(C)]} := \inf\{\|y\|;\ Cy = x\}.$$

We will write $Z \hookrightarrow X$ to mean that $Z \subset X$ and Z is continuously embedded in X, that is, the identity map, from Z to X, is continuous. By $A\big\lceil_Z$ we will mean the part of A in Z, that is,

$$\mathcal{D}\left(A\big\lceil_Z\right) := \{x \in \mathcal{D}(A) \cap Z;\ Ax \in Z\}, \qquad A\big\lceil_Z x = Ax, \quad \text{for all } x \in \mathcal{D}\left(A\big\lceil_Z\right).$$

R will denote the set of real numbers, N the set of positive integers, and $\mathbf{C}$ the set of complex numbers.

By a solution of (ACP), we mean a continuously differentiable X-valued function $u(t)$ such that $u(t) \in \mathcal{D}(A)$ for all $t \geq 0$ and (ACP) is satisfied.

By $G \in LT_\omega - \mathbf{L}(Y,\ X)$, for a function $G(\cdot) : (a,\ \infty) \longrightarrow \mathbf{L}(Y,\ X)$ for some $a > 0$, we will mean that there exists a strongly continuous and $O(e^{\omega t})$ mapping $H : [0,\ \infty) \longrightarrow \mathbf{L}(Y,\ X)$ such that for $\lambda > a$

$$G(\lambda)y = \int_0^\infty e^{-\lambda t} H(t)y\,dt, \qquad y \in Y.$$

2. Existence families

Let A be as in (ACP). Let $E_0 \in \mathbf{L}(Y, X)$, and let $U_0 \in \mathbf{L}(X)$ be injective.

Definition 2.1. (1) The strongly continuous family of operators $\{E(t)\}_{t\geq 0} \subset \mathbf{L}(Y, X)$ is called an E_0-existence family for A, if for each $y \in Y$, $t \geq 0$, $\int_0^t E(s)yds \in \mathcal{D}(A)$ and

$$A\left(\int_0^t E(s)yds\right) = E(t)y - E_0 y. \tag{2.1}$$

We also say that A has an E_0-existence family $\{E(t)\}_{t\geq 0}$.

(2) ([5]) The strongly continuous family of operators $\{U(t)\}_{t\geq 0} \subset \mathbf{L}(X)$ is called a U_0-uniqueness family for A, if for all $x \in \mathcal{D}(A)$, $t \geq 0$,

$$\int_0^t U(s)Axds = U(t)x - U_0 x. \tag{2.2}$$

We also say that A has a U_0-uniqueness family $\{U(t)\}_{t\geq 0}$.

Remark 2.2. Definition 2.1 (1) is motivated by the work of deLaubenfels [4]. When $Y = X$, the E_0-existence family here coincides with the mild E_0-existence family defined in [4, p.8]. Clearly, such a family for A gives, for each $x \in \mathcal{R}(E_0)$, a mild solution of (ACP), i.e., a map $u(\cdot) \in C([0, \infty), X)$ such that

$$u(t) = A\int_0^t u(s)ds + x, \qquad t \geq 0. \tag{2.3}$$

In [4], a C-existence family is defined as a strongly continuous family of operators $\{W(t)\}_{t\geq 0}$ $\subset \mathbf{L}([\mathcal{D}(A)])$ satisfying $\left(\int_0^t AW(s)xds\right) = W(t)x - Cx$ $(x \in \mathcal{D}(A),\ t \geq 0)$. It was shown ([4, Theorem 2.6]) that $\{W(t)\}_{t\geq 0}$ produces solutions of (ACP) for all $x \in C(\mathcal{D}(A))$. We will show that $E(\cdot)$ also yields solutions of (ACP) for a reasonable set $\mathcal{D}_{E_0}(A)$ (see the following definition) of initial data. For some advantages of $\mathcal{D}_{E_0}(A)$, cf. Example 2.5, Remarks 2.12 and 4.5.

Definition 2.3. $\mathcal{D}_{E_0}(A)$ is the subspace of $\mathcal{D}(A)$ that consists of those elements x satisfying $Ax \in \mathcal{R}(E_0)$.

