Approximate Solutions to the Abstract Cauchy Problem

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Abstract. *We introduce the notion of approximate solutions to deal with Cauchy problems with generalized solutions. We show the equivalence of this notion of solvability with the regularized solutions introduced by I. Cioranescu and G. Lumer. As an example, we show that the backwards heat equation in* $\Omega \subset \mathbb{R}^d$ *is well posed in the approximate (or regularized) sense if and only if* $d = 1$.

1. INTRODUCTION

The aim of this article is to give an alternative point of view on regularized solutions[(1)] to the abstract Cauchy problem

$$(ACP) \qquad u'(t) = Au(t); \qquad u(0) = x; \qquad 0 \le t < T \le \infty$$

as introduced by I. Cioranescu and G. Lumer [5],[6],[7]. A regularized solution is a continuous function v that satisfies

$$v = A(1 \star v) + (1 \star k)x$$

on $[0, T)$ for some regularizer $k \in L^1_{loc}[0, T)$ with $0 \in supp(k)$, where

$$f \star g := t \mapsto \int_0^t f(t-s)g(s)\,ds$$

[(1)]We investigate only scalar valued kernels in this article. Regularized solutions were defined with even more general operator valued kernels. Conditions on these kernels that still allow approximate solutions can be found in [2].

denotes the convolution of f and g on $\mathbb{R}^+$. If v is a regularized solution, it is the k-convoluted image of the "true" solution; i.e., if (ACP) admits a classical solution u, then $v = k \star u$.

Our approach uses the fact that we can always invert the process of convolution (see [2]): For any function $v \in C_0([0,T);X)$ and any regularizer k we can construct a sequence $u_n \in C([0,T);X)$ such that $k \star u_n \to v$ uniformly on compact intervals. This suggests the following definition of an approximate solution, converging to the "true" solution:

DEFINITION: A k-approximate solution to (ACP) for some closed[(2)] linear operator A on a Banach space X is a sequence of continuous functions u_n such that

(i) $k \star u_n \to v$ for some $v \in C_0([0,T);X)$,

(ii) $1 \star u_n \in C([0,T);[\mathcal{D}(A)])$,

(iii) $k \star (u_n - A(1 \star u_n) - x) \to 0$.

Clearly, any k-approximate solution $[u_n]$ gives rise to a k-regularized solution $v := \lim k \star u_n$. One can think of the sequences $[u_n]$ as elements in the generalized function space[(3)]

$$C^{[k]}([0,T);X) := \overline{C([0,T);X)}^{\|\cdot\|_k},$$

the completion of the space of continuous functions with respect to the seminorms $\|f\|_{k,T_n} := \|k \star f\|_{T_n} := \sup_{0 \le t \le T_n} \|\int_0^t k(t-s)f(s)\,ds\|$. In other words, we consider existence, uniqueness and continuous dependence on initial data of the problem

$$[u_n] = A[1 \star u_n] + x$$

for some $[u_n] \in C^{[k]}([0,T);X)$. Furthermore, for any regularized solution we will actually construct a sequence of exponential functions that form an approximate solution.

We say that a Cauchy problem is well posed if there exists $k \in L^1_{loc}[0,T)$ with $0 \in supp(k)$ such that (ACP) has a unique k-approximate solution $[u_n] \in C^{[k]}([0,T);X)$ for all $x \in X$. With this definition of wellposedness we show that the backwards heat equation on a regular set $\Omega \subset \mathbb{R}^d$ with Dirichlet or Neumann boundary conditions is well posed if and only if $d = 1$.

2. PRELIMINARIES

We briefly recall some theorems about asymptotic Laplace transforms and its inversion [1]. The asymptotic Laplace transform is an operator extension of the usual Laplace transform to all of L^1_{loc}, including functions that are not exponentially bounded. It was first introduced by J.C. Vignaux in 1939 [12] and further developed by G. Lumer and F. Neubrander [11]. The asymptotic Laplace transform

(2) Techniques for investigating the abstract Cauchy problem with nonclosed operators are found in [3] and [2].

(3) For a more detailed introduction to these generalized function spaces see [4] or [2].

