ASYMPTOTIC PROPERTIES OF MILD SOLUTIONS OF NONAUTONOMOUS EVOLUTION EQUATIONS WITH APPLICATIONS TO RETARDED DIFFERENTIAL EQUATIONS

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ABSTRACT. We investigate asymptotic properties of mild solutions of the inhomogeneous nonautonomous evolution equation $\frac{d}{dt}u(t) = (A+B(t))u(t)+f(t)$, $t \in \mathbb{R}$, where $(A, D(A))$ is a Hille-Yosida operator on a Banach space X, $B(t)$, $t \in \mathbb{R}$, is a family of operators in $\mathcal{L}(\overline{D(A)}, X)$ satisfying certain boundedness and measurability conditions, and $f \in L^1_{loc}(\mathbb{R}, X)$. The mild solutions of the corresponding homogeneous problem are represented by an evolution family $(U_B(t,s))_{t\geq s}$. For various function spaces $\mathcal{F}$ we derive conditions on $(U_B(t,s))_{t\geq s}$ and f which ensure the existence of a unique mild solution u contained in $\mathcal{F}$. In particular, if $(U_B(t,s))_{t\geq s}$ is p-periodic and f is bounded there is a unique bounded mild solution u subject to certain spectral assumptions on $U_B(p,0)$, f, and u. We apply our results to discuss asymptotic properties of nonautonomous retarded differential equations. For certain p-periodic retarded differential equations we derive a characteristic equation which makes it easier to verify the spectral conditions on the operator $U_B(p,0)$.

1. INTRODUCTION

Consider the inhomogeneous nonautonomous evolution equation

$$(INP)'_f \quad \frac{d}{dt}u(t) = A(t)u(t) + f(t), \quad t \in \mathbb{R},$$

where $A(t)$, $t \in \mathbb{R}$, are (unbounded, linear) operators on a Banach space X and $f \in L^1_{loc}(\mathbb{R}, X)$. Assume that the homogeneous equation

$$(NP)' \quad \frac{d}{dt}u(t) = A(t)u(t), \quad t \in \mathbb{R},$$

is well-posed in the sense that the solutions of $(NP)'$ define a uniquely determined evolution family $(U(t,s))_{t\geq s}$ of bounded operators on X. In that case solutions $u : \mathbb{R} \to X$ of the integral equation

$$(I) \quad u(t) = U(t,s)u(s) + \int_s^t U(t,\sigma)f(\sigma)\,d\sigma, \quad t \geq s,$$

can be interpreted as mild solutions of $(INP)'_f$. It has been shown in [5], [22], [23] that for each $f \in C_b(\mathbb{R}, X)$ resp. $C_0(\mathbb{R}, X)$ equation $(INP)'_f$ has a *unique* mild solution $u \in C_b(\mathbb{R}, X)$ resp. $C_0(\mathbb{R}, X)$ if and only if the evolution family $(U(t,s))_{t\geq s}$ has an exponential dichotomy (see also [10], [24] when the operators $A(t)$, $t \in \mathbb{R}$, are bounded). For a detailed account of the numerous other results in this direction we refer to [22]

1991 *Mathematics Subject Classification.* 34C25, 34C27, 34G10, 34K15, 34K30, 35B10, 35B15, 35B40, 47D06, 47H15.

Key words and phrases. Inhomogeneous, semilinear, nonautonomous, periodic, almost periodic, Hille-Yosida operator, retarded differential equation, mild solution.

Most of the work on this paper was done while the authors were visiting the Department of Mathematics at the University of Missouri-Columbia. The authors are most grateful for the warm hospitality they experienced during their visit. Support of the first author by "Friedrich-Ebert-Stiftung" is gratefully acknowledged.

and [7].
Now assume that equation $(NP)'$ is p-periodic, i.e., $A(t+p) = A(t)$, $t \in \mathbb{R}$. It has been shown in [30], [19], [27], [6] that under a certain spectral condition (*nonresonance condition*) on the monodromy operator $U(p,0)$ and the inhomogeneity f there is a p-periodic resp. almost periodic mild solution of $(INP)'_f$ provided that f has the same property. Moreover, u is unique subject to certain spectral assumptions. If $(U(t,s))_{t\geq s}$ has an exponential dichotomy, then the nonresonance condition is always satisfied and we obtain existence and uniqueness of a p-periodic resp. almost periodic mild solution of $(INP)'_f$ for every p-periodic resp. almost periodic inhomogeneity f. We point out that in [11], [31] related results are discussed for Volterra equations (see also [32]).
In the present paper we study the equation

$$(INP)_f \quad \frac{d}{dt}u(t) = (A + B(t))u(t) + f(t), \quad t \in \mathbb{R},$$

where $(A, D(A))$ is a Hille-Yosida operator on the Banach space X, $B(t)$, $t \in \mathbb{R}$, is a family of operators in $\mathcal{L}(\overline{D(A)}, X)$, and $f \in L^1_{loc}(\mathbb{R}, X)$. We stress that, in general, $X_0 = \overline{D(A)}$ is a proper subspace of X from which the main difficulties arise. Our approach is based on the theory of extrapolation spaces associated with the operator A (see Section 2 and [26]). In particular, it is used in our definition of mild solutions of $(INP)_f$. Moreover, it allows to show that under a certain boundedness and measurability condition on the family $B(t)$, $t \in \mathbb{R}$, there is a (unique) evolution family $(U_B(t,s))_{t\geq s}$ on X_0 associated with the homogeneous equation

$$(NP) \quad \frac{d}{dt}u(t) = (A + B(t))u(t), \quad t \in \mathbb{R}$$

(cf. [9], [33]). The evolution family $(U_B(t,s))_{t\geq s}$ is used to derive another representation of the mild solutions of $(INP)_f$ (see Theorem 2.2). This representation is crucial for the investigations in Section 3. There we extend the above-mentioned results on the existence and uniqueness of mild solutions of $(INP)'_f$ satisfying a particular asymptotic behavior to mild solutions of $(INP)_f$. We point out that in the autonomous case, i.e., $B(t) = B$, similar results are obtained in [2]. In Section 4 we discuss asymptotic properties of mild solutions of the semilinear nonautonomous equation

$$(SNP) \quad \frac{d}{dt}u(t) = (A + B(t))u(t) + F(t, u(t)), \quad t \in \mathbb{R},$$

where the nonlinearity $F : \mathbb{R} \times X_0 \to X$ satisfies a standard Lipschitz condition. In Section 5 the advantage of our approach becomes visible when we study inhomogeneous nonautonomous retarded differential equations

$$(RINP)_h \quad \frac{d}{dt}w(t) = Cw(t) + K(t)w_t + h(t), \quad t \in \mathbb{R},$$

on a Banach space Y. A standard procedure (see e.g. [16], [33], [40]) allows to transform $(RINP)_h$ into an equation of the form $(INP)_f$ on a different Banach space. Now the results of Section 3 can be applied to investigate asymptotic properties of mild solutions of $(RINP)_h$. For a special periodic retarded differential equation we derive a characteristic equation which makes it easier to verify the spectral conditions in our results (see Theorem 5.9). We point out that in finite dimensions asymptotic properties of solutions of inhomogeneous retarded differential equations have been studied in [38] under the assumption that the corresponding homogeneous equation admits an exponential dichotomy (see also [17, Section 6.6.2], [29]). Finally, we refer to [36], [34] and the references therein for other results on asymptotic behaviour of (nonlinear) retarded differential equations.