Theorem 2.4. (1) *If there is an E_0-existence family $\{E(t)\}_{t\geq 0}$ for A, then (ACP) admits a solution $u(\cdot)$ for all $x \in \mathcal{D}_{E_0}(A)$ satisfying*

$$\|u(t)\|_{[\mathcal{D}(A)]} \leq M(t)(\|Ax\|_{[\mathcal{R}(E_0)]} + \|x\|), \qquad t \geq 0, \tag{2.4}$$

for a certain locally bounded positive function $M(t)$ on $[0, \infty)$.

(2) *If there is a U_0-uniqueness family $\{U(t)\}_{t\geq 0}$ for A, then all solutions of (ACP) are unique.*

Proof. (1) Let $x \in \mathcal{D}_{E_0}(A)$. We take a $y_0 \in Y$ such that $Ax = E_0 y_0$. Define, for $t \geq 0$,

$$u(t) := x + \int_0^t E(s) y_0 ds. \tag{2.5}$$

Then we verify by (2.1) that $u(t)$ satisfies (ACP). Note that $\|E(t)\|$ is locally bounded on $[0, \infty)$ by making use of the Banach-Steinhaus theorem. So (2.4) follows by the arbitrariness of y_0.

(2) Let u be a solution of (ACP) with $x = 0$. Then by (2.2)

$$\begin{aligned} \frac{d}{ds}(U(t-s)u(s)) &= -U(t-s)Au(s) + U(t-s)u'(s) \\ &= 0, \quad t \geq s \geq 0. \end{aligned}$$

Accordingly,

$$U_0 u(t) = U(t)u(0) = 0, \quad t \geq 0$$

and therefore $u(t) \equiv 0$ by the injectivity of U_0.

The following is a simple example indicating that the choice of a Banach space Y different from X produces a larger set of initial data.

Example 2.5. Let A be a linear operator in $C_0(R)$, defined by

$$(Af)(x) := xf(x), \quad x \in R$$

for all

$$f \in \mathcal{D}(A) := \{f \in C_0(R);\ xf(x) \in C_0(R)\}.$$

Then it is easy to verify that A has an E_0-existence family $\{E(t)\}_{t\geq 0}$ of operators in $\mathbf{L}(C_b(R), C_0(R))$, as well as a U_0-uniqueness family $\{U(t)\}_{t\geq 0}$ of operators in $\mathbf{L}(C_0(R))$. Here for $x \in R$, $t \geq 0$,

$$\begin{aligned} &(E(t)f)(x) := e^{-x^2} e^{tx} f(x), \quad f \in C_b(R), \\ &U(t) := E(t)\restriction_{C_0(R)}, \\ &E_0 = E(0), \quad U_0 = U(0). \end{aligned}$$

By Theorem 2.4, we know that (ACP) admits a unique solution whenever the initial value is in

$$\mathcal{D}_{E_0}(A) = \{f \in C_0(R);\ xe^{x^2} f(x) \text{ is bounded on } R\}.$$

Certainly, $\{U(t)\}_{t\geq 0}$ is also a U_0-existence family in $\mathbf{L}(C_0(R))$ for A. But we remark that in this case, $\mathcal{D}_{E_0}(A)$ strictly contains

$$\mathcal{D}_{U_0}(A) = U_0(\mathcal{D}(A)) = \{f \in C_0(R);\ xe^{x^2}f(x) \in C_0(R)\}.$$

Now, we characterize the $O(e^{\omega t})$ E_0-existence families in the language of Laplace transforms.

Theorem 2.6. *Let $\omega \in R$ and let $\{E(t)\}_{t\geq 0}$ be a strongly continuous family of operators in $\mathbf{L}(Y,X)$ such that*

$$\|E(t)\| \leq \text{const } e^{\omega t}, \qquad t \geq 0.$$

Suppose that A is closed, and $\lambda - A$ is injective for $\lambda > a$ (for some $a > \max(\omega, 0)$). Then the following statements are equivalent.

