Stability of the equilibria for spatially periodic flows in porous media

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Abstract

We prove exponential stability of the equilibria for two moving boundary problems describing flows in porous media which are of Hele-Shaw and Stefan type, respectively. The main tool is the principle of linearized stability for fully nonlinear parabolic evolution equations. Crucial points are the assumption of spatial periodicity for the flows and the identification of conserved quantities.

Key words: fully nonlinear parabolic equation, exponential stability, moving interface

1 Introduction

The investigation of various mathematical models for flows in porous media has recently attracted a lot of interest from different points of view. This article is concerned with two such models that share the following basic properties:

- the flow is governed by Darcy's law,
- the porous medium is either completely saturated by the liquid or dry, and both phases are separated by a sharp interface,
- the main interest is in the determination of the motion of this interface, i.e. we are concerned with a moving boundary problem.

The use of Darcy's law together with linear constitutive equations leads to linear governing equations for the velocity potential u. From a mathematical point of view, the moving boundary problems that are formed by these equations together with appropriate initial and boundary conditions are equivalent to a (multidimensional) Hele-Shaw flow problem and a Stefan-type problem, respectively. We will consider a geometric setting and driving forces describing an infinite layer of liquid which moves under the influence of gravity; no other driving forces are considered. Moreover, the liquid domain is bounded from below by a fixed impermeable horizontal bottom. The corresponding free boundary problems have been discussed in [7, 8, 10, 11] for the case of an incompressible porous medium (the Hele-Shaw problem) and in [6] in the case of a compressible porous medium (the Stefan-type problem, see also [4, 5]).

Our analysis is based on the methods used in [6, 7, 8] which can be briefly and informally sketched as follows:

1. The problem is reformulated as a nonlinear initial boundary value problem on a fixed reference domain, using a transformation via an a priori unknown, time-dependent diffeomorphism.
2. By discussion of the linearization, this problem is identified as an abstract parabolic Cauchy problem.
3. Well-posedness of the problem and smoothness of its solution can be shown using the theory of maximal regularity for fully nonlinear parabolic equations, see e.g. [2, 3, 12].

In general, these methods can also be used to investigate the stability of stationary solutions on the basis of the "principle of linearized stability", i.e. by studying the spectrum of the linearized evolution equation near a stationary solution.

However, even in the simplest two-dimensional case of the Hele-Shaw type problem it turns out that the L^2-realization of the corresponding linearized evolution equation has a pure continuous spectrum, given by $[0, \infty)$. Moreover, there is a continuous family of equilibria given by all liquid layers of constant density.

It is the aim of this article to prove exponential stability for a modified version of the two moving boundary value problems above, overcoming these difficulties in the following way: By demanding periodicity of the flows in space, the unbounded domain $\mathbb{R}^n$ is replaced by the torus $\mathbb{T}^n$, and by the identification of a conserved quantity, the evolution is a priori restricted to a submanifold of the phase space which intersects the set of equilibria in a single point. It is shown that the equilibrium of these restrictions is exponentially stable.

2 Flows in incompressible porous media

In the description of the problem as well as in the notation, we closely follow [8].

Let $\alpha \in (0,1)$ and define

$$\mathcal{U} := \left\{ f \in BUC^{2+\alpha}(\mathbb{R}^n) \,|\, \inf_{x \in \mathbb{R}^n} f(x) > 0 \right\}.$$

Given $f \in \mathcal{U}$, let

$$\tilde{\Omega}_f := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \,|\, 0 < y < f(x),\, x \in \mathbb{R}^n\}$$

and denote the components of its boundary by

$$\tilde{\Gamma}_f := \{(x, f(x)) \,|\, x \in \mathbb{R}^n\}, \quad \tilde{\Gamma}_0 := \mathbb{R}^n \times \{0\}.$$

We consider the elliptic boundary value problem (BVP)

$$\left.\begin{array}{rcll} \Delta_{n+1} u &=& 0 & \text{in } \tilde{\Omega}_f, \\ \partial_{n+1} u &=& 0 & \text{on } \tilde{\Gamma}_0, \\ u &=& y & \text{on } \tilde{\Gamma}_f, \end{array}\right\} \tag{2.1}$$

where Δ_{n+1} is the $(n+1)$-dimensional Laplacian, ∂_{n+1} denotes the $(n+1)$st partial derivative and y is the $(n+1)$st coordinate function on $\mathbb{R}^{n+1} \cong \mathbb{R}^n \times \mathbb{R}$. This BVP is a simple model for the motion of an incompressible liquid in a rigid porous medium under the influence of gravity. The liquid occupies the domain $\tilde{\Omega}_f$ over the impermeable bottom $\tilde{\Gamma}_0$, and u is the velocity potential. (The well-posedness of this problem will be shown under additional assumptions later.) One is interested in the evolution of the interface separating the saturated from the dry part of the porous medium which is represented by the boundary component $\tilde{\Gamma}_{f(t)}$, where now $f: [0,\infty) \longrightarrow \mathcal{U}$. This evolution is described by the nonlinear Cauchy problem

$$\left.\begin{array}{rcl} \partial_t f(t,\cdot) + \sqrt{1+|\nabla_n f(t,\cdot)|^2}\,\partial_\nu u(\cdot, f(t,\cdot)) & = & 0, \\ f(0,\cdot) & = & f_0 \end{array}\right\} \tag{2.2}$$

on $\mathbb{R}^n$, where f_0 corresponds to the initial interface, ∇_n denotes the gradient on $\mathbb{R}^n$, u is the solution of (2.1), and $\partial_\nu u$ denotes the derivative of u with respect to the outer unit normal on $\tilde{\Gamma}_f$.

We remark that problem (2.1), (2.2) is mathematically equivalent to a multidimensional one-phase Hele-Shaw problem with mixed boundary conditions.

In our stability analysis we shall make the following crucial periodicity requirement on f and on u:

$$f(t, x + 2k\pi e_j) = f(t,x) \quad \forall x \in \mathbb{R}^n,\ j = 1,\dots,n,\ k \in \mathbb{Z},\ t \geq 0 \tag{2.3}$$

$$u(x + 2k\pi e_j, y) = u(x,y) \quad \forall (x,y) \in \tilde{\Omega}_{f(t)},\ j = 1,\dots,n,\ k \in \mathbb{Z},\ t \geq 0. \tag{2.4}$$

In other words, instead of (2.1), (2.2) we discuss

$$\left.\begin{array}{rcllr} \Delta_{n+1} u & = & 0 & \text{in } \Omega_{f(t)}, & t \geq 0, \\ \partial_{n+1} u & = & 0 & \text{on } \Gamma_0, & t \geq 0, \\ u & = & y & \text{on } \Gamma_{f(t)}, & t \geq 0, \\ \partial_t f(t,\cdot) + \sqrt{1+|\nabla_n f(t,\cdot)|^2}\,\partial_\nu u(\cdot, f(t,\cdot)) & = & 0, & \text{on } \mathbb{T}^n, & t > 0, \\ f(0,\cdot) & = & f_0 & \text{on } \mathbb{T}^n & \end{array}\right\} \tag{2.5}$$

where

$$\begin{array}{rcl} \Omega_{f(t)} & := & \{(x,y) \in \mathbb{T}^n \times \mathbb{R} \,|\, 0 < y < f(t,x),\ x \in \mathbb{T}^n\}, \\ \Gamma_{f(t)} & := & \{(x, f(t,x)) \,|\, x \in \mathbb{T}^n\}, \quad \Gamma_0 := \mathbb{T}^n \times \{0\}, \end{array}$$

for $t \geq 0$ and $\mathbb{T}^n$ denotes the n-dimensional torus. For the sake of simplicity, we identify periodic functions on $\mathbb{R}^n$ and functions on $\mathbb{T}^n$.

Remark: From a modeling point of view, it is unsatisfactory to demand periodicity both for f and for u. Actually, it seems reasonable to conjecture that (2.3) and $u \in BUC^{2+\alpha}(\tilde{\Omega}_f)$ imply (2.4) . To show this, however, one needs to know whether the solution of (2.1) is unique in $BUC^{2+\alpha}(\tilde{\Omega}_f)$. Unfortunately, we are only able to prove this under the additional assumption that f is constant or $n = 1$.