2. Mild Solutions and Extrapolation Spaces

We first recall some properties of Hille-Yosida operators and extrapolation spaces. For more details we refer to [26] and the references therein. Throughout the whole paper X denotes a Banach space and $(A, D(A))$ is a *Hille-Yosida operator* on X, i.e., A is linear and the *resolvent set* $\rho(A)$ of A contains a half-line (ω, ∞) such that

$$M = \sup\{\|(\lambda-\omega)^n R(\lambda, A)^n\| : \lambda > \omega;\, n \in \mathbb{N}\} < \infty,$$

where $R(\lambda, A) = (\lambda - A)^{-1}$ is the resolvent of A at λ. It is well-known that the part A_0 of A in $X_0 = \overline{D(A)}$ generates a C_0-semigroup $(T_0(t))_{t\geq 0}$ on X_0 and that $\|T_0(t)\| \leq Me^{\omega t}$, $t \geq 0$. For $\lambda \in \rho(A_0)$ the resolvent $R(\lambda, A_0)$ is the restriction of $R(\lambda, A)$ to X_0. Typical examples of Hille-Yosida operators appearing in partial differential equations can be found e.g. in [12], see also Section 5. On X_0 we introduce the norm $\|x\|_{-1} = \|R(\lambda_0, A_0)x\|$ where $\lambda_0 \in \rho(A)$ is fixed. A different choice of $\lambda_0 \in \rho(A)$ leads to an equivalent norm. The completion X_{-1} of X_0 with respect to $\|\cdot\|_{-1}$ is called the *extrapolation space* of X_0 with respect to A. The *extrapolated semigroup* $(T_{-1}(t))_{t\geq 0}$ consists of the unique continuous extensions $T_{-1}(t)$ of the operators $T_0(t)$, $t \geq 0$, to X_{-1}. The semigroup $(T_{-1}(t))_{t\geq 0}$ is strongly continuous and its generator A_{-1} is the unique continuous extension of A_0 to $\mathcal{L}(X_0, X_{-1})$. Moreover, X is continuously embedded in X_{-1} and $R(\lambda, A_{-1})$ is the unique continuous extension of $R(\lambda, A)$ to X_{-1} for $\lambda \in \rho(A)$. Finally, A_0 and A are the parts of A_{-1} in X_0 and X, respectively. It follows from [26, Proposition 3.3], that for $f \in L^1_{loc}(\mathbb{R}, X)$ and $t \geq s$

$$\int_s^t T_{-1}(t-\sigma) f(\sigma)\, d\sigma \in X_0,$$

$$(t,s) \mapsto \int_s^t T_{-1}(t-\sigma) f(\sigma)\, d\sigma \text{ is continuous, and} \tag{2.1}$$

$$\left\| \int_s^t T_{-1}(t-\sigma) f(\sigma)\, d\sigma \right\| \leq M_1 \int_s^t e^{\omega(t-\sigma)} \|f(\sigma)\|\, d\sigma \text{ for a constant } M_1 \geq 1.$$

We consider the inhomogeneous nonautonomous evolution equation

$$(INP)_f \quad \frac{d}{dt} u(t) = (A + B(t))u(t) + f(t), \quad t \in \mathbb{R}, \quad f \in L^1_{loc}(\mathbb{R}, X),$$

where $(A, D(A))$ is a Hille-Yosida operator on the Banach space X and $B(t) \in \mathcal{L}(X_0, X)$, $t \in \mathbb{R}$, is a family of operators such that $t \mapsto B(t)x$ is strongly measurable for every $x \in X_0$ and $\|B(\cdot)\| \leq b(\cdot)$ for a function $b \in L^1_{loc}(\mathbb{R})$. For our purposes the notion of a mild solution of equation $(INP)_f$ is most useful. We point out that our definition of a mild solution coincides with that given in [8], the F-solutions in [12], the weak solutions in [13] and the integral solutions in [40].

Definition 2.1. *If $f \in L^1_{loc}(\mathbb{R}, X)$ and $T \geq s$, then $u = u(\cdot, f) \in C([s,T], X_0)$ is called a* mild solution *of $(INP)_f$ on $[s,T]$ if*

$$u(t) = T_0(t-s)u(s) + \int_s^t T_{-1}(t-\sigma)\Big(B(\sigma)u(\sigma) + f(\sigma)\Big)\, d\sigma \quad \text{for } t \in [s,T]. \tag{2.2}$$

A function $u = u(\cdot, f) \in C(\mathbb{R}, X_0)$ that satisfies (2.2) for all $t \geq s$ in $\mathbb{R}$ is called a mild solution on $\mathbb{R}$ *of $(INP)_f$.*

Under our assumptions on A and $B(t), t \in \mathbb{R}$, it follows that for $f \in L^1_{loc}(\mathbb{R}, X)$ and $s \in \mathbb{R}$ there is a unique mild solution $u = u(\cdot, f, s, x) \in C([s,\infty), X_0)$ of

$$(INP)_{f,s,x} \quad \begin{cases} \frac{d}{dt}u(t) = & (A + B(t))u(t) + f(t), \quad t \geq s, \\ u(s) = & x \in X_0 \end{cases}$$

(see e.g. [15] or Theorem 2.2). Mild solutions of the homogeneous equation

$$(NP) \quad \frac{d}{dt}v(t) = (A + B(t))v(t), \quad t \in \mathbb{R},$$

have another representation. For that we need the following notion. A family $(U(t,s))_{t\geq s}$ in $\mathcal{L}(X_0)$ is called an *evolution family* on X_0 if $U(t,t) = Id$, $U(t,r)U(r,s) = U(t,s)$ for $t \geq r \geq s$ and $(t,s) \mapsto U(t,s)x$ is continuous for $t \geq s$ and $x \in X_0$. It is known (see for example [9, Theorem 2.3,], [33, Theorem 2.3], where a slightly more special situation is considered) that there exists a unique evolution family $(U_B(t,s))_{t\geq s}$ on X_0 that satisfies the variation-of-parameters formula

$$(2.3) \qquad U_B(t,s)x = T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)B(\sigma)U_B(\sigma,s)x\, d\sigma, \quad t \geq s,\ x \in X_0.$$

Thus $t \mapsto U_B(t,s)x$ is the unique mild solution on $[s,\infty)$ of the initial value problem

$$(NP)_{s,x} \quad \begin{cases} \frac{d}{dt}u(t) = & (A + B(t))u(t), \quad t \geq s, \\ u(s) = & x \in X_0. \end{cases}$$

Gronwall's inequality (see for example [1, Corollary II.6.2]), the estimate in (2.1), and (2.3) imply

$$(2.4) \qquad \|U_B(t,s)\| \leq M e^{\omega(t-s) + M_1 \int_s^t b(\sigma)\, d\sigma}, \quad t \geq s,$$

for the constants M, $M_1 \geq 1$. In particular, if $\|B(\cdot)\|$ is bounded from above by a function $b \in L^1_{loc,u}(\mathbb{R})$, i.e., $\|b\|_{1,loc,u} = \sup_{t\in\mathbb{R}} \int_{t-1}^t |b(\sigma)|\, d\sigma < \infty$, then the evolution family $(U_B(t,s))_{t\geq s}$ is *exponentially bounded*, i.e., $\|U_B(t,s)\| \leq N e^{\beta(t-s)}$ for $t \geq s$ and constants $N \geq 1$, $\beta \in \mathbb{R}$. In the following result we give a representation of mild solutions of $(INP)_f$ in terms of the evolution family $(U_B(t,s))_{t\geq s}$. A special case has been discussed in [16, Theorem 3.6].