(a) *$\{E(t)\}_{t\geq 0}$ is an E_0-existence family for A.*

(b) *$\mathcal{R}(E_0) \subset \mathcal{R}(\lambda - A)$ for $\lambda > a$ and*

$$(\lambda - A)^{-1}E_0 y = \int_0^\infty e^{-\lambda t}E(t)y\,dt, \qquad y \in Y,\ \lambda > a.$$

Proof. (a)$\Longrightarrow$(b). Taking Laplace transforms in (2.1), we obtain that for $\lambda > \max(\omega, 0)$,

$$(\lambda^{-1}A - 1)\int_0^\infty e^{-\lambda t}E(t)y\,dt = -\lambda^{-1}E_0 y, \qquad y \in Y.$$

The desired result follows immediately.

(b)$\Longrightarrow$ (a). Observe that for $\lambda > a$, $y \in Y$,

$$\lambda^{-1}(\lambda - A)^{-1}E_0 y = \int_0^\infty e^{-\lambda t}\left(\int_0^t E(s)y\,ds\right)dt,$$

$$\begin{aligned}\lambda^{-1}A(\lambda - A)^{-1}E_0 y &= -\lambda^{-1}E_0 y + (\lambda - A)^{-1}E_0 y \\ &= \int_0^\infty e^{-\lambda t}(E(t)y - E_0 y)\,dt.\end{aligned}$$

We deduce, by virtue of Lemma 2.7 below, that for all $t \geq 0$, $y \in Y$, $\int_0^t E(s)y\,ds \in \mathcal{D}(A)$ and (2.1) holds.

Lemma 2.7 ([15]). *Let $h_1,\ h_2 \in C([0,\infty), X)$ with $\|h_i(t)\| \leq$ const $e^{\omega t}$ $(t \geq 0,\ i = 1, 2)$ for some $\omega > 0$. Assume that A is a closed operator in X such that for all $\lambda > \omega$, $\int_0^\infty e^{-\lambda t}h_1(t)\,dt \in \mathcal{D}(A)$ and*

$$A\int_0^\infty e^{-\lambda t}h_1(t)\,dt = \int_0^\infty e^{-\lambda t}h_2(t)\,dt.$$

Then for every $t \geq 0$, $h_1(t) \in \mathcal{D}(A)$ and $Ah_1(t) = h_2(t)$.

Remark 2.8. When $X = Y$, Theorem 2.6 appeared in [5, p.1489], where the implication (b)$\Longrightarrow$(a) is proved by using the "integrated version of Widder's theorem" for Banach spaces (see [1, Theorem 1.1]).

Theorem 2.9. *Let* (a) *or* (b) *of Theorem* 2.6 *hold. Then for every $x \in \mathcal{D}_{E_0}(A)$, (ACP) admits a unique exponentially bounded solution $u(t)$ such that*

$$\|u(t)\|_{[\mathcal{D}(A)]} \leq const\ M(t)(\|x\| + \|Ax\|_{[\mathcal{R}(E_0)]}), \qquad t \geq 0,$$

where

$$M(t) := \begin{cases} e^{\max(\omega,0)t} & if \quad \omega \neq 0, \\ t, & if \quad \omega = 0; \end{cases}$$

and if $x \in \mathcal{R}(E_0)$ in addition,

$$\|u(t)\| \leq const\ e^{\omega t}\|x\|_{[\mathcal{R}(E_0)]}, \qquad t \geq 0. \tag{2.6}$$

Proof. Immediately Theorem 2.4, as well as its proof, shows the required results, except the uniqueness and the estimate (2.6). The uniqueness of exponentially bounded solutions is given by Proposition 2.9 of [4]. As to (2.6), we observe by (2.5) that for $\lambda > \max(\omega, 0)$,

$$\begin{aligned} \int_0^\infty e^{-\lambda t}u(t)dt &= \lambda^{-1}x + \lambda^{-1}\int_0^\infty e^{-\lambda t}E(s)y_0 ds \\ &= \lambda^{-1}x + \lambda^{-1}(\lambda - A)^{-1}E_0 y_0 \\ &= (\lambda - A)^{-1}x, \quad \text{noting } E_0 y_0 = Ax. \end{aligned} \tag{2.7}$$