As in [7, 8], problem (2.5) is transformed to a problem on a fixed reference manifold. Let $\Omega := \mathbb{T}^n \times (0,1)$,

$$c := \frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} f_0(x)\, dx, \tag{2.6}$$

set $g := f - c$, and define $\phi \in \mathrm{Diff}^{2+\alpha}(\Omega, \Omega_f)$ by

$$\phi(x,y) := \phi_g(x,y) = (x, (1-y)(c+g(x))), \qquad (x,y) \in \Omega.$$

Defining the push-forward and pull-back operators induced by ϕ_g,

$$\begin{array}{ll} \phi_g^*: & C(\overline{\Omega}_f) \longrightarrow C(\overline{\Omega}), \quad u \mapsto u \circ \phi_g, \\ \phi_*^g: & C(\overline{\Omega}) \longrightarrow C(\overline{\Omega}_f), \quad v \mapsto v \circ \phi_g^{-1}, \end{array}$$

we introduce the transformed operators $\mathcal{A}(g)$ and $\mathcal{B}_i(g)$, $i = 0, 1$, acting on $C^2(\overline{\Omega})$ by

$$\begin{array}{rcl} \mathcal{A}(g) & := & -\phi_g^* \circ \Delta_{n+1} \circ \phi_*^g, \\ \mathcal{B}_i(g)v & := & \phi_g^*(\gamma_i \nabla_{n+1}(\phi_*^g v) \cdot n_i), \quad i = 0, 1, \end{array}$$

where γ_0 and γ_1 stand for the trace operators on Γ_{c+g} and Γ_0, respectively. Moreover, $n_0 := (-\nabla_n g, 1)$ and $n_1 := (0, \dots, 0, -1)$. Observe that n_0 is normal to Γ_{c+g}. For an explicit coordinate representation we refer to [8], Lemma 2.1. Although an unbounded geometry is considered there, the expressions are formally the same an in [8].

Transformation of (2.5) to Ω and Γ_0, respectively, yields

$$\left.\begin{array}{rcll} \mathcal{A}(g)v & = & 0 & \text{in } \Omega \times [0, \infty), \\ v & = & g + c & \text{on } \Gamma_0 \times [0, \infty), \\ \mathcal{B}_1(g)v & = & 0 & \text{on } \Gamma_1 \times [0, \infty), \\ \partial_t g + \mathcal{B}_0(g)v & = & 0 & \text{on } \Gamma_0 \times (0, \infty), \\ g(0) & = & g_0, & \end{array}\right\} \tag{2.7}$$

where $v := \phi_g^* u$ and $g_0 := f_0 - c$.

In the following, we will employ the so-called little Hölder spaces which are defined as follows: For $k \in \mathbb{N}$, let $h^{k+\alpha}(\mathbb{T}^n)$ denote the closure of $C^\infty(\mathbb{T}^n)$ in $BUC^{k+\alpha}(\mathbb{T}^n)$ and let $h^{k+\alpha}(\Omega)$ denote the closure of $BUC^\infty(\Omega)$ in $BUC^{k+\alpha}(\Omega)$. The spaces $h^{k+\alpha}(\mathbb{T}^n)$ and $h^{k+\alpha}(\Omega)$ are Banach algebras. Hence, by arguments analogous to the proof of Lemma 2.3 in [8], it follows that for some open neighborhood $\mathcal{O}$ of 0 in $h^{2+\alpha}(\mathbb{T}^n)$, we have

$$\begin{array}{rcl} \mathcal{A} & \in & C^\omega\left(\mathcal{O}, \mathcal{L}\left(h^{2+\alpha}(\Omega), h^\alpha(\Omega)\right)\right), \\ \mathcal{B}_i & \in & C^\omega\left(\mathcal{O}, \mathcal{L}\left(h^{2+\alpha}(\Omega), h^{1+\alpha}(\mathbb{T}^n)\right)\right), \quad i = 0, 1, \end{array}$$

where we use the obvious identification $\mathbb{T}^n \times \{i\} \cong \mathbb{T}^n$, and where C^ω denotes analytic dependence.

Moreover, given $g \in \mathcal{O}$, we have

$$(\mathcal{A}(g), \gamma_0, \mathcal{B}_1(g)) \in \mathrm{Isom}\left(h^{2+\alpha}(\Omega), h^\alpha(\Omega) \times h^{2+\alpha}(\mathbb{T}^n) \times h^{1+\alpha}(\mathbb{T}^n)\right). \tag{2.8}$$

This can be proved analogously to Theorem 3.5 in [7], the only necessary modification being the use of the compactness of Ω to ensure that any bounded and uniformly continuous function v on Ω attains its maximum on $\overline{\Omega}$. In view of (2.8) we can define for any $g \in \mathcal{O}$ the solution operator

$$\mathcal{T}(g) := (\mathcal{A}(g), \gamma_0, \mathcal{B}_1(g))^{-1} | \{0\} \times h^{2+\alpha}(\mathbb{T}^n) \times \{0\},$$

i.e., given $h \in h^{2+\alpha}(\mathbb{T}^n)$, we write $\mathcal{T}(g)h = v$ for the unique solution of the BVP

$$\left.\begin{array}{rcll} \mathcal{A}(g)v &=& 0 & \text{in } \Omega, \\ v &=& h & \text{on } \Gamma_0, \\ \mathcal{B}_1(g)v &=& 0 & \text{on } \Gamma_1. \end{array}\right\}$$

The same arguments as in [8], Lemma 2.5, show that

$$\mathcal{T} \in C^\omega\left(\mathcal{O}, \mathcal{L}\left(h^{2+\alpha}(\mathbb{T}^n), h^{2+\alpha}(\Omega)\right)\right).$$

Using this operator, we can reduce (2.7) to a single nonlocal Cauchy problem on $\mathbb{T}^n$:

$$\left.\begin{array}{rcl} \partial_t g + \Phi(g) &=& 0, \\ g(0) &=& g_0, \end{array}\right\} \tag{2.9}$$

where

$$\Phi(g) := \mathcal{B}_0(g)\mathcal{T}(g)(g+c).$$

It follows immediately from the definitions that $\mathcal{T}(g)$ commutes with the operator $h \mapsto h + c$ and that $\mathcal{B}_0(g)$ vanishes on constants. Hence

$$\Phi(g) = \mathcal{B}_0(g)(\mathcal{T}(g)g + c) = \mathcal{B}_0(g)\mathcal{T}(g)g. \tag{2.10}$$

From the analyticity of $\mathcal{T}$ and $\mathcal{B}_0$ we get

$$\Phi \in C^\omega\left(\mathcal{O}, h^{1+\alpha}(\mathbb{T}^n)\right). \tag{2.11}$$

The following result on the solvability of (2.9) holds:

Theorem 2.1 *(Existence, uniqueness, and smoothness of solutions* [7, 8]*)*
Given $g_0 \in \mathcal{O}$, the Cauchy problem (2.9) *has a unique maximal solution*

$$g \in C(J, \mathcal{O}) \cap C^\omega(\dot{J} \times \mathbb{T}^n),$$

where $J = [0, t^+(g_0))$ is the maximal interval of existence.

Remark: As we are concerned here with stability of the equilibrium only, we refrain from choosing $\mathcal{O}$ maximally large. We refer to [7, 8] for this.

The proof of Theorem 2.1 can be given in analogy to [7, 8], based on the theory of maximal regularity for fully nonlinear parabolic equations, see also [2, 3]. The only necessary changes are due to the fact that instead of an unbounded domain in $\mathbb{R}^{n+1}$ we consider here a bounded $(n+1)$-dimensional manifold. However, both the parameter-dependent estimates for the occurring elliptic boundary value problems and the interpolation results for little Hölder spaces can be carried over to this case without essential difficulties.

We start our discussion of the equilibrium $g = 0$ of (2.9) with an observation expressing essentially the conservation of volume for the domain saturated by the liquid. On $L^2(\mathbb{T}^n)$ we introduce a continuous projection P by

$$Pu = u - \frac{1}{(2\pi)^n}\int_{\mathbb{T}^n} u\,dx.$$

Obviously, the restriction of P to $h^{k+\alpha}(\mathbb{T}^n)$ is continuous on these spaces, too. On all these spaces, $\ker P$ is given by the constants and $\operatorname{Im} P$ is given by the functions with mean 0. For the latter we will write $\tilde{L}^2(\mathbb{T}^n)$, $\tilde{h}^{k+\alpha}(\mathbb{T}^n)$ etc.

Lemma 2.2 *(Conservation of volume)*
Given $g \in \mathcal{O}$, we have $\Phi(g) \in \tilde{h}^{1+\alpha}(\mathbb{T}^n)$.