Theorem 2.2. *Let $f \in L^1_{loc}(\mathbb{R}, X)$, $s \in \mathbb{R}$, and $x \in X_0$. Then there is a unique mild solution $u \in C([s,\infty), X_0)$ of $(INP)_{f,s,x}$ given by*

$$u(t) = U_B(t,s)x + \lim_{\lambda\to\infty} \int_s^t U_B(t,\sigma)\lambda R(\lambda, A) f(\sigma)\, d\sigma \quad \text{for } t \geq s.$$

Moreover, $\lim_{\lambda\to\infty} \int_s^t U_B(t,\sigma)\lambda R(\lambda, A) f(\sigma)\, d\sigma \in X_0$ exists uniformly for $t \geq s$ in compact sets in $\mathbb{R}$.

Proof. Let $\lambda > \omega$ and set

$$w_\lambda(t,s) = \int_s^t U_B(t,\sigma)\lambda R(\lambda, A) f(\sigma)\, d\sigma, \quad t \geq s.$$

Then (2.3) leads to

(2.5)

$$\begin{aligned}
w_\lambda(t,s) = &\int_s^t T_0(t-\sigma)\lambda R(\lambda,A)f(\sigma)\,d\sigma \\
&+\int_s^t \Big(\int_\sigma^t T_{-1}(t-\tau)B(\tau)U_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma)\,d\tau\Big)\,d\sigma \\
=&\,\lambda R(\lambda,A_0)\int_s^t T_{-1}(t-\sigma)f(\sigma)\,d\sigma \\
&+\int_s^t \Big(\int_s^\tau T_{-1}(t-\tau)B(\tau)U_B(\tau,\sigma)\lambda R(\lambda,A)f(\sigma)\,d\sigma\Big)\,d\tau \\
=&\,\lambda R(\lambda,A_0)\int_s^t T_{-1}(t-\sigma)f(\sigma)\,d\sigma+\int_s^t T_{-1}(t-\sigma)B(\sigma)w_\lambda(\sigma,s)\,d\sigma,\ t\ge s.
\end{aligned}$$

If $z(t,s)=\int_s^t T_{-1}(t-\sigma)f(\sigma)\,d\sigma$, $t\ge s$, then by (2.1) for $\lambda,\mu>\omega$

$$\begin{aligned}
\|w_\lambda(t,s)-w_\mu(t,s)\| \le &\,\|(\lambda R(\lambda,A_0)-\mu R(\mu,A_0))z(t,s)\| \\
&+M_1\int_s^t e^{\omega(t-\sigma)}b(\sigma)\|w_\lambda(\sigma,s)-w_\mu(\sigma,s)\|\,d\sigma.
\end{aligned}\tag{2.6}$$

From (2.1) it follows that z is a continuous mapping into X_0. Hence

$$\lim_{\lambda,\mu\to\infty}\|(\lambda R(\lambda,A_0)-\mu R(\mu,A_0))z(t,s)\|=0$$

uniformly for $t\ge s$ in compact intervals. Thus if $\epsilon>0$ and $I\subseteq\mathbb{R}$ is a compact interval, then by (2.6) there is a constant $\tilde M$ depending only on the length of I such that

$$\|w_\lambda(t,s)-w_\mu(t,s)\|\le\epsilon+\tilde M\int_s^t b(\sigma)\|w_\lambda(\sigma,s)-w_\mu(\sigma,s)\|\,d\sigma$$

for $t\ge s$ in I and $\lambda,\mu>\omega$ sufficiently large. An application of Gronwall's inequality (see [1, Corollary II.6.2]) leads to the estimate

$$\|w_\lambda(t,s)-w_\mu(t,s)\|\le\epsilon e^{\tilde M\int_s^t b(\sigma)\,d\sigma}$$

for $t\ge s$ in I and $\lambda,\mu>\omega$ sufficiently large. Hence $w(t,s)=\lim_{\lambda\to\infty}w_\lambda(t,s)$ exists uniformly for $t\ge s$ in compact intervals.
Since A is a Hille-Yosida operator it follows from the definition of w_λ that $\sup\{\|w_\lambda(t,s)\|:\lambda>\omega+1;\,t\ge s\text{ in }I\}<\infty$. Hence, by (2.5) and Lebesgue's Dominated Convergence Theorem, we have

$$w(t,s)=\int_s^t T_{-1}(t-\sigma)B(\sigma)w(\sigma,s)\,d\sigma+\int_s^t T_{-1}(t-\sigma)f(\sigma)\,d\sigma,\quad t\ge s.\tag{2.7}$$

Now consider the function

$$u(t)=U_B(t,s)x+\lim_{\lambda\to\infty}\int_s^t U_B(t,\sigma)\lambda R(\lambda,A)f(\sigma)\,d\sigma=U_B(t,s)x+w(t,s),\quad t\ge s.$$

By (2.7) and (2.3), we obtain

$$\begin{aligned}
u(t) =& U_B(t,s)x + \int_s^t T_{-1}(t-\sigma)B(\sigma)w(\sigma,s)\,d\sigma + \int_s^t T_{-1}(t-\sigma)f(\sigma)\,d\sigma \\
=& T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)B(\sigma)\Big(U_B(\sigma,s)x + w(\sigma,s)\Big)\,d\sigma \\
&+ \int_s^t T_{-1}(t-\sigma)f(\sigma)\,d\sigma \\
=& T_0(t-s)x + \int_s^t T_{-1}(t-\sigma)\Big(B(\sigma)u(\sigma) + f(\sigma)\Big)\,d\sigma.
\end{aligned}$$

Hence u is a mild solution of $(INP)_f$.
If $\tilde{u} \in C([s,\infty), X_0)$ is another mild solution of $(INP)_{f,s,x}$ we obtain

$$u(t) - \tilde{u}(t) = \int_s^t T_{-1}(t-\sigma)B(\sigma)(u(\sigma) - \tilde{u}(\sigma))\,d\sigma, \quad t \geq s,$$

and an application of Gronwall's inequality yields $u = \tilde{u}$. □

Remark 2.3. *If in Theorem 2.2 we assume that $f \in L^1_{loc}(\mathbb{R}, X_0)$, then the function $u \in C([s,\infty], X_0)$ is a mild solution of $(INP)_{f,s,x}$ if and only if*

$$u(t) = U_B(t,s)x + \int_s^t U_B(t,\sigma)f(\sigma)\,d\sigma, \quad t \geq s.$$

Theorem 2.2 has the following immediate consequence.

Corollary 2.4. *If $f \in L^1_{loc}(\mathbb{R}, X)$, then $u \in C(\mathbb{R}, X_0)$ is a mild solution of $(INP)_f$ if and only if*

$$u(t) = U_B(t,s)u(s) + \lim_{\lambda\to\infty} \int_s^t U_B(t,\sigma)\lambda R(\lambda, A)f(\sigma)\,d\sigma \quad \textit{for } t \geq s.$$

In our next result we improve the convergence of the integrals considered in Theorem 2.2. By $BUC_r(\mathbb{R}, X)$ we denote the space of bounded, uniformly continuous functions f from $\mathbb{R}$ into X such that f has relatively compact range.

Proposition 2.5. *Let $\|B(\cdot)\| \leq b(\cdot)$ for some $b \in L^1_{loc,u}(\mathbb{R})$ and let $f \in BUC_r(\mathbb{R}, X)$. Then, for fixed $s > 0$, the limit*

$$\lim_{\lambda\to\infty} \int_{t-s}^t U_B(t,\sigma)\lambda R(\lambda, A)f(\sigma)\,d\sigma$$

exists uniformly for t in $\mathbb{R}$.