When $x \in \mathcal{R}(E_0)$, we take a $y_1 \in Y$ such that $E_0 y_1 = x$. Then for $\lambda > a$,

$$(\lambda - A)^{-1}x = (\lambda - A)^{-1}E_0 y_1 = \int_0^\infty e^{-\lambda t}E(t)y_1 dt.$$

This, together with (2.7), implies $u(t) \equiv E(t)y_1$, according to the uniqueness theorem for Laplace transforms. Now (2.6) follows by the arbitrariness of y_1. The proof is then complete.

Next is a characterization of a $O(e^{\omega t})$ U_0-uniqueness family for A in terms of Laplace transforms.

Theorem 2.10. *Suppose that $\{U(t)\}_{t\geq 0}$ is a strongly continuous family of operators in* $\mathbf{L}(X)$ *such that*

$$\|U(t)\| \leq \text{const } e^{\omega t} \quad (t \geq 0) \quad for\ some\ \omega \geq 0.$$

Then the following statements are equivalent.

(a) $\{U(t)\}_{t\geq 0}$ *is a* U_0*-uniqueness family for* A.

(b) *For* $\lambda > \omega$,

$$U_0x = \int_0^\infty e^{-\lambda t}U(t)(\lambda - A)x\,dt, \qquad x \in \mathcal{D}(A).$$

Proof. Taking Laplace transforms in (2.2) leads to the implication (a)$\Longrightarrow$(b). Using the uniqueness theorem for Laplace transforms verifies the implication (b)$\Longrightarrow$ (a).

Remark 2.11. When $X = Y$, E_0 is injective and $E_0A \subset AE_0$, it is evident that the $\{E(t)\}_{t\geq 0}$ in Theorem 2.6 is also an exponentially bounded E_0-uniqueness family for A. In this situation, one is getting what is called a (exponentially bounded) E_0-regularized semigroup $\{E(t)\}_{t\geq 0}$ generated by $E_0^{-1}AE_0$ (see, e.g. [5, Theorem 3.12], [14, Section 3]).

Remark 2.12. From Example 2.5, we have seen the nice behavior of $\mathcal{D}_{E_0}(A)$. We would like to emphasize that only in the case of $X = Y$, does $E_0\mathcal{D}(A)$ make sense. Now, we clarify the relationship between $\mathcal{D}_{E_0}(A)$ and $E_0\mathcal{D}(A)$ in the case of $X = Y$ and $E_0A \subset AE_0$. It is not difficult to see that

$$E_0\mathcal{D}(A) \subset \mathcal{D}_{E_0}(A) \bigcap \mathcal{R}(E_0),$$

while the opposite inclusion holds true if and only if $\{x;\ AE_0x \in \mathcal{R}(E_0)\} \subset \mathcal{D}(A)$; that

$$E_0\mathcal{D}(A) = \mathcal{D}_{E_0}(A), \tag{2.8}$$

if A is a one to one mapping of $\mathcal{D}(A)$ onto X. The following two simple counterexamples indicate that the equality (2.8) may fail if A is not injective or surjective.

(a) Let X be an infinite-dimensional Banach space, and A, any linear operator in X, be surjective but not injective. Let $E_0 \equiv 0$ in X. Then

$$E_0A \subset AE_0, \quad E_0\mathcal{D}(A) = \{0\},$$

$$\mathcal{D}_{E_0}(A) = \{x \in X;\ Ax = 0\} \neq \{0\}.$$

(b) Let $X = l^2$ be the set of all sequences $u = \{u_m\}_{m\in N}$ of complex numbers with

$$\|\{u_m\}\|^2 := \sum_{m=1}^{\infty} |u_m|^2 < \infty.$$

Let A, E_0 be the operators defined by

$$A\{u_m\} = \{\widetilde{u}_m\}, \qquad \text{for all } \{u_m\} \in X,$$

$$E_0\{u_m\} = \{\overline{u}_m\}, \qquad \text{for all } \{u_m\} \in X,$$

where $\widetilde{u}_1 = u_1$, $\overline{u}_1 = 0$ and

$$\widetilde{u}_m = \overline{u}_m = \begin{cases} 0 & \text{if } m \text{ is even} \\ u_{i+1} & \text{if } m = 2i+1 \ (i = 1, 2, \cdots). \end{cases}$$