Proof: Let $f := g + c$ and denote by u the solution of $(2.5)_1$–$(2.5)_3$. By definition, we have $\mathcal{T}(g)g = \phi_g^* u$ and Gauss' theorem and (2.10) give

$$\begin{aligned}
\int_{\mathbb{T}^n} \Phi(g)\, dx &= \int_{\mathbb{T}^n \times \{0\}} \mathcal{B}_0(g)\mathcal{T}(g)g\, dx = \int_{\mathbb{T}^n \times \{0\}} \phi_g^*(\gamma_0(\nabla_{n+1} u) \cdot n_0)\, dx \\
&= \int_{\mathbb{T}^n \times \{0\}} \phi_g^* \left(\sqrt{1 + |\nabla_n g|^2} \frac{\partial u}{\partial n} \right) dx = \int_{\Gamma_f} \frac{\partial u}{\partial n}\, d\sigma \\
&= \int_{\Omega_f} \Delta u\, dx - \int_{\Gamma_0} \frac{\partial u}{\partial n}\, d\sigma = -\int_{\Gamma_0} \partial_{n+1} u\, d\sigma = 0.
\end{aligned}$$

■

Observe that due to (2.6) we have

$$g_0 \in \tilde{h}^{2+\alpha}(\mathbb{T}^n), \tag{2.12}$$

and due to Lemma 2.2 the evolution proceeds completely in $\tilde{h}^{2+\alpha}(\mathbb{T}^n)$.

Next, we calculate the Fréchet derivative

$$A := \Phi'(0) \in \mathcal{L}\left(h^{2+\alpha}(\mathbb{T}^n), (h^{1+\alpha}(\mathbb{T}^n)\right)$$

of Φ at 0. One finds

$$\begin{aligned}
\Phi'(0)[h] &= \mathcal{B}_0'(0)[h]\mathcal{T}(0)0 + \mathcal{B}_0(0)\mathcal{T}'(0)[h]0 + \mathcal{B}_0(0)\mathcal{T}(0)h \\
&= \mathcal{B}_0(0)\mathcal{T}(0)h,
\end{aligned}$$

where we have used the linearity of the mappings $\mathcal{T}(0)$, $\mathcal{B}_0'(0)[h]$, and $\mathcal{T}'(0)[h]$. More explicitly,

$$\Phi'(0)[h] = \phi_0^* \left(\left. \frac{\partial u}{\partial n} \right|_{\Gamma_c} \right),$$

where u is the solution of

$$\left.\begin{aligned}
\Delta u &= 0 \quad \text{in } \mathbb{T}^n \times (0, c), \\
u &= h \quad \text{on } \mathbb{T}^n \times \{c\}, \\
\frac{\partial u}{\partial n} &= 0 \quad \text{on } \mathbb{T}^n \times \{0\},
\end{aligned}\right\} \tag{2.13}$$

i.e. A is essentially the Dirichlet-Neumann operator on $\mathbb{T}^n \times \{c\}$ for the Laplacian with homogeneous Neumann boundary condition on $\mathbb{T}^n \times \{0\}$. Due to the product structure of $\mathbb{T}^n \times (0, c)$, (2.13) is solvable by separation of variables. Thereby one obtains complete information on the closure A_{L^2} of A to $L^2(\mathbb{T}^n)$.

Lemma 2.3 *(The linearized operator on L^2)*
The operator A_{L^2} has a complete orthogonal system of eigenfunctions e_k, $k \in \mathbb{Z}^n$, given by

$$\mathsf{e}_k(\theta) = e^{ik\cdot\theta}, \qquad \theta \in \mathbb{T}^n.$$

These eigenfunctions correspond to the eigenvalues $\lambda_k := |k| \tanh c|k|$, where $|k| = |k_1| + \ldots + |k_n|$.

The proof rests on the well-known completeness of the system $\{\mathsf{e}_k \,|\, k \in \mathbb{Z}^n\}$ in $L^2(\mathbb{T}^n)$ and on the explicit solution of (2.13) for $h = \mathsf{e}_k$, cf. (2.18) below. Observe that, in particular, the operators A_{L^2} and P commute, $D(A_{L^2}) = H^1(\mathbb{T}^n)$, $0 \in \sigma(A_{L^2})$, and all eigenvalues of A_{L^2} are nonnegative.

By standard arguments from functional analysis together with earlier results on A, the information on the spectrum of A_{L^2} can be carried over to A.

Lemma 2.4 *(The linearized operator in $h^{1+\alpha}$)*
Consider A as an unbounded, densely defined operator in $h^{1+\alpha}(\mathbb{T}^n)$ with $D(A) = h^{2+\alpha}(\mathbb{T}^n)$. The spectra of A and A_{L^2} coincide.

Proof: It can be shown as in the proof of Corollary 6.3 in [7] that

$$A \in \mathcal{H}\left(h^{2+\alpha}(\mathbb{T}^n), h^{1+\alpha}(\mathbb{T}^n)\right). \tag{2.14}$$

In particular, the resolvent set of A is not empty. Moreover, the embedding $h^{2+\alpha}(\mathbb{T}^n) \hookrightarrow\hookrightarrow h^{1+\alpha}(\mathbb{T}^n)$ is compact. Thus, A is an operator with compact resolvent and therefore ([9] Theorem III.8.29) its spectrum consists only of eigenvalues having finite multiplicity. Because of $\mathsf{e}_k \in h^{2+\alpha}(\mathbb{T}^n)$ for all $k \in \mathbb{Z}^n$, these functions are eigenfunctions of A corresponding to the eigenvalues λ_k. On the other hand, due to the embedding $h^{2+\alpha}(\mathbb{T}^n) \hookrightarrow H^1(\mathbb{T}^n)$, all eigenfunctions of A are also eigenfunctions of A_{L^2}. This implies the assertion. ■

The only obstacle for a proof of exponential stability of $g = 0$ in (2.9) is the fact that $0 \in \sigma(A)$. However, the eigenspace E_0 corresponding to this eigenvalue consists precisely of the constants, and by Lemma 2.2 it is sufficient to consider the restriction $\tilde{A}$ of A to $\tilde{h}^{1+\alpha}(\mathbb{T}^n)$ which is a complement of E_0 in $h^{1+\alpha}$. The following lemma provides the necessary results on $\tilde{A}$. To begin, observe that $\tilde{h}^{2+\alpha}(\mathbb{T}^n)$ is dense in $\tilde{h}^{1+\alpha}(\mathbb{T}^n)$.

Lemma 2.5 *(The linearized operator on $\tilde{h}^{1+\alpha}(\mathbb{T}^n)$)*
Consider $\tilde{A}$ as an unbounded, densely defined operator on $\tilde{h}^{1+\alpha}(\mathbb{T}^n)$ with $D(\tilde{A}) = \tilde{h}^{2+\alpha}(\mathbb{T}^n)$. Then

$$\tilde{A} \in \mathcal{H}\left(\tilde{h}^{2+\alpha}(\mathbb{T}^n), \tilde{h}^{1+\alpha}(\mathbb{T}^n)\right) \tag{2.15}$$

and

$$\sigma(\tilde{A}) = \{\lambda_k \,|\, k \in \mathbb{Z}^n \backslash \{0\}\}.$$

Proof: Using the direct decomposition

$$h^{k+\alpha}(\mathbb{T}^n) = \tilde{h}^{k+\alpha}(\mathbb{T}^n) \oplus E_0, \qquad k = 1, 2,$$

and the fact that $AP \supset PA$ we can interpret A as a matrix operator

$$A = \begin{bmatrix} \tilde{A} & 0 \\ 0 & 0 \end{bmatrix}$$

and conclude (2.15) from [1], Corollary 1.6.3. For $k \in \mathbb{Z}^n \backslash \{0\}$, one has $\mathsf{e}_k \in \tilde{h}^{2+\alpha}(\mathbb{T}^n)$ and hence $\lambda_k \in \sigma(\tilde{A})$. On the other hand, given $\lambda \in \rho(A)$ and $f \in \tilde{h}^{1+\alpha}(\mathbb{T}^n)$, the equation

$$(A - \lambda I)u = f \tag{2.16}$$

is uniquely solvable in $h^{2+\alpha}(\mathbb{T}^n)$. Applying P to both sides yields

$$P(A - \lambda I)u = (A - \lambda I)Pu = Pf = f,$$

i.e. Pu solves (2.16), hence $u = Pu \in \tilde{h}^{2+\alpha}(\mathbb{T}^n)$. Thus $\lambda \in \rho(\tilde{A})$, and consequently $\sigma(\tilde{A}) \subset \sigma(A)$.