Proof. We claim that the function

$$\psi : \mathbb{R} \to X_0 : t \mapsto z(t, t-s) = \int_{t-s}^t T_{-1}(t-\sigma)f(\sigma)\,d\sigma = \int_0^s T_{-1}(\sigma)f(t-\sigma)\,d\sigma$$

has relatively compact range. In fact, fix $\epsilon > 0$. There exists $\delta = \frac{s}{n} > 0$ for an $n \in \mathbb{N}$ and a function $g : \mathbb{R} \to X$ such that g is constant on each interval $[k\delta, (k+1)\delta)$, $k \in \mathbb{Z}$, the range of g is contained in a finite set $K \subseteq X$, and $\|f - g\|_\infty \leq \epsilon$. From (2.1) it follows that the mapping

$$(r, x) \mapsto \int_0^r T_{-1}(\sigma)x\,d\sigma$$

from $\mathbb{R}_+ \times X$ into X_0 is continuous. The range of

$$\phi : \mathbb{R} \to X_0 : t \mapsto \int_0^s T_{-1}(\sigma) g(t-\sigma)\, d\sigma$$

is contained in

$$K_0 = \Big\{ \sum_{k=1}^{n+1} \Big(T_0(\tau_k) \int_0^{r_k} T_{-1}(\sigma) x_k \, d\sigma \Big) : 0 \le \tau_k \le s;\ 0 \le r_k \le s;\ x_k \in K \Big\},$$

and hence, K_0 is compact. On the other hand, by (2.1), there is a constant N independent of $t \in \mathbb{R}$ such that

$$\| \int_0^s T_{-1}(\sigma)(f(t-\sigma) - g(t-\sigma))\, d\sigma \| \le N \int_0^s \|f(t-\sigma) - g(t-\sigma)\| \, d\sigma \le N s \epsilon.$$

Thus the range of ψ is contained in $K_0 + Ns\epsilon B_{X_0}$, where B_{X_0} denotes the closed unit ball of X_0. In particular, the range of ψ is totally bounded, which proves the claim. Since ψ has relatively compact range we obtain

$$\lim_{\lambda \to \infty} \Big(\lambda R(\lambda, A_0) - \mu R(\mu, A_0) \Big) z(t, t-s) = 0 \quad \text{uniformly for } t \in \mathbb{R}. \tag{2.8}$$

If $w_\lambda(t, t-s) = \int_{t-s}^t U_B(t,\sigma) \lambda R(\lambda, A) f(\sigma)\, d\sigma$, $t \in \mathbb{R}$, then as in the proof of Theorem 2.2 we derive from (2.8) and (2.6) that

$$\lim_{\lambda, \mu \to \infty} \|w_\lambda(t, t-s) - w_\mu(t, t-s)\| = 0$$

uniformly for $t \in \mathbb{R}$. This completes the proof. □

The following Lemma will be used in Section 3.

Lemma 2.6. *Let $f \in L^1_{loc}(\mathbb{R}, X)$ and let $u \in C(\mathbb{R}, X_0)$ be a mild solution of $(INP)_f$. If $\phi \in C^1(\mathbb{R})$, then ϕu is a mild solution of $(INP)_{\phi' u + \phi f}$.*

Proof. If $t \ge s$, then the representation of u obtained in Theorem 2.2 leads to

$$\begin{aligned}
&\int_s^t U_B(t,\sigma)\phi'(\sigma) u(\sigma)\, d\sigma \\
&\quad = \int_s^t U_B(t,\sigma)\phi'(\sigma) U_B(\sigma, s) u(s)\, d\sigma \\
&\qquad + \lim_{\lambda \to \infty} \int_s^t U_B(t,\sigma)\phi'(\sigma) \int_s^\sigma U_B(\sigma,\tau) \lambda R(\lambda, A) f(\tau)\, d\tau d\sigma \\
&\quad = (\phi(t) - \phi(s)) U_B(t,s) u(s) \\
&\qquad + \lim_{\lambda \to \infty} \int_s^t \int_\tau^t \phi'(\sigma) U_B(t,\tau) \lambda R(\lambda, A) f(\tau)\, d\sigma d\tau \\
&\quad = \phi(t) \Big(U_B(t,s) u(s) + \lim_{\lambda \to \infty} \int_s^t U_B(t,\tau) \lambda R(\lambda, A) f(\tau)\, d\tau \Big) \\
&\qquad - U_B(t,s)\phi(s) u(s) - \lim_{\lambda \to \infty} \int_s^t U_B(t,\tau) \lambda R(\lambda, A) \phi(\tau) f(\tau)\, d\tau.
\end{aligned}$$

Another application of Theorem 2.2 establishes the result. □

3. Asymptotic properties of solutions of inhomogeneous equations

In this section we discuss conditions on the evolution family $(U_B(t,s))_{t\geq s}$ and the inhomogeneity $f \in L^1_{loc}(\mathbb{R}, X)$ which ensure that $(INP)_f$ has a (unique) mild solution u with a prescribed asymptotic behavior. For the rest of the paper we impose the following condition on the perturbation $(B(t))_{t\in\mathbb{R}}$.

(B) $\quad \|B(\cdot)\| \leq b(\cdot)$ for some $b \in L^1_{loc,u}(\mathbb{R})$.

Note that (B) implies exponential boundedness of the evolution family $(U_B(t,s))_{t\geq s}$ (see (2.4)).
At first we discuss the case where $(U_B(t,s))_{t\geq s}$ has an exponential dichotomy. We recall the following notion (see [10], [18], [21], [23], [24], [25], [37]).

Definition 3.1. *An evolution family $(U(t,s))_{t\geq s}$ on the Banach space Z has an* exponential dichotomy *with constants $\alpha > 0$, $L \geq 1$ if there exists a bounded, strongly continuous family of projections $(P(t))_{t\in\mathbb{R}} \subseteq \mathcal{L}(Z)$ such that for $t \geq s$*

(i) *$P(t)U(t,s) = U(t,s)P(s)$,*
(ii) *the map $U_|(t,s) : (Id - P(s))Z \to (Id - P(t))Z : z \mapsto U(t,s)z$ is invertible,*
(iii) *$\|U(t,s)z\| \leq Le^{-\alpha(t-s)}\|z\|$ for $z \in P(s)Z$,*
(iv) *$\|U_|(t,s)^{-1}z\| \leq Le^{-\alpha(t-s)}\|z\|$ for $z \in (Id - P(t))Z$.*

In that case the family $(\Gamma(t,s))_{(t,s)\in\mathbb{R}^2} \subseteq \mathcal{L}(Z)$ given by

$$\Gamma(t,s) = \begin{cases} P(t)U(t,s)P(s), & t \geq s, \\ -(Id - P(t))[U_|(s,t)]^{-1}(Id - P(s)), & t < s, \end{cases}$$

is called the corresponding Green's operator function.

Remark 3.2. *It is shown in* [39, Lemma VI.9.15] *that $(t,s) \mapsto [U_|(t,s)]^{-1}(Id - P(t))$ is strongly continuous for $t \geq s$.*

The existence of an exponential dichotomy for the evolution family $(U(t,s))_{t\geq s}$ on the Banach space Z allows to connect asymptotic properties of the solution $u(\cdot,f) \in C(\mathbb{R}, Z)$ of the integral equation

$$u(t,f) = U(t,s)u(s,f) + \int_s^t U(t,\tau)f(\tau)\,d\tau, \quad t \geq s, \tag{3.1}$$

with asymptotic properties of the function $f \in C(\mathbb{R}, Z)$. We recall the following result in [22, Theorem 2.1], (see also [23, 10.2, Theorem 1], [5, Theorem 4]). By $C_b(\mathbb{R}, Z)$ we denote the set of all bounded, continuous, Z-valued functions on $\mathbb{R}$, and $C_0(\mathbb{R}, Z)$ is the space of all functions in $C_b(\mathbb{R}, Z)$ vanishing at $\pm\infty$.