Clearly $A,\ E_0 \in \mathbf{L}(X)$, A is injective but not surjective, $E_0 A = A E_0$, and

$$\begin{aligned} \mathcal{D}_{E_0}(A) &= \{\{v_m\} \in X;\ \ v_1 = 0\}, \\ E_0 \mathcal{D}(A) &= \{\{v_m\} \in X;\ \ v_1 = v_m = 0 \text{ for any even } m\}. \end{aligned}$$

3. An interpolation theorem for existence families

Theorem 3.1. *Let $E_0 \in \mathbf{L}(Y, X)$, $C \in \mathbf{L}(X, Z)$, $\omega \in R$, and A a linear operator in X such that $(a, \infty) \subset \rho(A)$ for some $a > \max(\omega, 0)$. Suppose that A has a $O(e^{\omega t})$ E_0-existence family of operators in $\mathbf{L}(Y, X)$. If $CA \subset AC$ and $Z \hookrightarrow X$, then $A\lceil_Z$ possesses a $O(e^{\omega t})$ CE_0-existence family of operators in $\mathbf{L}(Y, Z)$.*

Proof. Denote by $E(\cdot) : [0, \infty) \longrightarrow \mathbf{L}(Y, X)$ the E_0-existence family for A. Then we have by Theorem 2.6 that

$$\|E(t)y\|_X \leq \text{const } e^{\omega t}\|y\|_Y, \qquad t \geq 0,\ y \in Y,$$

$$(\lambda - A)^{-1} E_0 y = \int_0^\infty e^{-\lambda t} E(t) y dt, \qquad y \in Y,\ \lambda > a.$$

Therefore

$$\|CE(t)y\|_Z \leq \text{const } \|E(t)y\|_X \leq \text{const } e^{\omega t}\|y\|_Y, \qquad y \in Y,\ t \geq 0,$$

and

$$\lambda \longmapsto C(\lambda - A)^{-1} E_0 \in LT_\omega - \mathbf{L}(Y, Z). \tag{3.1}$$

For each $\lambda > a$, $x \in X$, write $z_{\lambda,x} := (\lambda - A)^{-1} C x$. Then

$$z_{\lambda,x} = C(\lambda - A)^{-1} x \in \mathcal{R}(C) \subset Z$$

and therefore

$$A z_{\lambda,x} = (A - \lambda) z_{\lambda,x} + \lambda z_{\lambda,x} = -Cx + \lambda z_{\lambda,x} \in Z.$$

Whence $z_{\lambda,x} \in \mathcal{D}\left(A\lceil_Z\right)$, which implies

$$\mathcal{R}(C) \subset \mathcal{R}\left(\lambda - A\lceil_Z\right) \qquad \text{for all } \lambda > a.$$

Thus we have

$$\left(\lambda - A\big\lceil_{Z}\right)^{-1} CE_0 = (\lambda - A)^{-1}CE_0 = C(\lambda - A)^{-1}E_0, \quad \lambda > a.$$

This combined with (3.1) yields that

$$\lambda \longmapsto \left(\lambda - A\big\lceil_{Z}\right)^{-1} CE_0 \in LT_\omega - \mathbf{L}(Y, Z).$$

This ends the proof by Theorem 2.6.