$\mathcal{L}_A(f)$ of a function $f \in L^1_{loc}([0,T);X)$ is defined to be an equivalence class of analytic functions defined in a post sectorial region Ω of the complex plane;i.e.,

$$\mathcal{L}_A(f) = \{r \in \mathcal{A}(\Omega, X) : r \approx_t \lambda \mapsto \int_0^t e^{-\lambda s} f(s)\, ds \text{ for all } 0 < t < T\}$$

where $r \approx_t q$ if $r - q$ is of exponential decay[(4)] t. This set is always nonempty, meaning that $\mathcal{L}_A(f)$ exists and is well defined for all f, and the properties of the Laplace transform extend fully (see [11] or [2]).

The asymptotic Laplace transform can be extended to transform generalized functions[(5)] $f \in C^{[k]}([0,T);X)$ for $k \in L^1_{loc}[0,T)$ with $0 \in supp(k)$. These spaces are the completions of the space of continuous functions with respect to the Frechét topology that is obtained by measuring the convolution image ($\|f\|_k := \|k \star f\|$). Thus we can define $k \star f := k \star [f_n] := [k \star f_n] = g$ for any element $f := [f_n] \in C^{[k]}([0,T);X)$, where $g \in C_0([0,T);X)$. The asymptotic Laplace transform is defined via

$$\mathcal{L}_A(f) := \frac{\mathcal{L}_A(k \star f)}{\mathcal{L}_A(k)}.$$

Thus we obtain co-classes of (in general) meromorphic functions. They have the property that for each element $\hat{f} \in \mathcal{L}_A(f)$ we can construct a Müntz sequence[(6)] $(\beta_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{f}(\beta_n)\| = -S \leq 0,$$

where $S := \min\{x; x \in supp(k \star f)\}$. For a proof see [1].

The following theorem is the backbone of this article. The proof is a straightforward adjustment of [1], Thm. 9.

THEOREM 1 (The Phragmén-Mikusiński Inversion of the Laplace Transform). *Let $f \in C_0([0,T);X)$, or let f be a generalized function in one of the Frechét spaces $C^{[k]}([0,T);X)$ for some regularizer $k \in L^1_{loc}[0,T)$ with $0 \in supp(k)$. Let $\hat{f}(\lambda) \in \mathcal{L}_A(f)$. Let $(\beta_n)_{n\in\mathbb{N}}$ be a Müntz sequence such that $\lim_{n\to\infty} \frac{1}{\beta_n} \ln \|\hat{f}(\beta_n)\| \leq 0$. Let $\epsilon_n = \max\{\frac{2}{\sqrt{n}}, \frac{2}{\beta_n} \ln \|\beta_n \hat{f}(\beta_n)\|\}$. Let N_n be such that $\sum_{i=1}^{N_n} \frac{1}{\beta_{ni}} \to T$, as $n \to \infty$. Define*

$$\alpha_{n,i} := \beta_{ni} e^{-\beta_{ni} \sum_{j=1}^{N_n} \frac{1}{\beta_{nj}}} \prod_{\substack{j=1 \\ j\neq i}}^{N_n} \frac{\beta_{nj}}{\beta_{nj} - \beta_{ni}}.$$

Then

$$f = \lim_{n\to\infty} \sum_{i=1}^{N_n} \alpha_{n,i} \hat{f}(\beta_{ni}) e^{\beta_{ni}(\cdot - \epsilon_n)},$$

where the limit is uniform on compact sets, or in $C^{[k]}([0,T);X)$ respectively. $\diamondsuit$

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(4) A function $r \in \mathcal{A}(\Omega, X)$ is of exponential decay t if $\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \|r(\lambda)\| \leq -t$

(5) We use $C^{[k]}([0,T);X)$ for simplicity of notation, of course the same construction also works for $f \in L^{p,[k]}_{loc}([0,T);X)$.

(6) $(\beta_n)_{n\in\mathbb{N}} \subset \mathbb{R}^+$ is a Müntz sequence if $\beta_{n+1} - \beta_n \geq 1$ and $\sum_{n=1}^{\infty} \frac{1}{\beta_n} = \infty$

3. APPROXIMATE SOLUTIONS

In this section we apply the Laplace transform results to the abstract Cauchy problem. We show the equivalence of the notions of approximate solutions and the notion of regularized solutions for scalar valued kernels. Furthermore, for any problem that is well posed in the regularized sense, we construct a sequence of exponential functions forming an approximate solution that converges in the best possible topology.