Theorem 3.3. *Let $(U(t,s))_{t\geq s}$ be an exponentially bounded evolution family on the Banach space Z and let $\mathcal{F}(\mathbb{R}, Z)$ be the space $C_0(\mathbb{R}, Z)$ or $C_b(\mathbb{R}, Z)$. Then $(U(t,s))_{t\geq s}$ has an exponential dichotomy if and only if for every $f \in \mathcal{F}(\mathbb{R}, Z)$ there exists a unique solution $u(\cdot,f) \in \mathcal{F}(\mathbb{R}, Z)$ of equation (3.1). In that case $u(\cdot,f)$ is given by*

$$u(t,f) = \int_{-\infty}^{\infty} \Gamma(t,\sigma)f(\sigma)\,d\sigma, \quad t \in \mathbb{R}. \tag{3.2}$$

We will show a corresponding result on asymptotic properties of the mild solutions of the inhomogeneous equation $(INP)_f$. We stress that in our case the evolution family $(U_B(t,s))_{t\geq s}$ given by equation (2.3) consists of operators on the Banach space X_0 whereas the inhomogeneity f has values in the larger space X. The following lemma plays a central role. By $L^1_{loc,u}(\mathbb{R}, X)$ we denote the space of uniformly locally integrable functions from $\mathbb{R}$ into X equipped with the norm $\|f\|_{1,loc,u} = \sup_{t\in\mathbb{R}} \int_{t-1}^t \|f(\sigma)\|\,d\sigma$.

Lemma 3.4. *Assume that $(U_B(t,s))_{t\geq s}$ has an exponential dichotomy with constants $\alpha > 0$, $L \geq 1$, and projections $(P_B(t))_{t\geq 0}$. For $f \in L^1_{loc,u}(\mathbb{R},X)$ and $\lambda > \omega$ define $u_\lambda(\cdot,f) \in C(\mathbb{R},X_0)$ by*

$$u_\lambda(t,f) = \int_{-\infty}^{\infty} \Gamma_B(t,\sigma)\lambda R(\lambda,A)f(\sigma)\,d\sigma, \quad t \in \mathbb{R},$$

where $(\Gamma_B(t,s))_{(t,s)\in\mathbb{R}^2}$ is the Green's operator function corresponding to $(U_B(t,s))_{t\geq s}$. Then:

(i) *$\|u_\lambda(\cdot,f)\| \leq C\|f\|_{1,loc,u}$ for a constant C independent of $\lambda \geq \omega+1$ and f.*
(ii) *$(u_\lambda(\cdot,f))$ is uniformly convergent on compact intervals in $\mathbb{R}$ as $\lambda \to \infty$.*
(iii) *If $f \in BUC_r(\mathbb{R},X)$, then $(u_\lambda(\cdot,f))$ is uniformly convergent on $\mathbb{R}$ as $\lambda \to \infty$.*

Proof. Let $Q_B(t) = Id - P_B(t)$, $t \in \mathbb{R}$. Since $(U_B(t,s))_{t\geq s}$ has an exponential dichotomy and A is a Hille-Yosida operator we obtain for $t \in \mathbb{R}$ and $\lambda \geq \omega+1$

$$\begin{aligned}
\|u_\lambda(t,f)\| \leq & \int_t^{\infty} \|[U_{B|}(\sigma,t)]^{-1}Q_B(\sigma)\lambda R(\lambda,A)f(\sigma)\|\,d\sigma \\
& + \int_{-\infty}^{t} \|U_B(t,\sigma)P_B(\sigma)\lambda R(\lambda,A)f(\sigma)\|\,d\sigma \\
\leq & \sum_{k\geq 0} Le^{-\alpha k}\|\lambda R(\lambda,A)\| \int_{t+k}^{t+k+1} \|Q_B(\sigma)\|\|f(\sigma)\|\,d\sigma \\
& + \sum_{k\geq 0} Le^{-\alpha k}\|\lambda R(\lambda,A)\| \int_{t-k-1}^{t-k} \|P_B(\sigma)\|\|f(\sigma)\|\,d\sigma \\
\leq & C\|f\|_{1,loc,u},
\end{aligned}$$

where C is a constant independent of f. From this assertion (i) and the continuity of u_λ follow.
In order to show (ii) note that

$$u_\lambda(t,f) = U_B(t,s)u_\lambda(s,f) + \int_s^t U_B(t,\sigma)\lambda R(\lambda,A)f(\sigma)\,d\sigma \ \text{ for } t \geq s$$

(see [22, Proof of Proposition 1.2]). For $\lambda, \mu > \omega+1$, $t \in \mathbb{R}$, and $r > 0$ we have

$$\begin{aligned}
&\|P_B(t)(u_\lambda(t,f) - u_\mu(t,f))\| \\
&\qquad \leq \|U_B(t,t-r)P_B(t-r)(u_\lambda(t-r,f) - u_\mu(t-r,f))\| \\
&\qquad\quad + \|P_B(t)\int_{t-r}^{t} U_B(t,\sigma)(\lambda R(\lambda,A) - \mu R(\mu,A))f(\sigma)\,d\sigma\| \\
&\qquad \leq Le^{-\alpha r}C_1 + \|P_B(t)\int_{t-r}^{t} U_B(t,\sigma)(\lambda R(\lambda,A) - \mu R(\mu,A))f(\sigma)\,d\sigma\|,
\end{aligned} \tag{3.3}$$

where $C_1 = \sup\{\|P_B(t)\|\|u_\lambda(t,f) - u_\mu(t,f)\| : t \in \mathbb{R};\ \lambda,\mu > \omega+1\}$. By Theorem 2.2, $\lim_{\lambda\to\infty} \lambda \int_s^t U_B(t,\sigma)\lambda R(\lambda,A)f(\sigma)\,d\sigma$ exists uniformly for $t \geq s$ in compact intervals in $\mathbb{R}$. Thus, if in (3.3) we choose $r > 0$ sufficiently large and then consider $\lambda,\mu \to \infty$ we obtain

$$\lim_{\lambda,\mu\to\infty} \|P_B(t)(u_\lambda(t,f) - u_\mu(t,f))\| = 0 \tag{3.4}$$

uniformly for t in compact intervals in $\mathbb{R}$.
On the other hand, for $\lambda, \mu > \omega + 1$, $t \in \mathbb{R}$, and $r > 0$ we obtain

$$\begin{aligned}
&\|[U_{B|}(t+r,t)]^{-1}Q_B(t+r)(u_\lambda(t+r,f) - u_\mu(t+r,f))\| \\
&\quad \geq \|Q_B(t)(u_\lambda(t,f) - u_\mu(t,f))\| \\
&\qquad - \|[U_{B|}(t+r,t)]^{-1}Q_B(t+r)\int_t^{t+r} U_B(t+r,\sigma)(\lambda R(\lambda,A) - \mu R(\mu,A))f(\sigma)\,d\sigma\|.
\end{aligned}$$