4. differential operators relating to existence families

Let Ω be a bounded open set in the n-dimensional Euclidean space R^n, with boundary $\partial\Omega$. Let $a_{ij}(\cdot)$, $1 \le i, j \le n$, be real-valued functions belonging to $W^{1,2}_{loc}(\Omega)$ such that $a_{ij} = a_{ji}$ and the following ellipticity condition is satisfied

$$c_1|\eta|^2 \le \sum_{i,j=1}^{n} a_{ij}(x)\eta_i\overline{\eta}_j \le c_2|\eta|^2, \qquad \eta \in \mathbf{C}^n, \text{ a.e. in } \Omega,$$

for some c_1, $c_2 > 0$. We consider the nonnegative symmetric quadratic form a:

$$a(u, v) = \sum_{i,j=1}^{n} \left(a_{ij}(x)\frac{\partial u}{\partial x_i}, \frac{\partial v}{\partial x_j}\right)_{L^2(\Omega)},$$

from $H^1_0(\Omega) \times H^1_0(\Omega)$ (resp. $H^1(\Omega) \times H^1(\Omega)$) to $\mathbf{C}$. The form corresponds to Dirichlet (resp. Neumann) boundary condition. We assume $\partial\Omega$ is of class C^1 when Neumann boundary condition is concerned. Denote by A_2 the associated self-adjoint and positive operator in $L^2(\Omega)$. It is known that

$$\mathcal{D}\left(A_2^{\frac{n}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\right) \subset L^p(\Omega), \qquad 2 \le p < \infty. \tag{4.1}$$

Define

$$\begin{aligned} A_p &= A_2\big\lceil_{L^p(\Omega)}, \qquad 2 < p < \infty, \\ A_q &= A_p^*, \qquad 1 < q < 2,\ p = \frac{q}{q-1}. \end{aligned}$$

Then for $1 < p < \infty$, $-A_p$ is the generator of a holomorphic semigroup $T_p(\cdot)$ of angle $\frac{\pi}{2}$ on $L^p(\Omega)$.

For the original material, see [11] and references therein.

Theorem 4.1. *Let $p \in [2, \infty)$. Then each of $\pm iA_p$ has a uniformly bounded $(\mu + A_2)^{-r}$-existence family of operators in $\mathbf{L}\left(L^2(\Omega), L^p(\Omega)\right)$ for all $r \ge \frac{n}{2}\left(\frac{1}{2} - \frac{1}{p}\right)$, where $\mu \in \rho(-A_2)$.*

Proof. Clearly, we have

$$\lambda \longmapsto (\lambda \pm iA_2)^{-1} \in LT_0 - \mathbf{L}(L^2(\Omega)), \tag{4.2}$$

since iA_2 generates a strongly continuous group of unitary operators on $L^2(\Omega)$ by Stone's theorem. Observe

$$\begin{aligned} \mathcal{R}\left((\mu + A_2)^{-r}\right) &= \mathcal{D}\left((\mu + A_2)^r\right) \\ &\subset \mathcal{D}\left(A_2^{\frac{n}{2}\left(\frac{1}{2}-\frac{1}{p}\right)}\right) \\ &\subset L^p(\Omega) \end{aligned}$$

by (4.1). It follows that $(\mu + A_2)^{-r}$ is a closed operator of $L^2(\Omega)$ to $L^p(\Omega)$ noting $(\mu + A_2)^{-r} \in \mathbf{L}\left(L^2(\Omega)\right)$ and

$$L^p(\Omega) \hookrightarrow L^2(\Omega). \tag{4.3}$$

So

$$(\mu + A_2)^{-r} \in \mathbf{L}\left(L^2(\Omega),\ L^p(\Omega)\right) \tag{4.4}$$

by the closed graph theorem. Thus combining (4.2) – (4.4) together and applying Theorem 3.1 leads to the result as desired. The proof is complete.

Corollary 4.2. *Let $1 < p < \infty$. Then each of $\pm iA_p$ generates a uniformly bounded $(\omega + A_p)^{-r}$-regularized semigroup on $L^p(\Omega)$ for all $r \geq \frac{n}{2}\left|\frac{1}{2} - \frac{1}{p}\right|$, where ω is large enough such that*

$$\|T_p(t)\| \leq \text{const } e^{(\omega-1)t}, \qquad t \geq 0.$$

Proof. This is an easy consequence of Theorem 4.1 for the case of $2 \leq p < \infty$, upon observing that

$$(\omega + A_2)^{-r}\Big\lceil_{L^p(\Omega)} \quad = \quad (\omega + A_p)^{-r}.$$

Let now $1 < p < 2$. Then

$$(iA_p)^* = iA_q, \qquad (q = \frac{p}{p-1} > 2);$$

each of $\pm iA_q$ generates a uniformly bounded $(\omega + A_q)^{-r}$-regularized semigroup on $L^q(\Omega)$ by the preceding justification. According to this, an appeal to the generation theorem of regularized semigroups (see [4, Corollary 17.5]) completes the proof.