THEOREM 2. *Let A be a closed linear operator on a Banach space X. Let $k \in L^1_{loc}[0,T)$ with $0 \in supp(k)$. The following are equivalent.*

(i) u_n is a k-approximate solution of (ACP).

(ii) $v := \lim k \star u_n$ is a k-regularized solution of (ACP).

PROOF. $(i) => (ii)$. Obvious by the closedness of A.
$(ii) => (i)$. Since $1 \star v \in C_0([0,t);[\mathcal{D}(A)])$ there exists $\hat{v} \in \mathcal{L}_A(v)$ with $\hat{v}(\lambda) \in \mathcal{D}(A)$ for all $\lambda > 0$. Let

$$u_n(t) := \sum_{i=1}^{N_n} \alpha_{n,i} \frac{\hat{v}(\beta_{ni})}{\hat{k}(\beta_{ni})} e^{\beta_{ni}(t-\epsilon_n)},$$

where β_n is such that $\lim_{n\to\infty} \frac{1}{\beta_n} \ln |\hat{k}(\beta_n)| = 0$ and the variables N_n and $\alpha_{n,i}$ are chosen as in Theorem 1. Then $u_n \in C^\infty([0,t);[\mathcal{D}(A)])$. By Theorem 1, $k \star u_n \to v$ uniformly in X on compact sets, and $k \star 1 \star u_n \to 1 \star v$ uniformly in $[\mathcal{D}(A)]$ on compact sets. Hence $k \star (u_n - A(1 \star u_n) - x) \to v - A(1 \star v) - k \star x = 0$. ♢

Thus, the existence and uniqueness results for regularized solutions carry over to approximate solutions.

COROLLARY 3. *Let $\emptyset \neq \rho(A)$. Let $k \in L^1_{loc}[0,T)$ with $0 \in supp(k)$. Let $f \in C^{[k]}([0,T);X)$. Suppose $\lambda \mapsto \hat{k}(\lambda)R(\lambda,A)\lambda\hat{f}(\lambda) \in \mathcal{L}_A(g)$ for some $g \in C_0([0,T);X)$. Then*

$$\left[\sum_{i=1}^{N_n} \alpha_{n,i}\beta_{ni}R(\beta_{ni},A)\hat{f}(\beta_{ni})e^{\beta_{ni}(t-\epsilon_n)}\right],$$

is a k-approximate solution to $u = A(1 \star u) + f$.

PROOF. By assumption there exists $\hat{g} \in \mathcal{L}_A(g)$ with

$$(\lambda - A)\hat{g}(\lambda) = \hat{k}(\lambda)\lambda\hat{f}(\lambda).$$

Thus g is a regularized solution, and with the same argument as above, the assertion follows. ♢

Sufficient conditions on when an analytic function in some region in the complex plane is in the asymptotic Laplace transform of a continuous function can be found in [11] or [2].

4. WELLPOSEDNESS

In this section we explore some of the consequences and conditions for wellposedness in the following non-traditional sense. We say that (ACP) is well posed if there exists a regularizer k such that (ACP) admits a unique k-approximate solution for all $x \in X$ that depend continuously on the initial data (Uniqueness and continuity in the sense of $[u_n] \in C^{[k]}([0,T);X)$). We will show that existence and uniqueness already imply continuous dependence on the initial data and a nonempty resolvent set. Furthermore we obtain a sharper version of Lyubich's uniqueness theorem on the growth of the resolvent. These theorems are extensions of results obtained by G. Lumer and F. Neubrander in [11].