It remains to show that $0 \in \rho(\tilde{A})$. Recall that the equation $\tilde{A}u = f$ is equivalent to the existence of a $v \in h^{2+\alpha}(\mathbb{T}^n \times (0,c))$ such that

$$\left.\begin{array}{rcll} \Delta v &=& 0 & \text{in } \mathbb{T}^n \times (0,c), \\ \dfrac{\partial v}{\partial n} &=& f & \text{on } \mathbb{T}^n \times \{c\}, \\ \dfrac{\partial v}{\partial n} &=& 0 & \text{on } \mathbb{T}^n \times \{0\}, \\ v &=& u & \text{on } \mathbb{T}^n \times \{c\}. \end{array}\right\} \tag{2.17}$$

If $f = 0$ then it follows from the maximum principle that any $v \in h^{2+\alpha}(\mathbb{T}^n \times (0,c))$ satisfying $(2.17)_1$– $(2.17)_3$ is a constant, hence u is a constant and thus $u = 0$ since $u \in \tilde{h}^{2+\alpha}(\mathbb{T}^n)$. This proves the injectivity of $\tilde{A}$. To show surjectivity, we fix $f \in \tilde{h}^{1+\alpha}(\mathbb{T}^n)$ and approximate f in the $BUC^{1+\alpha}$-norm by smooth f^j, $j \in \mathbb{N}$. We have $f^j = \sum_{k\in\mathbb{Z}^n\setminus\{0\}} a_k^j \mathsf{e}_k$ where the a_k^j are rapidly decreasing in k for any fixed j. Setting

$$v^j(\theta, y) := \sum_{k\in\mathbb{Z}^n\setminus\{0\}} \frac{a_k^j}{|k| \sinh c|k|} \mathsf{e}_k(\theta) \cosh |k| y \tag{2.18}$$

and $u^j := v^j|_{\mathbb{T}^n\times\{c\}}$ we find by standard arguments that $u^j \in \tilde{C}^\infty(\mathbb{T}^n)$ and

$$\left.\begin{array}{rcll} \Delta v^j &=& 0 & \text{in } \mathbb{T}^n \times (0,c), \\ \dfrac{\partial v^j}{\partial n} &=& f^j & \text{on } \mathbb{T}^n \times \{c\}, \\ \dfrac{\partial v^j}{\partial n} &=& 0 & \text{on } \mathbb{T}^n \times \{0\}, \\ v^j &=& u^j & \text{on } \mathbb{T}^n \times \{c\}, \end{array}\right\} \tag{2.19}$$

hence $\tilde{A}u^j = f^j$. For $j, l \in \mathbb{N}$, the well-known Schauder estimate for solutions of $(2.19)_1$–$(2.19)_3$ yields

$$\|v^j - v^l\|_{BUC^{2+\alpha}(\mathbb{T}^n\times(0,c))} \leq C \left(\|f^j - f^l\|_{BUC^{1+\alpha}(\mathbb{T}^n)} + \|v^j - v^l\|_{L^2(\mathbb{T}^n\times(0,c))} \right).$$

(Usually, in the last term on the right the $BUC^0(\mathbb{T}^n \times (0,c))$-norm is used. However, it follows from [14], Theorem 12.12, that both formulations are equivalent.) Clearly, one has

$$\|u^j - u^l\|_{BUC^{2+\alpha}(\mathbb{T}^n)} \leq C \|v^j - v^l\|_{BUC^{2+\alpha}(\mathbb{T}^n\times(0,c))}.$$

Moreover, the obvious estimate

$$\|v^j - v^l\|_{L^2(\mathbb{T}^n\times(0,c))} \leq C \|f^j - f^l\|_{L^2(\mathbb{T}^n)} \leq C \|f^j - f^l\|_{BUC^{1+\alpha}(\mathbb{T}^n)}$$

shows that $\{v^j\}$ and $\{u^j\}$ are Cauchy sequences in $h^{2+\alpha}(\mathbb{T}^n \times (0,c))$ and $\tilde{h}^{2+\alpha}(\mathbb{T}^n)$, converging to some $v \in h^{2+\alpha}(\mathbb{T}^n \times (0,c))$ and $u \in \tilde{h}^{2+\alpha}(\mathbb{T}^n)$, respectively. Passage to the limit in (2.19) yields $\tilde{A}u = f$. ■

We are ready now for the proof of the main result of this section:

Theorem 2.6 *(Exponential stability)*

For any $\omega < \lambda_1 = \tanh c$, there are positive constants r and C such that for any $g_0 \in \mathcal{O}$ with $\|g_0\|_{BUC^{2+\alpha}(\mathbb{T}^n)} < r$ we have $t^+(g_0) = \infty$ and for the solution g of (2.9) the estimate

$$\|g(t)\|_{BUC^{2+\alpha}(\mathbb{T}^n)} + \|g'(t)\|_{BUC^{1+\alpha}(\mathbb{T}^n)} \leq Ce^{-\omega t}\|g_0\|_{BUC^{2+\alpha}(\mathbb{T}^n)} \qquad \forall t \geq 0$$

holds.

Proof: Setting $X := \tilde{h}^{1+\alpha}(\mathbb{T}^n)$, $D := \tilde{h}^{2+\alpha}(\mathbb{T}^n)$, $G := \tilde{A} - \Phi$, $\omega_0 := -\sup\{\operatorname{Re}\lambda \mid \lambda \in \sigma(-\tilde{A})\} = \lambda_1$, we find that all assumptions of Theorem 9.1.2. in [12] are satisfied because of (2.11), (2.12), and Lemmas 2.2 and 2.5. Our result follows from this theorem. ∎

3 Flows in compressible porous media

In the description of the problem as well as in the notation, we will follow [6] in this section.

We consider a free boundary flow problem in a deformable porous medium. The geometric setting is the same as before, i.e. we assume that for fixed time the liquid occupies the domain

$$\tilde{\Omega}_f := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \,|\, 0 < y < f(x)\}\,,$$

where $f \in C^1(\mathbb{R}^n)$ is a uniformly positive function whose graph

$$\tilde{\Gamma}_f := \{(x,y) \in \mathbb{R}^n \times \mathbb{R} \,|\, y = f(x)\}$$

represents the interface between the saturated and the dry part of the porous medium.

Accordingly, we define

$$\tilde{\mathcal{A}} := \left\{ f \in C^1(\mathbb{R}^n) \,|\, \inf_{x\in\mathbb{R}^n} f(x) > 0 \right\}$$

and for $f \in C^1([0,T], \tilde{\mathcal{A}})$

$$\tilde{\Omega}_{f,T} := \left\{ (t,z) \in [0,T] \times \mathbb{R}^{n+1} \,|\, z \in \tilde{\Omega}_{f(t)} \right\}.$$

Note that $\tilde{\Omega}_{f,T}$ is an unbounded striplike domain in $[0,T] \times \mathbb{R}^{n+1}$. We are looking for a pair (u,f) of (sufficiently smooth) functions $u : \tilde{\Omega}_{f,T} \longrightarrow \mathbb{R}$, $f : [0,T] \longrightarrow \tilde{\mathcal{A}}$ satisfying the system of equations

$$\left.\begin{array}{rcll}
\partial_t u - \Delta u & = & 0 & \text{in } \tilde{\Omega}_{f(t)}, \quad t > 0, \\
\partial_{n+1} u & = & 0 & \text{on } \tilde{\Gamma}_0, \quad t \geq 0, \\
u & = & f & \text{on } \tilde{\Gamma}_{f(t)}, \quad t \geq 0, \\
\partial_t f + \sqrt{1 + |\nabla_n f|^2}\partial_\nu u & = & 0 & \text{on } \tilde{\Gamma}_{f(t)}, \quad t > 0, \\
u(0,\cdot) & = & u_0 & \text{in } \tilde{\Omega}_{f_0}, \\
f(0) & = & f_0 & \text{on } \mathbb{R}^n
\end{array}\right\} \qquad (3.1)$$

with given initial data (u_0, f_0). As in the previous section, $\partial_\nu u$ denotes the derivative of $u(t,\cdot)$ with respect to the outer unit normal on $\tilde{\Gamma}_{f(t)}$ and the Laplacian involves the spatial coordinates only.