Remark 4.3. The integrated semigroup relative of Corollary 4.2 appeared in [11, Theorem 4.2].

Example 4.4. Assume that $\Omega \subset R^3$ is a bounded open set. Consider the initial-boundary value problem for the Schrödinger equation

$$\begin{cases} \dfrac{\partial u(t,\ x)}{\partial t} = i\Delta u(t,\ x), \quad \text{in } [0,\infty)\times\Omega, \\ u(0,\ x) = u_0(x), \quad \text{in } \Omega, \\ u(t,\ x) = 0, \quad \text{in } [0,\infty)\times\partial\Omega, \end{cases} \tag{4.5}$$

in the space $L^6(\Omega)$.

Applying Theorem 4.1 to $p = 6$ and $A_2 = -\Delta$ with Dirichlet boundary condition, we obtain that iA_6 has a uniformly bounded $(\mu + A_2)^{-\frac{1}{2}}$-existence family ($\mu \in \rho(-A_2)$), since $\frac{3}{2}\left(\frac{1}{2} - \frac{1}{6}\right) = \frac{1}{2}$. On the other hand, Corollary 4.2 indicates that iA_6 has a $(\mu_0 + A_6)^{-\frac{1}{2}}$-uniqueness family ($\mu_0$ large enough) as well. Let $E_0 = (\mu + A_2)^{-\frac{1}{2}}$. In view of Theorem 2.2 in [7, §IV.2], we get $[\mathcal{R}(E_0)] = H_0^1(\Omega)$. Moreover, it is not hard to see

$$\mathcal{D}(A_2) \supset H^2(\Omega) \cap H_0^1(\Omega).$$

Note $\mathcal{R}(E_0) \subset L^6(\Omega)$ by (4.1), which implies $\mathcal{D}_{E_0}(A_2) = \mathcal{D}_{E_0}(A_6)$. Therefore, we conclude by Theorem 2.9 that for any $u_0 \in H_0^3(\Omega) \subset L^6(\Omega)$, the initial-boundary value problem (4.5) admits a unique solution $u \in C^1([0,\infty),\ L^6(\Omega))$ satisfying

$$\begin{aligned} \|u\|_{L^6(\Omega)} &\leq \text{const } \|u_0\|_{[\mathcal{R}(E_0)]} \\ &\leq \text{const } \|u_0\|_{H^1(\Omega)}. \end{aligned} \tag{4.6}$$

Remark 4.5. In the above example, iA_6 is also the generator of a uniformly bounded $(\mu_0 + A_6)^{-\frac{1}{2}}$-regularized semigroup ($\mu_0$ large enough) in view of Corollary 4.2. Yet the set of initial data given by the regularized semigroup is $(\mu_0 + A_6)^{-\frac{1}{2}}\mathcal{D}(A_6)$, which is smaller than $\mathcal{D}_{E_0}(A_6)$. Moreover the estimate (4.6) is better than that given by the regularized semigroup, i.e.,

$$\|u\|_{L^6(\Omega)} \leq \text{const } \|u_0\|_{W^{1,6}(\Omega)}.$$

Example 4.6. Assume that Ω is of class C^2. We denote by Δ_p the Laplacian on $L^p(\Omega)$ ($1 < p < \infty$). It is well known that Δ_p with the domain $W^{2,p}(\Omega) \bigcap W_0^{1,p}(\Omega)$ is the generator of a strongly continuous semigroup. Let $p \in [2,\infty)$, $q \in (1,p]$ and let B be a second order differential operator with coefficients in $C^2\left(\overline{\Omega}\right)$. We look at the matrix operator on $L^q(\Omega) \times L^p(\Omega)$

$$\mathcal{A} = \begin{pmatrix} \Delta_q & B \\ 0 & i\Delta_p \end{pmatrix},$$

with domain

$$\mathcal{D}(\mathcal{A}) := \left(W^{2,q}(\Omega) \bigcap W_0^{1,q}(\Omega)\right) \times \left(W^{2,p}(\Omega) \bigcap W_0^{1,p}(\Omega)\right).$$