THEOREM 4 (Existence+Uniqueness => Cont. Dependence). *Suppose the abstract Cauchy problem has a unique approximate solution $u(\cdot,x) \in C^{[k]}([0,T);X)$ for all $x \in X$. Then (ACP) is well posed; i.e., for all $0 < t < T$ there exists $M_t > 0$ such that $\|u(\cdot,x)\|_k \leq M_t\|x\|$ on $[0,t]$.*

PROOF. Let $S(\cdot)x := k \star u(\cdot,x)$. Thus, by Theorem 2,

$$Sx = A(1 \star Sx) + (1 \star k)x.$$

By the existence and uniqueness property, we know that $S : X \to C([0,T);X)$ is a linear operator. Furthermore, S is a closed operator, since $x_n \to x$ and $Sx_n \to v$ implies that $1 \star Sx_n \to 1 \star v$ and, by the closedness of A, that

$$A(1 \star v) = v - (1 \star k)x.$$

By the uniqueness property, $Sx = v$, and hence S is a closed linear operator. By the closed graph theorem, S is bounded and hence for all $0 \leq t < T$ there exists $M_t > 0$ such that

$$\|u(\cdot,x)\|_{k,t} = \|Sx\|_t = \sup_{s\in[0,t]} \|S(s)x\| \leq M_t\|x\|.$$

$\diamondsuit$

Recall the fact shown by G. Lumer and F. Neubrander [11] that any function that is analytic in a postsectorial region and satisfies a certain growth condition is in the asymptotic Laplace transform of a continuous function. Suppose a meromorphic function on such a region multiplied by the asymptotic Laplace transform of a regularizer yields an analytic function satisfying the growth conditions. Then, this meromorphic function is an element of the asymptotic Laplace transform of a generalized function. Thus, the existence of a meromorphic resolvent on a postsectorial region with some growth condition implies wellposedness.

The next theorem shows that wellposedness has implications on the resolvent set and on the growth behavior of the resolvent of the operator A.

THEOREM 5 (Existence of Resolvent). *If (ACP) is well posed, then every Müntz sequence has a subsequence $\left(\beta_{n_j}\right)_{j\in\mathbb{N}}$ such that β_{n_j} is in the resolvent set of A and* $\limsup_{j\to\infty} \frac{1}{\beta_{n_j}} \ln \|R(\beta_{n_j}, A)\| \le 0$.

PROOF. If (ACP) is well posed, then there exists a strongly continuous operator family $(S(t))_{t\in[0,T)}$ such that

$$Sx = A(1 \star Sx) + (1 \star k)x.$$

Furthermore, let $0 < t < T$, and

$$\tilde{S}(\lambda)x := \int_0^t e^{-\lambda s} S(s)x\, ds.$$

By the previous Theorem, $(1 \star S)(t) \in \mathcal{L}(X, [\mathcal{D}(A)])$ for all $0 \le t < T$, and thus $\tilde{S}(\lambda) \in \mathcal{L}(X, [\mathcal{D}(A)])$ and $\left(\tilde{S}(\lambda)\right)_{\lambda\in\mathbb{C}}$ is a strongly analytic operator family. Let $\hat{k} \in \mathcal{L}_A(k)$. Then

$$(\lambda - A)\tilde{S}(\lambda)x = \hat{k}(\lambda)x + r_x(\lambda),$$

where $r_x \approx_t 0$ for all $x \in X$. Furthermore, $r_{(\cdot)}(\lambda) \in \mathcal{L}(X)$. Let $0 < \epsilon < t/2$. Since $r_x \approx_t 0$, there exists M_x such that

$$e^{\lambda(t-\epsilon)}\|r_x(\lambda)\| \le M_x$$

for all $\lambda > 0$. By the uniform boundedness principle there exists a constant M such that $r(\lambda) : x \mapsto r_x(\lambda)$ satisfies

$$\|r(\lambda)\| \le M e^{-\lambda(t-\epsilon)}.$$

Let $(\beta_n)_{n\in\mathbb{N}}$ be a Müntz sequence. Then there exists a subsequence $\left(\beta_{n_j}\right)_{j\in\mathbb{N}}$ such that $\lim_{j\to\infty} \frac{1}{\beta_{n_j}} \ln|\hat{k}(\beta_{n_j})| = 0$ and thus there exists a constant J such that

$$|\hat{k}(\beta_{n_j})| \ge e^{-\beta_{n_j}\epsilon} > 2\|r(\beta_{n_j})\|$$

for all $j > J$. Then $\|\frac{r(\beta_{n_j})}{\hat{k}(\beta_{n_j})}\| < 1/2$. Thus $Id + \frac{r(\beta_{n_j})}{\hat{k}(\beta_{n_j})}$ is an invertible operator that maps onto X and

$$\| \left(Id + \frac{r(\beta_{n_j})}{\hat{k}(\beta_{n_j})} \right)^{-1} \| = \| \sum_{i=0}^{\infty} (-1)^i \left(\frac{r(\beta_{n_j})}{\hat{k}(\beta_{n_j})} \right)^i \| \le 2.$$