The crucial difference of (3.1) to the problem (2.1),(2.2) is the fact that the governing equation $(3.1)_1$ is parabolic, hence it is not possible here to determine $u(t,\cdot)$ for given $f(t)$ at the fixed time $t \in [0,T]$. Instead, one has to consider a pair of coupled Cauchy problems for u and f. The system (3.1) occurs as a model for the gravity-driven motion of a compressible Newtonian liquid in a linearly elastic porous medium, where u represents the velocity potential of the flow. For the derivation of this model we refer to [6] and the references therein. Note, moreover, that (3.1) can also be interpreted as a one-phase Stefan problem with mixed boundary conditions.

As in the incompressible case, we demand spatial periodicity of f and u, i.e. (2.3) and (2.4), and consider (3.1) with $\tilde{\mathcal{A}}$, $\tilde{\Omega}_{f(t)}$, $\tilde{\Gamma}_{f(t)}$, $\tilde{\Omega}_{f,T}$ replaced by

$$
\begin{aligned}
\mathcal{A} &:= \left\{ f \in C^1(\mathbb{T}^n) \,|\, \min_{x\in\mathbb{T}^n} f(x) > 0 \right\}, \\
\Omega_{f(t)} &:= \left\{ (x,y) \in \mathbb{T}^n \times \mathbb{R} \,|\, 0 < y < f(t,x) \right\}, \\
\Gamma_{f(t)} &:= \left\{ (x,y) \in \mathbb{T}^n \times \mathbb{R} \,|\, y = f(t,x) \right\}, \\
\Omega_{f,T} &:= \left\{ (t,z) \in [0,T] \times \mathbb{T}^n \times \mathbb{R} \,|\, z \in \Omega_{f(t)} \right\}.
\end{aligned}
$$

Let us transform (3.1) to a problem on the fixed reference domain $\Omega := \mathbb{T}^n \times (0,1)$ using a diffeomorphism depending on the unknown function f. Following [6], we fix $p > n+2$, $s \in \left(0, \frac{1}{p}\right)$, $\alpha \in \left[\frac{1}{p}, 1 - \frac{n+1}{p}\right]$. Furthermore, we write $\Gamma_i := \mathbb{T}^n \times \{i\}$, $i = 0, 1$, and denote by γ_i the trace operator from Ω to the boundary component Γ_i. Throughout this section we write $W_p^t(\Omega)$ and $W_p^t(\Gamma_0)$ for the Sobolev-Slobodeckii spaces over Ω and Γ_0, respectively.

It is shown in [6], Section 2, that there is an extension operator

$$
\rho_0 \in \mathcal{L}\left(W_p^{k+s-\frac{1}{p}}(\Gamma_0), W_p^{k+s}(\Omega) \right), \quad k = 1, 2,
$$

such that $\gamma_0 \rho_0$ is the identity on $W_p^{k+s-\frac{1}{p}}(\Gamma_0)$ and $\rho_0 g$ vanishes near Γ_1 for all $g \in W_p^{1+s-\frac{1}{p}}(\Gamma_0)$. (Here and in the following, references to results in [6] are to be understood with the necessary changes due to our periodicity assumptions, i.e. with $\mathbb{R}^n$ replaced by $\mathbb{T}^n$. The results remain valid, and there are no essential new difficulties in the proofs.)

We set

$$
Q_0 := \int_{\Omega_{f_0}} u_0 \, dx - \frac{1}{2} \int_{\mathbb{T}^n} f_0^2 \, dx' + \int_{\mathbb{T}^n} f_0 \, dx'. \tag{3.2}
$$

It will be shown later that for our purposes it is natural to demand $Q_0 > 0$. Making this assumption, we define $d > 0$ as the (unique) solution of

$$
\frac{(2\pi)^n}{2} d(d+2) = Q_0 \tag{3.3}
$$

(see the remark after Lemma 3.3 below). Moreover, with

$$
\mathcal{D} := \left\{ g \in W_p^{2+s-\frac{1}{p}}(\Gamma_0) \,|\, \inf_{(x,y)\in\Omega} (d + \rho_0 g) > 0, \right.
$$

$$\sup_{(x,y)\in\Omega}\sqrt{|\nabla_x(\rho_0 g)(x,y)|^2+|(1-y)\partial_{n+1}(\rho_0 g)(x,y)-(\rho_0 g)(x,y)|^2}<(1\wedge\delta)\Big\}$$

and $\delta\in(0,d)$ fixed, we define for any $g\in\mathcal{D}$ the mapping $\theta_g:\Omega\longrightarrow\Omega_{g+d}$ by

$$\theta_g(x,y):=(x,(1-y)(d+(\rho_0 g)(x,y)).$$

The corresponding pull-back and push-forward operators will be denoted by θ_g^* and θ_*^g, respectively. It is shown in [6], Lemma 2.2.a,b and Corollary 2.3, that $\mathcal{D}$ is an open neighborhood of 0 in $W_p^{2+s-\frac{1}{p}}(\Gamma_0)$, and $\theta_g\in\mathrm{Diff}^{1+\alpha}(\Omega,\Omega_{g+d})$ with

$$\theta_*^g\in\mathcal{L}\left(W_p^{k+s}(\Omega),W_p^{k+s}(\Omega_{g+d})\right),\qquad k=0,1,2,$$

for any $g\in\mathcal{D}$.

This justifies the following approach: Using the obvious identification of Γ_0 and $\mathbb{T}^n$ we demand $f(t,\cdot)-d\in\mathcal{D}$, $u(t,\cdot)\in W_p^{2+s}(\Omega_{f(t)})$ for $t\in[0,T]$ and define functions g on $[0,T]\times\Gamma_0$ and v on $[0,T]\times\Omega$ by

$$g:=f-d,\quad v(t,\cdot):=\theta_{g(t)}^*(u(t,\cdot)-d).$$

The problem (3.1) for (u,f) on the unknown domain $\Omega_{f,T}$ can be transformed now to a problem for (v,g) on the fixed cylindrical domain $[0,T]\times\Omega$, namely,

$$\left.\begin{array}{rcll}\partial_t v+\mathcal{A}(g)v+\mathcal{R}(v,g)&=&0&\text{in }(0,T]\times\Omega,\\ v&=&g&\text{on }[0,T]\times\Gamma_0,\\ \mathcal{B}_1(g)v&=&0&\text{on }[0,T]\times\Gamma_1,\\ \partial_t g+\mathcal{B}_0(g)v&=&0&\text{on }(0,T]\times\Gamma_0,\\ v(0,\cdot)&=&v_0&\text{in }\Omega,\\ g(0,\cdot)&=&g_0&\text{on }\Gamma_0,\end{array}\right\}\tag{3.4}$$

where $g_0:=f_0-d\in\mathcal{D}$ and $v_0:=\theta_{g_0}^*(u_0-d)\in W_p^{2+s}(\Omega)$ are the given initial data,

$$\begin{array}{rcl}\mathcal{A}(g)v&:=&-\theta_g^*\Delta(\theta_*^g v),\\ \mathcal{B}_0(g)&:=&\theta_g^*(\gamma_{g+d}\nabla(\theta_*^g v)\cdot n_0),\\ \mathcal{B}_1(g)&:=&\theta_g^*(\gamma_0\nabla(\theta_*^g v)\cdot n_1),\\ \mathcal{R}(v,g)&:=&\dfrac{(y-1)\partial_{n+1}v}{G(g)}\rho_0(\mathcal{B}_0(g)v),\\ G(g)&:=&d+\rho_0 g+(y-1)\partial_{n+1}(\rho_0 g),\end{array}$$

$n_0:=(-\nabla_x g,1)$ and $n_1:=(0,-1)$ denote outer normals on Γ_f and Γ_0, respectively, and γ_{g+d} is the trace operator from Ω_{g+d} to Γ_{g+d}. Note that $g\in\mathcal{D}$ implies that $G(g)$ has a positive lower bound on Ω and

$$G(g)=|\det D\theta_g|.\tag{3.5}$$

For an explicit coordinate representation of the transformed operators $\mathcal{A}(g)$ and $\mathcal{B}_i(g)$ we refer to [6], Lemma 2.5. By transformation to Ω_{g+d} and Gauss' theorem one has

$$-\int_\Omega\mathcal{A}(g)v|\det D\theta_g|\,dx\;=\;\int_{\Omega_{g+d}}\Delta(\theta_*^g v)\,dx=\int_{\partial\Omega_{g+d}}\partial_\nu(\theta_*^g v)\,d\sigma$$

$$
\begin{aligned}
&= \int_{\Gamma_{g+d}} \frac{\gamma_{g+d}\nabla(\theta_*^g v)\cdot n_0}{\sqrt{1+|\nabla_x g|^2}}\,d\sigma + \int_{\Gamma_0} \gamma_0 \nabla(\theta_*^g v)\cdot n_1\,dx' \\
&= \int_{\Gamma_0} \mathcal{B}_0(g)v\,dx' + \int_{\Gamma_1} \mathcal{B}_1(g)v\,dx' \qquad (3.6)
\end{aligned}
$$

for $v \in W_p^{2+s}(\Omega)$, $g \in \mathcal{D}$.