Thus

$$\begin{aligned}
&\|Q_B(t)(u_\lambda(t,f) - u_\mu(t,f))\| \\
&\qquad \leq Le^{-\alpha r}\Big(C_2 \\
&\qquad\qquad + \|Q_B(t+r)\int_t^{t+r} U_B(t+r,\sigma)(\lambda R(\lambda,A) - \mu R(\mu,A))f(\sigma)\,d\sigma\|\Big),
\end{aligned} \tag{3.5}$$

where $C_2 = \sup\{\|Q_B(t)\|\|u_\lambda(t,f) - u_\mu(t,f)\| : t \in \mathbb{R};\ \lambda, \mu > \omega + 1\}$. As above, if we choose $r > 0$ sufficiently large and apply Theorem 2.2 we obtain

$$\lim_{\lambda,\mu\to\infty} \|Q_B(t)(u_\lambda(t,f) - u_\mu(t,f))\| = 0 \tag{3.6}$$

uniformly for t in compact intervals of $\mathbb{R}$. Assertion (ii) is now an immediate consequence of (3.4) and (3.6).
Finally, if $f \in BUC_r(\mathbb{R}, X)$, then (3.3) and (3.5) together with Proposition 2.5 imply

$$\lim_{\lambda,\mu\to\infty} \|P_B(t)(u_\lambda(t,f) - u_\mu(t,f))\| = 0 \text{ and } \lim_{\lambda,\mu\to\infty} \|Q_B(t)(u_\lambda(t,f) - u_\mu(t,f))\| = 0$$

uniformly for $t \in \mathbb{R}$. This proves (iii). □

We come to our first main result. It is an analogue of Theorem 3.3 and connects asymptotic properties of mild solutions of $(INP)_f$ with the existence of an exponential dichotomy for the evolution family $(U_B(t,s))_{t\geq s}$. In the special case where $B(t) = B$ is constant a similar result has been shown in [2] by completely different methods.

Theorem 3.5. *The following assertions are equivalent.*

(i) *The evolution family $(U_B(t,s))_{t\geq s}$ has an exponential dichotomy.*
(ii) *For every $f \in L^1_{loc,u}(\mathbb{R}, X)$ there is a unique mild solution $u \in C_b(\mathbb{R}, X_0)$ of $(INP)_f$.*
(iii) *For every $f \in C_b(\mathbb{R}, X)$ there is a unique mild solution $u \in C_b(\mathbb{R}, X_0)$ of $(INP)_f$.*
(iv) *For every $f \in C_0(\mathbb{R}, X)$ there is a unique mild solution $u \in C_0(\mathbb{R}, X_0)$ of $(INP)_f$.*

In that case the function $u = u(\cdot, f)$ is given by

$$u(t,f) = \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma_B(t,\sigma)\lambda R(\lambda,A)f(\sigma)\,d\sigma, \quad t \in \mathbb{R}, \tag{3.7}$$

where $(\Gamma_B(t,s))_{(t,s)\in\mathbb{R}^2}$ is the Green's operator function corresponding to $(U_B(t,s))_{t\geq s}$.

Proof. (i) ⇒ (ii): Assume that $(U_B(t,s))_{t\geq s}$ has an exponential dichotomy and let $f \in L^1_{loc,u}(\mathbb{R}, X)$. Lemma 3.4 implies that the limit function $u = u(\cdot, f)$ in (3.7) is defined and $u \in C_b(\mathbb{R}, X)$. We claim that $u(\cdot, f)$ is a mild solution of $(INP)_f$. In fact, if $t \geq s$, then

$$
\begin{aligned}
&u(t,f) - U_B(t,s)u(s,f) \\
&= \lim_{\lambda\to\infty} \lambda\Big(\int_{-\infty}^{\infty} \Gamma_B(t,\sigma)R(\lambda,A)f(\sigma)\,d\sigma - \int_{-\infty}^{s} U_B(t,\sigma)P_B(\sigma)R(\lambda,A)f(\sigma)\,d\sigma \\
&\quad + \int_s^t U_B(t,\sigma)Q_B(\sigma)R(\lambda,A)f(\sigma)\,d\sigma + \int_t^{\infty} [U_{|B}(\sigma,t)]^{-1}Q_B(\sigma)R(\lambda,A)f(\sigma)\,d\sigma\Big) \\
&= \lim_{\lambda\to\infty} \lambda\Big(\int_{-\infty}^{\infty} \Gamma_B(t,\sigma)R(\lambda,A)f(\sigma)\,d\sigma - \int_{-\infty}^{t} U_B(t,\sigma)P_B(\sigma)R(\lambda,A)f(\sigma)\,d\sigma \\
&\quad + \int_s^t U_B(t,\sigma)R(\lambda,A)f(\sigma)\,d\sigma + \int_t^{\infty} [U_{|B}(\sigma,t)]^{-1}Q_B(\sigma)R(\lambda,A)f(\sigma)\,d\sigma\Big) \\
&= \lim_{\lambda\to\infty} \int_s^t U_B(t,\sigma)\lambda R(\lambda,A)f(\sigma)\,d\sigma.
\end{aligned}
$$

By Theorem 2.2, $u(\cdot,f)$ is a mild solution of $(INP)_f$. To show that $u(\cdot,f)$ is the only mild solution of $(INP)_f$ belonging to $C_b(\mathbb{R},X)$ we can assume that $f\equiv 0$ and repeat the arguments in [22, Proof of Proposition 1.2].
Since $C_b(\mathbb{R},X)\subseteq L^1_{loc,u}(\mathbb{R},X)$ the implication (ii) $\Rightarrow$ (iii) is obvious.
(iii) $\Rightarrow$ (iv): From the definition of a mild solution it follows immediately that the operator G assigning to each $f\in C_b(\mathbb{R},X)$ the unique mild solution $u=u(\cdot,f)\in C_b(\mathbb{R},X_0)$ of $(INP)_f$ is closed. Hence, G is bounded. Now let $f\in C_0(\mathbb{R},X)$. We have to show that also $u(\cdot,f)\in C_0(\mathbb{R},X)$. Let $n\in\mathbb{N}$ and choose $t_n>n$ such that $\sup_{|t|>t_n-n}\|f(t)\|<\frac{1}{n}$. For $|t|>t_n$ choose $\phi_t\in C^1(\mathbb{R})$ such that $0\leq\phi_t\leq 1$, $\phi_t(t)=1$, $\mathrm{supp}\phi_t\subseteq[t-n,t+n]$, and $\|\phi_t'\|\leq\frac{2}{n}$. By Lemma 2.6, $G(\phi_t'u+\phi_tf)=\phi_tu$. Hence

$$\|\phi_tu\|_\infty \leq \|G\|\|\phi_t'u+\phi_tf\|_\infty \leq n^{-1}\|G\|(2\|u\|_\infty+1).$$

In particular, $\|u(t)\|=\|\phi_t(t)u(t)\|\leq n^{-1}\|G\|(2\|u\|_\infty+1)$ for $|t|>t_n$. Hence $u\in C_0(\mathbb{R},X_0)$.
Since $C_0(\mathbb{R},X_0)\subseteq C_0(\mathbb{R},X)$ implication (iv) $\Rightarrow$ (i) follows from Theorem 3.3. ☐

Remark 3.6. *The arguments in the proof of (iii) $\Rightarrow$ (iv) can be used to simplify parts of the proof of* [22, Theorem 2.1] *considerably.*