Since $\hat{k}(\beta_{n_j}) \ne 0$ is scalar-valued, the operator

$$(\beta_{n_j} - A)\tilde{S}(\beta_{n_j}) = \hat{k}(\beta_{n_j})(Id + \frac{r(\beta_{n_j})}{\hat{k}(\beta_{n_j})})$$

is a continuously invertible operator. Let

$$Q(\beta_{n_j}) := \left(\hat{k}(\beta_{n_j})(Id + \frac{r(\beta_{n_j})}{\hat{k}(\beta_{n_j})})\right)^{-1}$$

and define

$$R(\beta_{n_j}) := \tilde{S}(\beta_{n_j})Q(\beta_{n_j}).$$

Then, $\|Q(\beta_{n_j})\| \leq \frac{2}{\hat{k}(\beta_{n_j})}$ and $(\beta_{n_j} - A)R(\beta_{n_j}) = Id$.

We show next that $(\beta_{n_j} - A)$ is one-to-one. Suppose $\beta_{n_j}x = Ax$ for some $x \in \mathcal{D}(A)$. Then

$$(e^{\beta_{n_j}t} - 1)x = A\frac{1}{\beta_{n_j}}(e^{\beta_{n_j}t} - 1)x = A\int_0^t e^{\beta_{n_j}s}x\,ds.$$

Since the solutions to ACP are unique, we deduce that $S(t)x = (k \star e^{\beta_{n_j}(\cdot)})(t)x$. Thus $\tilde{S}(\lambda)x$ is a scalar multiple of x for all $\lambda \in \mathbb{C}$. Hence,

$$0 = (\beta_{n_j} - A)\tilde{S}(\beta_{n_j})x = \hat{k}(\beta_{n_j})(x + \frac{r(\beta_{n_j})x}{\hat{k}(\beta_{n_j})}).$$

Therefore, $\|-x\| = \|\frac{r(\beta_{n_j})x}{\hat{k}(\beta_{n_j})}\| \leq \|x\|/2$, and thus $x = 0$. Hence $(\beta_{n_j} - A)$ is one-to-one and thus $R(\beta_{n_j}) = R(\beta_{n_j}, A)$ is the resolvent of A at β_{n_j}.

Since $\|Q(\beta_{n_j})\| \leq 2/|\hat{k}(\beta_{n_j})|$, we know that

$$\limsup_{j\to\infty} \frac{1}{\beta_{n_j}} \ln \|Q(\beta_{n_j})\| = 0$$

and hence the same holds for $R(\beta_{n_j}) = \tilde{S}(\beta_{n_j})Q(\beta_{n_j})$. ◊

As a Corollary to the Theorem 1 we obtain that existence and growth of the resolvent yield the following extension of Lyubich's uniqueness condition, see [10] or [3].

COROLLARY 6 (Lyubich's Uniqueness Theorem). *Suppose there exists a Müntz sequence $(\beta_n)_{n\in\mathbb{N}}$ such that $\beta_n \in \rho(A)$ for all n and*

$$\limsup_{n\to\infty} \frac{1}{\beta_n} \ln \|R(\beta_n, A)\| \leq 0.$$

Then every approximate solution is unique.