In order to discuss (3.4) in a suitable abstract framework we set

$$
\begin{aligned}
E_0 &:= W_p^s(\Omega) \times W_p^{1+s-\frac{1}{p}}(\Gamma_0), \\
E_1 &:= \left\{ (v,g) \in W_p^{2+s}(\Omega) \times W_p^{2+s-\frac{1}{p}}(\Gamma_0) \,|\, \gamma_0 v = g, \gamma_1 \partial_{n+1} v = 0 \right\}, \\
D_1 &:= \{(v,g) \in E_1 \,|\, g \in \mathcal{D}\}
\end{aligned}
$$

and define $\Pi : D_1 \longrightarrow E_0$ by

$$
\Pi(v,g) := (\mathcal{A}(g)v + \mathcal{R}(v,g), \mathcal{B}_0(g)v).
$$

Following [6], one finds that D_1 is open in E_1 and $\Pi \in C^\omega(D_1, E_0)$. With these definitions, we reformulate (3.4) as an abstract Cauchy problem

$$
\left.
\begin{aligned}
\partial_t z + \Pi(z) &= 0, \\
z(0) &= z_0 := (v_0, g_0)
\end{aligned}
\right\} \qquad (3.7)
$$

in E_0 and it is clear that any solution $z \in C([0,T], D_1) \cap C^1([0,T], E_0)$ yields a solution of (3.4) via $z = (v,g)$.

Moreover, for $\rho > 0$, $\sigma > 0$ we define

$$
W_{\rho,\sigma} := \left\{ (v,g) \in D_1 \;\middle|\; \|v\|_{W_p^{2+s}(\Omega)} < \rho,\ \|\partial_{n+1} v\|_{C^\alpha(\Omega)} < \sigma,\ \|\gamma_0 \partial_{n+1} v\|_{W_p^{1+s-\frac{1}{p}}(\Gamma_0)} < \sigma \right\}.
$$

It is easily seen that $W_{\rho,\sigma}$ is an open neighborhood of 0 in D_1.

The crucial result on the parabolic character of (3.7) is the following ([6] Corollary 3.6):

Lemma 3.1 *(Parabolicity of* (3.7)*)*

For any $\rho > 0$ there is a $\sigma > 0$ such that

$$
\partial \Pi(z) \in \mathcal{H}(E_1, E_0) \qquad \forall z \in W_{\rho,\sigma}.
$$

This is the crucial fact in the proof of the classical solvability of (3.7). More precisely, the following results are proved in [6], cf. Theorem 3.8.:

Theorem 3.2 *(Solvability of* (3.7) *and time regularity of the solution)*

Let $\rho > 0$ be given. There exists $\sigma > 0$ such that for each $z_0 \in W_{\rho,\sigma}$ there is a $t^+ := t^+(z_0) > 0$ and a unique maximal solution

$$
z \in C([0,t^+), W_{\rho,\sigma}) \cap C^1([0,t^+), E_0)
$$

of (3.7)*. Moreover,*

$$
z \in C^\infty\left((0,t^+), W_p^{2+\tilde{s}}(\Omega) \times W_p^{2+\tilde{s}-\frac{1}{p}}(\Gamma_0)\right)
$$

for any $\tilde{s} \in (0,s)$.

It is easily checked that $z = (0,0)$ is a stationary solution of (3.7). As in the previous section, the first step in the investigation of its solvability is the identification of a conserved quantity.

Lemma 3.3 *(A conserved quantity)*
Let $z = (v,g)$ be a solution of (3.7) *in the sense of Theorem* 3.2. *Then the functional*

$$I(z) := \int_\Omega (v+d)|\det D\theta_g|\,dx - \frac{1}{2}\int_{\Gamma_0} (g+d)^2\,dx' + \int_{\Gamma_0} (g+d)\,dx'$$

is constant on $[0,t^+)$.

Proof: Due to the continuous embeddings $E_1 \hookrightarrow BUC^{1+\alpha}(\Omega) \times BUC^{1+\alpha}(\Gamma_0)$ and $W_p^{2+\tilde{s}}(\Omega) \times W_p^{2+\tilde{s}-\frac{1}{p}}(\Gamma_0) \hookrightarrow BUC^{1+\alpha}(\Omega) \times BUC^{1+\alpha}(\Gamma_0)$ we have

$$z \in C\left([0,t^+), BUC^{1+\alpha}(\Omega) \times BUC^{1+\alpha}(\Gamma_0)\right) \cap C^1\left((0,t^+), BUC^{1+\alpha}(\Omega) \times BUC^{1+\alpha}(\Gamma_0)\right),$$

and the mapping $t \mapsto I(z(t))$ is continuous on $[0,t^+)$ and differentiable on $(0,t^+)$ with derivative

$$\begin{aligned}
\dot{I}(z) &= \int_\Omega \partial_t v|\det D\theta_g|\,dx + \int_{\Gamma_0}\int_0^1 (v+d)\partial_y[(y-1)\rho_0(\partial_t g)]\,dy dx' \\
&\quad - \int_{\Gamma_0} \partial_t g(g+d)\,dx' + \int_{\Gamma_0} \partial_t g\,dx' \\
&= \int_\Omega \partial_t v|\det D\theta_g|\,dx + \int_{\Gamma_0}\int_0^1 \partial_y[(v+d)(y-1)\rho_0(\partial_t g)]\,dy dx' \\
&\quad - \int_{\Gamma_0}\int_0^1 \frac{\partial_y v(y-1)\rho_0(\partial_t g)}{G(g)}|\det D\theta_g|\,dy dx' - \int_{\Gamma_0} \partial_t g(g+d)\,dx' + \int_{\Gamma_0} \partial_t g\,dx' \\
&= \int_\Omega (\partial_t v + \mathcal{R}(v,g))|\det D\theta_g|\,dx - \int_{\Gamma_0} (g+d)\mathcal{B}_0(g)v\,dx' \\
&\quad - \int_{\Gamma_0} \partial_t g(g+d)\,dx' + \int_{\Gamma_0} \partial_t g\,dx' \\
&= -\int_\Omega \mathcal{A}(g)v|\det D\theta_g|\,dx + \int_{\Gamma_0} \partial_t g\,dx' \\
&= \int_{\Gamma_0} \mathcal{B}_0(g)v\,dx' + \int_{\Gamma_1} \mathcal{B}_1(g)v\,dx' + \int_{\Gamma_0} \partial_t g\,dx' = 0,
\end{aligned}$$

where (3.5) and (3.6) have been used. ∎

Remark: It is natural to reinterpret Lemma 3.3 in terms of the original problem (3.1). For solutions of this problem, the quantity

$$Q(u,f) = \int_{\Omega_{f(t)}} u\,dx - \frac{1}{2}\int_{\mathbb{T}^n} f^2\,dx + \int_{\mathbb{T}^n} f\,dx$$

is conserved during the evolution. For positive $Q(u,f)$, this fact determines to which of the possible equilibria $(u,f) = (c,c)$, $c > 0$, a solution with given initial data (u_0, f_0)

may tend. In view of (3.2), (3.3), and $Q(c,c) = \frac{(2\pi)^n}{2} c(c+2)$ we find $c = d$. On the other hand, for initial data with

$$Q(u_0, f_0) < 0 \tag{3.8}$$

one obtains that a corresponding solution cannot tend to any equilibrium. An elementary calculation shows that (3.8) can hold only if $\partial_{n+1} u_0$ is relatively large. This gives rise to the conjecture that solutions of (3.1) with (3.8) will cease to exist at a finite time at which the strict positivity of f is lost, i.e. at which the upper boundary component of the liquid domain will touch the bottom. (Of course, it is by no means clear whether there are further blowup mechanisms.) Note, moreover, that if $Q_0 > 0$ then for any $\rho > 0$ there is a $\sigma > 0$ such that $I(v,g) > 0$ for all $(v,g) \in W_{\rho,\sigma}$.