It is clear that $\mathcal{A}$ is closed. Furthermore we obtain, in view of Theorem 4.1 and the following Proposition 4.7, that $\mathcal{A}$ possesses an exponentially bounded $\begin{pmatrix} I & 0 \\ 0 & (I-\Delta_2)^{-\frac{n}{2}\left(\frac{1}{2}-\frac{1}{p}\right)-1} \end{pmatrix}$-existence family of operators in $\mathbf{L}\left(L^q(\Omega) \times L^2(\Omega), L^q(\Omega) \times L^p(\Omega)\right)$.

Proposition 4.7. *Assume that A is a linear operator in X such that $\lambda - A$ is injective for $\lambda > a$ (for some $a > 0$), and that $B : \mathcal{D}(B) \subset X \to Z$ is a closable linear operator with $\mathcal{D}(B) \supset \mathcal{D}(A)$. Suppose that G is the generator of a strongly continuous semigroup $T(\cdot)$ on Z, and that A has an exponentially bounded E_0-existence family of operators in $\mathbf{L}(Y, X)$. Let $\omega \in \rho(A)$. Then the matrix operator on the space $Z \times X$*

$$\mathcal{A} := \begin{pmatrix} G & B \\ 0 & A \end{pmatrix}, \qquad \text{with } \mathcal{D}(\mathcal{A}) := \mathcal{D}(G) \times \mathcal{D}(A),$$

possesses a C-existence family $W(\cdot)$ of operators in $\mathbf{L}(Z \times Y, Z \times X)$. Here

$$C := \begin{pmatrix} I & 0 \\ 0 & (\omega - A)^{-1}E_0 \end{pmatrix},$$

$$W(t) := \begin{pmatrix} T(t) & \int_0^t T(t-s)B(\omega - A)^{-1}E(s)ds \\ 0 & (\omega - A)^{-1}E(t) \end{pmatrix}, \qquad t \geq 0.$$

Proof. By hypothesis, we have that for λ large enough

$$(\lambda - G)^{-1}z = \int_0^\infty e^{-\lambda t}T(t)zdt, \qquad z \in Z,$$
$$(\lambda - A)^{-1}E_0 y = \int_0^\infty e^{-\lambda t}E(t)ydt, \qquad y \in Y,$$

and $B(\omega - A)^{-1} \in \mathbf{L}(X, Z)$. So for λ large enough,

$$\begin{aligned} &\int_0^\infty e^{-\lambda t}\left(\int_0^t T(t-s)B(\omega - A)^{-1}E(s)yds\right)dt \\ = \;& (\lambda - G)^{-1}B(\omega - A)^{-1}(\lambda - A)^{-1}E_0 y, \qquad y \in Y. \end{aligned}$$

Accordingly, we get that for $z \in Z$, $y \in Y$ and for λ large enough,

$$\begin{aligned}
&\int_0^\infty e^{-\lambda t} W(t) \begin{pmatrix} z \\ y \end{pmatrix} dt \\
= &\begin{pmatrix} (\lambda - G)^{-1} & (\lambda - G)^{-1} B(\lambda - A)^{-1}(\omega - A)^{-1} E_0 \\ 0 & (\lambda - A)^{-1}(\omega - A)^{-1} E_0 \end{pmatrix} \begin{pmatrix} z \\ y \end{pmatrix} \\
= &(\lambda - \mathcal{A})^{-1} C.
\end{aligned}$$

Therefore, using Theorem 2.6 and noting the closedness of $\mathcal{A}$ yields the result as desired.

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