Now we assume that the evolution family $(U_B(t,s))_{t\geq s}$ is *p-periodic*, in the sense that there exists $p>0$ such that $U_B(t+p,s+p)=U_B(t,s)$ for $t\geq s$. From formula (2.3) we see that $(U_B(t,s))_{t\geq s}$ is p-periodic provided that $t\mapsto B(t)$ is p-periodic, i.e., $B(t)=B(t+p)$. We call $U_B(p,0)$ the *monodromy operator* of the evolution family $(U_B(t,s))_{t\geq s}$. On $C(\mathbb{R},X_0)$ we define the operator T by

$$Th(t)=U_B(t,t-p)h(t-p),\quad h\in C_(\mathbb{R},X_0),\ t\in\mathbb{R}. \tag{3.8}$$

If $u\in C(\mathbb{R},X_0)$ is a mild solution of $(INP)_f$, then the representation formula for u obtained in Theorem 2.2 leads to

$$(Id-T)u(t)=\lim_{\lambda\to\infty}\int_{t-p}^{t} U_B(t,\sigma)\lambda R(\lambda,A)f(\sigma)\,d\sigma,\quad t\in\mathbb{R}. \tag{3.9}$$

We need the notion of the *spectrum* $sp(f)$ of a Banach space-valued function $f:\mathbb{R}\to Z$ (cf. [3], [20], [23], [32], [35]). If $f\in C_b(\mathbb{R},Z)$ we set

$$
\begin{aligned}
sp(f)=\{\xi\in\mathbb{R} : &\text{ for every } \epsilon>0 \text{ there exists } \phi\in L^1(\mathbb{R}), \text{ such that} \\
&supp(\hat{\phi})\subseteq[\xi-\epsilon,\xi+\epsilon] \text{ and } \phi\star f\neq 0\},
\end{aligned}
$$

where $\hat{\phi}$ denotes the Fourier transform of ϕ and $\phi \star f$ is the convolution of ϕ and f. Moreover, we set

$$\Sigma_p(f) = \overline{sp(f) + (2\pi/p)\mathbb{Z}} \subseteq \mathbb{R}.$$

We obtain the following extension of [6, Theorem 3.8].

Theorem 3.7. *Assume that the evolution family $(U_B(t,s))_{t\geq s}$ is p-periodic. Let $f \in C_b(\mathbb{R}, X)$ and suppose that $\sigma(U_B(p,0)) \cap \overline{\{e^{i\eta p} : \eta \in sp(f)\}} = \emptyset$. Then:*

(a) *There is at most one mild solution $u \in C_b(\mathbb{R}, X_0)$ of $(INP)_f$ such that $sp(u) \subseteq \Sigma_p(f)$.*

(b) *Let $\mathcal{F}(\mathbb{R}, X_0)$ be a closed, translation-invariant subspace of $BUC(\mathbb{R}, X_0)$ such that $s \mapsto e^{2\pi ins/p} Rh(s)$ belongs to $\mathcal{F}(\mathbb{R}, X_0)$ whenever $h \in \mathcal{F}(\mathbb{R}, X_0)$, $R \in \mathcal{L}(X_0)$, and $n \in \mathbb{Z}$. Suppose that $f \in BUC_r(\mathbb{R}, X)$ such that $\lambda R(\lambda, A)f(\cdot) \in \mathcal{F}(\mathbb{R}, X_0)$ for $\lambda > \omega$. Then there exists a mild solution $u \in \mathcal{F}(\mathbb{R}, X_0)$ of $(INP)_f$, and u has relatively compact range.*

Proof. In order to prove (a) consider

$$\mathcal{M} = \{h \in C_b(\mathbb{R}, X_0) : \, sp(h) \subseteq \Sigma_p(f)\}.$$

In [6, Proof of Theorem 3.8] it is shown that the operator T defined in (3.8) maps $\mathcal{M}$ into itself and the restriction $T_{|\mathcal{M}}$ of T to $\mathcal{M}$ is bounded and satisfies $1 \in \rho(T_{|\mathcal{M}})$. The invertibility of $Id - T_{|\mathcal{M}}$ and (3.9) show that there is at most one mild solution u of $(INP)_f$ contained in $\mathcal{M}$.
For the proof of (b) let

$$\mathcal{N} = \{h \in \mathcal{F}(\mathbb{R}, X_0) : sp(h) \subseteq \Sigma_p(f)\}.$$

In [6, Proof of Theorem 3.8] it is shown that $\mathcal{N}$ is T-invariant and $1 \in \rho(T_{|\mathcal{N}})$. For $\lambda > \omega$ set $f_\lambda = \lambda R(\lambda, A)f(\cdot)$. Note that $sp(f_\lambda) \subseteq sp(f)$. By [6, Theorem 3.8] for each $\lambda > \omega$ there is a (unique) mild solution $u_\lambda \in \mathcal{F}(\mathbb{R}, X_0)$ of $(INP)_{f_\lambda}$ such that $sp(u_\lambda) \subseteq \Sigma_p(f_\lambda) \subseteq \Sigma_p(f)$, and u_λ has relatively compact range. Let

$$w_\lambda(t) = \int_{t-p}^{t} U_B(t,\sigma)\lambda R(\lambda, A)f(\sigma)\, d\sigma, \quad t \in \mathbb{R}, \, \lambda > \omega.$$

Since $f \in BUC_r(\mathbb{R}, X)$ Proposition 2.5 implies that $w(t) = \lim_{\lambda\to\infty} w_\lambda(t)$ exists uniformly for t in $\mathbb{R}$. From (3.9) we obtain $(Id - T_{|\mathcal{N}})u_\lambda = w_\lambda$, $\lambda > \omega$. In particular, $w_\lambda \in \mathcal{N}$ for $\lambda > \omega$, and $u_\lambda = (Id - T_{|\mathcal{N}})^{-1}w_\lambda$ converges uniformly to $u = (Id - T_{|\mathcal{N}})^{-1}w \in \mathcal{N}$ as $\lambda \to \infty$. From Theorem 2.2 and the fact that each u_λ is a mild solution of $(INP)_{f_\lambda}$ it follows that the limit function u is a mild solution of $(INP)_f$. Moreover, since each u_λ has relatively compact range also u has relatively compact range. This completes the proof. □

Recall that a function $h \in BUC(\mathbb{R}, Z)$ is *almost periodic* if the set of translates $\{h(\cdot + t) : t \in \mathbb{R}\}$ is relatively compact in $BUC(\mathbb{R}, Z)$. By $AP(\mathbb{R}, Z)$ we denote the space of almost periodic, Z-valued functions. Theorem 3.7 has the following immediate consequence (cf. [6, Corollary 3.9]).

Corollary 3.8. *Assume that $(U_B(t,s))_{t\geq s}$ is p-periodic. Let $f \in AP(\mathbb{R}, X)$, and suppose that $\sigma(U_B(p,0)) \cap \overline{\{e^{i\eta p} : \eta \in sp(f)\}} = \emptyset$. Then there is a unique $u \in C_b(\mathbb{R}, X_0)$ such that u is a mild solution of $(INP)_f$ and $sp(u) \subseteq \Sigma_p(f)$. Moreover, $u \in AP(\mathbb{R}, X_0)$.*

Let $S^1 = \{\lambda \in \mathbb{C} : \, |\lambda| = 1\}$ be the unit circle.

Corollary 3.9. *If the evolution family $(U_B(t,s))_{t\geq s}$ is p-periodic, then the following assertions are equivalent.*

(i) $S^1 \subseteq \rho(U_B(p,0))$.
(ii) *For every* $f \in AP(\mathbb{R}, X)$ *there is a unique mild solution* $u \in AP(\mathbb{R}, X_0)$ *of* $(INP)_f$.