In a next step, we derive the necessary information on the spectrum of the operator $A := \partial\Pi(0)$. Due to the dense embedding $E_1 \hookrightarrow E_0$, A can and will be considered as an unbounded, densely defined operator on E_0 with $D(A) = E_1$. Lemma 3.3 in [6] applied to $z = (v,g) = 0$ shows that

$$A(w,h) = (\mathcal{A}(0)w, \mathcal{B}_0(0)w).$$

Lemma 3.4 *(The linearized operator)*

The spectrum of A consists only of nonnegative real eigenvalues.

Proof: From Lemma 3.1 we know that $A \in \mathcal{H}(E_1, E_0)$. As in the proof of Lemma 2.4, it follows from this together with the compactness of the embedding $E_1 \hookrightarrow E_0$ that $\sigma(A)$ consists only of eigenvalues. On E_1 we define the scalar product $(\cdot|\cdot)$ by

$$\big((v,g)\big|(w,h)\big) := d\int_\Omega vw\,dx + \int_{\Gamma_0} gh\,dx' = \int_{\Omega_d} \theta_*^0 v \theta_*^0 w\,dx + \int_{\Gamma_d} \theta_*^0 g \theta_*^0 h\,dx'.$$

The operator A is symmetric and positive semidefinite with respect to this scalar product. Indeed, for arbitrary $(v,g),(w,h) \in E_1$,

$$\begin{aligned}
\big(A(v,g)\big|(w,h)\big) &= -\int_{\Omega_d} \Delta(\theta_*^0 v)\theta_*^0 w\,dx + \int_{\Gamma_d} \partial_\nu(\theta_*^0 v)\theta_*^0 h\,dx' \\
&= -\int_{\Omega_d} \theta_*^0 v \Delta(\theta_*^0 w)\,dx + \int_{\Gamma_d} \big(\theta_*^0 v \partial_\nu(\theta_*^0 w) - \partial_\nu(\theta_*^0 v)\theta_*^0 w\big)\,dx' \\
&\quad + \int_{\Gamma_0} \big(\theta_*^0 v \partial_\nu(\theta_*^0 w) - \partial_\nu(\theta_*^0 v)\theta_*^0 w\big)\,dx' + \int_{\Gamma_d} \partial_\nu(\theta_*^0 v)\theta_*^0 h\,dx' \\
&= -\int_{\Omega_d} \theta_*^0 v \Delta(\theta_*^0 w)\,dx + \int_{\Gamma_d} \theta_*^0 g \partial_\nu(\theta_*^0 w)\,dx' = \big((v,g)\big|A(w,h)\big)
\end{aligned}$$

and

$$\begin{aligned}
&\big(A(v,g)\big|(v,g)\big) = -\int_{\Omega_d} \Delta(\theta_*^0 v)\theta_*^0 v\,dx + \int_{\Gamma_d} \partial_\nu(\theta_*^0 v)\theta_*^0 g\,dx' \\
= &\int_{\Omega_d} |\nabla(\theta_*^0 v)|^2\,dx - \int_{\Gamma_d} \partial_\nu(\theta_*^0 v)\theta_*^0 v\,dx' - \int_{\Gamma_0} \partial_\nu(\theta_*^0 v)\theta_*^0 v\,dx' + \int_{\Gamma_d} \partial_\nu(\theta_*^0 v)\theta_*^0 g\,dx' \\
= &\int_{\Omega_d} |\nabla(\theta_*^0 v)|^2\,dx \geq 0,
\end{aligned}$$

where we have used that $\gamma_0 v = g$, $\gamma_0 w = h$, $\gamma_1 \partial_{n+1} v = \gamma_1 \partial_{n+1} w = 0$.

On the complexification of E_1 we introduce the scalar product $[\cdot|\cdot]$ by

$$[z_1|z_2] = (x_1|x_2) + (y_1|y_2) + i\big((y_1|x_2) - (y_2|x_1)\big),$$

where $z_j = x_j + iy_j$, $x_j, y_j \in E_1$, $j = 1,2$. Suppose now λ is an eigenvalue of A with eigenfunction $z \neq 0$. This implies

$$\lambda[z|z] = [Az|z] = [z|Az] = \overline{\lambda}[z|z],$$

hence $\overline{\lambda} = \lambda \in \mathbb{R}$, and z can and will be chosen to be real. Then $(Az|z) = \lambda(z|z) \geq 0$, hence $\lambda \geq 0$. ∎

As in the incompressible case discussed above, it is not hard to see that 0 is an eigenvalue of A. Therefore, by exploiting the conservation of I, one again has to consider a reduced system in which this eigenvalue does not occur. In the present situation, however, the evolution does not proceed in an invariant subspace but, according to Lemma 3.3, in the invariant subset

$$\mathcal{M}' := \{z \in E_1 \,|\, I(z) = Q_0\}.$$

Using (3.5), it is straightforward to calculate the Fréchet derivative of $I \in C^\omega(D_1, \mathbb{R})$ at $(v,g) \in D_1$ as

$$\begin{aligned} \partial I(v,g)[w,h] &= \int_\Omega w|\det D\theta_g|\,dx + \int_{\Gamma_0}\int_0^1 (v+d)\partial_y[(y-1)\rho_0 h]\,dydx' \\ &\quad + \int_{\Gamma_0} h(1-(g+d))\,dx', \end{aligned}$$

$(w,h) \in E_1$. This linear functional is nondegenerate, and it follows that $\mathcal{M} := \mathcal{M}' \cap D_1$ is a smooth hypersurface in E_1. Note that 0 is the only stationary point of (3.7) in $\mathcal{M}$. Our strategy is to describe the restriction of the dynamical system given by (3.7) to $\mathcal{M}$ using a local diffeomorphism mapping $\mathcal{M}$ near 0 onto its tangent space $\tilde{E}_1 := T_0\mathcal{M} = \ker \partial I(0,0)$ where

$$\begin{aligned} \partial I(0,0) &= d\int_\Omega w\,dx + d\int_{\Gamma_0}\int_0^1 \partial_y[(y-1)\rho_0 h]\,dydx' + (1-d)\int_{\Gamma_0} h\,dx' \\ &= d\int_\Omega w\,dx + \int_{\Gamma_0} h\,dx'. \end{aligned}$$

Let us define the projection $P \in \mathcal{L}(E_0) \cap \mathcal{L}(E_1)$ by

$$P(v,g) := (v,g) - \frac{(1,1)}{(2\pi)^n(d+1)}\big((v,g)\big|(1,1)\big).$$

Clearly, $P(E_1) = \tilde{E}_1$ and $\ker P = \{(c,c)\,|\, c \in \mathbb{R}\}$. Defining $\tilde{E}_0 := P(E_0)$, we obtain the direct decomposition

$$E_i = \tilde{E}_i \oplus \ker P, \qquad i = 0,1.$$

The operator A vanishes on $\ker P$, hence $A = AP$. Furthermore,

$$(I-P)A(w,h) = \frac{(1,1)}{(2\pi)^n(d+1)}\big(A(w,h)\big|(1,1)\big) = \frac{(1,1)}{(2\pi)^n(d+1)}\big((w,h)\big|A(1,1)\big) = 0$$

for $(w,h) \in E_1$, hence $A = AP = PA$ on E_1. We define the restriction $\tilde{A} := A|\tilde{E}_1$ and collect the properties of $\tilde{A}$ that will be important for our purpose.

Lemma 3.5 *(The linearized operator on the restricted space)*
$\tilde{A} \in \mathcal{H}(\tilde{E}_1, \tilde{E}_0)$ and the spectrum of $\tilde{A}$ consists only of positive eigenvalues.