PROOF. Suppose u is a k-approximate solution of $u' = Au; u(0) = 0$. Let $\hat{u} \in \mathcal{L}_A(u)$. Then there exists $\hat{r} \in \mathcal{L}_A(0)$ such that $\hat{u}(\beta_n) = R(\beta_n, A)r(\beta_n)$ and thus

$$\|\hat{u}(\beta_n)\| \leq \|R(\beta_n, A)\|\|r(\beta_n)\| \approx_T 0.$$

Thus $u = 0$. ◊

5. EXAMPLE: THE BACKWARDS HEAT EQUATION

Consider the Laplace operator $-\Delta$ on a bounded open set $\Omega \subset \mathbb{R}^d$ with smooth boundary and Dirichlet or Neumann boundary conditions. Then it is well known (see for example [8]) that the eigenvalues λ_n grow like $n^{2/d}$. Thus, if $d \geq 2$, the point spectrum is forming a Müntz sequence. This implies that any regularizer $\hat{k}$ that regularizes the resolvent would have to be zero on a Müntz sequence; otherwise $\lambda \mapsto \hat{k}(\lambda)R(\lambda, A)$ can not be analytic on some region containing (ω, ∞). But that in turn implies that $k = 0$.

In one dimension, consider $\Omega = (0, \pi)$. We use the eigenfunctions of the Laplacian; i.e., let $e_n(x) := \sin(nx)$ for the Dirichlet problem, and $e_n(x) := \cos(nx)$ for the Neumann problem. Every function $f \in L^2[0, \pi]$ can be expressed in terms of its eigenfunction expansion $f = \sum_{n=0}^{\infty} a_n e_n$ and thus $R(\lambda, A)f = \sum_{n=0}^{\infty} \frac{a_n}{\lambda - n^2} e_n$. Let

$$\hat{k}(\lambda) := \frac{1}{\lambda^2} \prod_{n=0}^{\infty} \frac{n^2 - \lambda}{n^2 + \lambda}.$$

Clearly, $\lambda^2 \hat{k}$ is bounded and analytic in a right halfplane and therefore $\hat{k}$ is the Laplace transform of a continuous function (see, for example, [3]). In order to show that k satisfies the conditions for a regularizer we have to show that $0 \in supp(k)$. We use the basic fact of Laplace transform theory that $\min\{x : x \in supp(k)\} = -\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln |\hat{k}(\lambda)|$. See, for example, [9] Satz 14.3.1 or [1]. Since

$$\begin{aligned} 0 \leq -\limsup_{\lambda\to\infty} \frac{1}{\lambda} \ln \left| \prod_{n=0}^{\infty} \frac{n^2 - \lambda}{n^2 + \lambda} \right| &= \liminf_{\lambda\to\infty} \frac{1}{\lambda} \sum_{n=0}^{\infty} \ln \left| \frac{n^2 + \lambda}{n^2 - \lambda} \right| \\ &\leq \liminf_{\lambda\to\infty} \frac{1}{\lambda} \sum_{n=0}^{\infty} \ln(1 + \frac{2\lambda}{|n^2 - \lambda|}) \\ &\leq \liminf_{\lambda\to\infty} \sum_{n=0}^{\infty} \frac{2}{|n^2 - \lambda|} = 0, \end{aligned}$$

we obtain that $0 \in supp(k)$. Since

$$\|\hat{k}(\lambda)R(\lambda, A)\| \leq \frac{1}{|\lambda^2|} \sum_{n=0}^{\infty} \left| \frac{1}{n^2 + \lambda} \prod_{j\neq n} \frac{j^2 - \lambda}{j^2 + \lambda} \right| \leq \frac{c + \frac{1}{|\lambda|}}{|\lambda^2|},$$

for λ in the open right halfplane, $\hat{k}(\lambda)R(\lambda, A)$ is the Laplace transform of a strongly continuous operator family.

Thus we can apply Corollary 3. Let $(\beta_n)_{n\in\mathbb{N}}$ be the sequence of positive integers excluding the square numbers. Then

$$u_n(t) := \sum_{i=1}^{N_n} \alpha_{n,i} \beta_{ni} R(\beta_{ni}, A) f e^{\beta_{ni}(t - 2/\sqrt{n})} = \sum_{i=1}^{N_n} \alpha_{n,i} \beta_{ni} \sum_{j=0}^{\infty} \frac{a_j}{\beta_{ni} - j^2} e_j e^{\beta_{ni}(t - 2/\sqrt{n})}$$

is a sequence of exponential functions that converges in the best possible topology towards the solution of the backwards heat equation

$$u'(t, x) = -\Delta u(t, x); \qquad u(0, x) = f(x).$$

Notice that the sequence is independent of our choice of k, which might not have been the optimal choice.

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