Proof: By the same arguments as in the proof of Lemma 2.5, we obtain $\tilde{A} \in \mathcal{H}(\tilde{E}_1, \tilde{E}_0)$ and $\sigma(\tilde{A}) \subset \sigma(A)$. It remains to show that $\tilde{A} \in \mathrm{Isom}(\tilde{E}_1, \tilde{E}_0)$. This will be done by discussing the Neumann BVP

$$\left.\begin{array}{rcll} -\Delta u &=& \phi & \text{in } \Omega_d, \\ \partial_\nu u &=& \psi & \text{on } \Gamma_d, \\ \partial_\nu u &=& 0 & \text{on } \Gamma_0, \\ \int_{\Omega_d} u\,dx + \int_{\Gamma_d} u\,dx' &=& 0. & \end{array}\right\} \tag{3.9}$$

We will show that that for any $(\phi,\psi) \in W_p^s(\Omega_d) \times W_p^{s+1-\frac{1}{p}}(\Gamma_d)$ satisfying

$$\int_{\Omega_d} \phi\,dx + \int_{\Gamma_d} \psi\,dx' = 0 \tag{3.10}$$

there exists precisely one solution of (3.9) in $W_p^{s+2}(\Omega_d)$. To show uniqueness, we set $(\phi,\psi) = 0$ and obtain from $(3.9)_1$– $(3.9)_3$ that $\int_{\Omega_d} |\nabla u|^2\,dx = 0$, hence u is constant, and it follows from $(3.9)_4$ that $u = 0$. To show existence, we first remark that $(\phi,\psi) \in L^p(\Omega_d) \times L^p(\Gamma_d) \hookrightarrow L^2(\Omega_d) \times L^2(\Gamma_d)$, and a discussion of the weak formulation shows that (3.9) has a unique solution $u \in H^1(\Omega_d)$ satisfying an estimate

$$\|u\|_{L^2(\Omega_d)} \le C\left(\|\phi\|_{L^2(\Omega_d)} + \|\psi\|_{L^2(\Gamma_d)}\right) \le C\left(\|\phi\|_{W_p^s(\Omega_d)} + \|\psi\|_{W_p^{s+1-\frac{1}{p}}(\Gamma_d)}\right). \tag{3.11}$$

Consider the case where ϕ and ψ are smooth. In this case, it follows from standard elliptic regularity results that u is smooth, and the a priori estimate given in [13] Theorem 5.3.4 together with Theorem 12.12 in [14] and (3.11) yields

$$\begin{aligned} \|u\|_{W_p^{s+2}(\Omega_d)} &\le C\left(\|\phi\|_{W_p^s(\Omega_d)} + \|\psi\|_{W_p^{s+1-\frac{1}{p}}(\Gamma_d)} + \|u\|_{L^2(\Omega_d)}\right) \\ &\le C'\left(\|\phi\|_{W_p^s(\Omega_d)} + \|\psi\|_{W_p^{s+1-\frac{1}{p}}(\Gamma_d)}\right). \end{aligned}$$

The solvability of (3.9) in $W_p^{s+2}(\Omega_d)$ in the general case follows now by a standard approximation argument.

To complete the proof, we set $(\phi,\psi) := (\theta_*^0 w, \theta_*^0 h)$ for $(w,h) \in \tilde{E}_0$. Then (3.10) is satisfied, and $A(v,g) = (w,h)$ with $(v,g) \in \tilde{E}_1$ if and only if $v = \theta_0^* u$, $g = \gamma_0 v$, i.e. this operator equation is always uniquely solvable. ■

Lemma 3.6 *(A local chart for $\mathcal{M}$)*
There are neighborhoods $\mathcal{U}_1$ and $\mathcal{V}_1$ of 0 in $\tilde{E}_1$ and $\mathcal{M}$, respectively, and a diffeomorphism $\Phi \in \mathrm{Diff}^\infty(\mathcal{U}_1, \mathcal{V}_1)$ such that $\Phi \circ P = \mathrm{id}_{\mathcal{V}_1}$ and $\partial\Phi(0) = \mathrm{id}_{\tilde{E}_1}$.

Proof: Define the function $F \in C^\infty(\tilde{E}_1 \times E_1, \tilde{E}_1 \times \mathbb{R})$ by

$$F(x,z) := (Pz - x, I(z) - Q_0).$$

Clearly $F(0,0) = 0$, the partial Fréchet derivatives in $(0,0)$ are given by

$$\begin{aligned} \partial_1 F(0,0)[h] &= (-h, 0), \\ \partial_2 F(0,0)[k] &= (Pk, \partial I(0,0)[k]), \end{aligned}$$

and $\partial_2 F(0,0)$ is invertible with

$$-(\partial_2 F(0,0))^{-1} \partial_1 F(0,0)[h] = h, \qquad h \in \tilde{E}_1.$$

By this, all assertions follow from the Implicit Function Theorem. ∎

Let $\rho > 0$ be given and choose $\sigma > 0$ according to Theorem 3.2. Furthermore, we choose open neighborhoods $\mathcal{V} \subset \mathcal{V}_1$ and $\mathcal{U} \subset \mathcal{U}_1$ of 0 in $\mathcal{M}$ and $\tilde{E}_1$, respectively, such that $\overline{\mathcal{V}} \subset W_{\rho,\sigma} \cap \mathcal{V}_1$, $\mathcal{U} = P[\mathcal{V}]$ and Φ is Lipschitz continuous on $\mathcal{U}$. Assume $z_0 \in \mathcal{V}$ and define

$$\begin{aligned} \tilde{t}^+ &:= \sup\{t \in [0, t^+) \,|\, z(t) \in \mathcal{V}\}, \\ \tilde{z}(t) &:= Pz(t), \qquad t \in [0, \tilde{t}^+), \\ \tilde{\Pi} &:= P \circ \Pi \circ \Phi \end{aligned}$$

with z and t^+ from Theorem 3.2. Note that $\tilde{\Pi} \in C^\infty(\mathcal{U}_1, \tilde{E}_0)$, $\tilde{z} \in C([0,\tilde{t}^+), \mathcal{U}) \cap C^1([0,\tilde{t}^+), \tilde{E}_0)$ and due to

$$P\Pi(z(t)) = P\Pi(\Phi(Pz(t))) = \tilde{\Pi}(\tilde{z}(t))$$

it satisfies

$$\left.\begin{aligned} \partial_t \tilde{z} + \tilde{\Pi}(\tilde{z}) &= 0, \\ \tilde{z}(0) &= Pz_0 \end{aligned}\right\} \tag{3.12}$$

on $[0, \tilde{t}^+)$.

Theorem 3.7 *(Exponential stability for the projected problem)*
There are positive constants r, M, and ω such that for any z_0 with $\|Pz_0\|_{E_1} < r$ the following holds:

i) $\tilde{z}$ is the only solution of (3.12) in $C([0,\tilde{t}^+), \mathcal{U}) \cap C^1([0,\tilde{t}^+), \tilde{E}_0)$,

ii) $\tilde{z}$ can be uniquely extended to a solution of (3.12) in $C([0,\infty), \mathcal{U}) \cap C^1([0,\infty), \tilde{E}_0)$, still denoted by $\tilde{z}$,

iii) $\tilde{z}$ satisfies the estimate

$$\|\tilde{z}(t)\|_{E_1} + \|\partial_t \tilde{z}(t)\|_{E_0} \le M e^{-\omega t} \|Pz_0\|_{E_1}.$$

Proof: Using Lemma 3.6 we find $\partial \tilde{\Pi}(0) = P\partial\Pi(0) = \tilde{A}$. As $\sigma(A)$ is closed, we obtain from Lemma 3.5 that

$$\omega_0 := \inf\{\operatorname{Re}\lambda \,|\, \lambda \in \sigma(\tilde{A})\} > 0.$$

The results follow immediately from this and $\tilde{A} \in \mathcal{H}(\tilde{E}_1, \tilde{E}_0)$ by [12], Theorem 9.1.2. ∎

Corollary 3.8 *(Exponential stability for the original problem)*
There are positive constants r, M, and ω such that for any z_0 with $\|z_0\|_{E_1} < r$ one has $t^+ = \infty$ in Theorem 3.2 and the estimate

$$\|z(t)\|_{E_1} \leq M e^{-\omega t} \|z_0\|_{E_1}, \qquad t \geq 0,$$

holds.

Proof: Let $r := \frac{\tilde{r}}{\|P\|_{\mathcal{L}(E_1)}}$ where $\tilde{r}$ denotes the constant r from Theorem 3.7 and assume $\|z_0\|_{E_1} < r$. This implies the statements of Theorem 3.7, in particular, $\tilde{t}^+ = t^+$ and thus

$$\{z(t) \mid t \in [0, t^+)\} = \Phi[\{\tilde{z}(t) \mid t \in [0, t^+)\}],$$

i.e. the trajectory of the maximal solution of (3.7) is the image of a relatively compact set under a continuous map, hence it is relatively compact. By [12], Proposition 8.2.1 and Lemma 8.2.2., this implies $\lim_{t \to t^+} z(t) \in \partial W_{\rho,\sigma}$ or $t^+ = \infty$. The first possibility is excluded because $\overline{\mathcal{V}} \subset W_{\rho,\sigma}$ and $W_{\rho,\sigma}$ is open, hence $t^+ = \infty$. The estimate of the corollary is an immediate consequence of Theorem 3.7 and the Lipschitz continuity of Φ and P. ∎

Due to the smoothness of z in t, the stability results can also be translated back to the original problem (3.1).

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