Proof. Note that (i) is equivalent to the existence of an exponential dichotomy for $(U_B(t,s))_{t\geq s}$ (see [19, Theorem 3.2.2], see also [18, Theorem 7.2.3]). Hence if (i) is satisfied and $f \in AP(\mathbb{R}, X)$, the existence of a mild solution $u \in AP(\mathbb{R}, X_0)$ of $(INP)_f$ follows from Corollary 3.8, whereas the uniqueness is a consequence of Theorem 3.5. The converse implication (ii) $\Rightarrow$ (i) follows immediately from [27, Lemma 4]. ☐

By $P_p(\mathbb{R}, Z)$ we denote the space of p-periodic, continuous, Z-valued functions on $\mathbb{R}$.

Corollary 3.10. *If the evolution family* $(U_B(t,s))_{t\geq s}$ *is* p*-periodic, then the following assertions are equivalent:*
(i) $1 \in \rho(U_B(p,0))$.
(ii) *For every* $f \in P_p(\mathbb{R}, X)$ *there exists a unique mild solution* $u \in P_p(\mathbb{R}, X_0)$ *of* $(INP)_f$.

Proof. Assume that (i) is satisfied. If $f \in P_p(\mathbb{R}, X)$, then $sp(f) \subseteq \frac{2\pi}{p}\mathbb{Z}$ (see [32, Example 0.1]). Hence, by Theorem 3.7, there is a unique mild solution $u \in P_p(\mathbb{R}, X_0)$ of $(INP)_f$. The implication (ii) $\Rightarrow$ (i) follows immediately from [19, Theorem 3.3.4](see also [27, Proposition 1]). ☐

Remark 3.11. *The operator* T_{P_p} *on* $P_p(\mathbb{R}, X_0)$ *satisfies* $T_{P_p}h(t) = U_B(t, t-p)h(t)$, $h \in P_p(\mathbb{R}, X_0)$, $t \in \mathbb{R}$. *Moreover* $1 \in \rho(U_B(p,0))$ *implies* $1 \in \rho(U_B(t+p,t))$ *for all* $t \in \mathbb{R}$ *(see* [18, Lemma 7.2.2]*). From this we obtain* $(Id - T_{P_p})^{-1}h(t) = (Id - U(t,t-p))^{-1}h(t)$, $h \in P_p(\mathbb{R}, X_0)$, $t \in \mathbb{R}$. *In particular, by (3.9), the mild solution* u *obtained in Corollary 3.10 (ii) has the representation*

$$u(t) = \lim_{\lambda\to\infty} (Id - U_B(t,t-p))^{-1} \int_{t-p}^{t} U_B(t,\sigma)\lambda R(\lambda, A) f(\sigma)\, d\sigma, \quad t \in \mathbb{R}.$$

4. The semilinear equation

In this section we apply the results of Section 3 to the semilinear equation

$$(SNP) \quad \frac{d}{dt}u(t) = (A + B(t))u(t) + F(t, u(t)), \quad t \in \mathbb{R},$$

where A and $B(t)$, $t \in \mathbb{R}$, are as in the previous sections and $F : \mathbb{R} \times X_0 \to X$ is jointly continuous and Lipschitz continuous in the second variable with Lipschitz constant l independent of t and x. Moreover, we assume that $t \mapsto F(t,0)$ is a bounded function on $\mathbb{R}$. Our definition of a mild solution of (SNP) is similar to Definition 2.1.

Definition 4.1. *A function* $u \in C(\mathbb{R}, X_0)$ *is called a* mild solution *of* (SNP) *if*

$$u(t) = T_0(t-s)u(s) + \int_s^t T_{-1}(t-\sigma)\Big(B(\sigma)u(\sigma) + F(\sigma, u(\sigma))\Big)\, d\sigma \quad \text{for } t \geq s.$$

The following conditions will be needed.

(H1) The evolution family $(U_B(t,s))_{t\geq s}$ has an exponential dichotomy with constants $\alpha > 0$, $L \geq 1$, and projections $(P_B(t))_{t\in\mathbb{R}}$, and $l < \dfrac{\alpha}{2LC}$, where $C = \sup_{t\in\mathbb{R}} \sup_{\lambda>\omega} \{\|\lambda P_B(t)R(\lambda, A)\|, \|\lambda(Id - P_B(t))R(\lambda, A)\|\} < \infty$.
(H2) The evolution family $(U_B(t,s))_{t\geq s}$ is p-periodic, $1 \in \rho(U_B(p,0))$, and $l < (\tilde{C}pC)^{-1}$, where $C = \sup_{t\in\mathbb{R}} \|(Id - U(t,t-p))^{-1}\|$ and $\tilde{C} = \sup_{t\in\mathbb{R}} \|U(t,t-p)\|$.

Theorem 4.2. *If condition (H1) holds, then there exists exactly one mild solution $u \in C_b(\mathbb{R}, X_0)$ of (SNP).*

Proof. For $f \in C_b(\mathbb{R}, X_0)$ set

$$Sf(t) = \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma_B(t,\sigma)\lambda R(\lambda, A)F(\sigma, f(\sigma))\, d\sigma, \quad t \in \mathbb{R}.$$

By Lemma 3.4 and the boundedness of $F(\cdot, 0)$, S is well-defined and maps $C_b(\mathbb{R}, X_0)$ into itself. If $f, g \in C_b(\mathbb{R}, X_0)$, then

$$\begin{aligned}\|Sf - Sg\|_\infty &= \sup_{t\in\mathbb{R}} \left\| \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma_B(t,\sigma)\lambda R(\lambda, A)\Big(F(\sigma, f(\sigma)) - F(\sigma, g(\sigma))\Big)\, d\sigma \right\| \\ &\le \sup_{t\in\mathbb{R}} CL \int_{-\infty}^{\infty} e^{-\alpha|t-\sigma|} l \|f-g\|_\infty \, d\sigma \le \frac{2CL}{\alpha} l \|f-g\|_\infty .\end{aligned}$$

By our assumption $\frac{2CL}{\alpha} l < 1$. Hence S is a contraction, and by Banach's Fixed Point Theorem there is a unique function $u \in C_b(\mathbb{R}, X_0)$ such that

$$u(t) = \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} \Gamma(t,\sigma)\lambda R(\lambda, A)F(\sigma, u(\sigma))\, d\sigma, \quad t \in \mathbb{R}.$$

Theorem 3.5 implies that u is the unique mild solution of (SNP) contained in $C_b(\mathbb{R}, X_0)$. □

In the same way the following two results can be derived from Theorem 3.5 and Corollary 3.9, respectively.

Proposition 4.3. *Assume that condition (H1) holds and that $\lim_{t\to\pm\infty} F(t, y) = 0$ uniformly for y in compact sets in X_0. Then there exists exactly one mild solution $u \in C_0(\mathbb{R}, X_0)$ of (SNP).*

Proposition 4.4. *Assume that condition (H1) holds and that the evolution family $(U_B(t,s))_{t\ge s}$ is p-periodic. If $F(\cdot, x)$ is almost periodic uniformly for x in compact sets in X_0, i.e. for every compact set K in X_0 and every sequence (t_n) in $\mathbb{R}$ there is a subsequence (s_n) of (t_n) such that $(F(t+s_n, x))$ converges uniformly for (t,x) in $\mathbb{R}\times K$, then there is exactly one mild solution $u \in AP(\mathbb{R}, X_0)$ of (SNP).*

The following result is the semilinear version of Corollary 3.10.

Theorem 4.5. *Assume that condition (H2) holds and that $F(t+p, x) = F(t, x)$ for every $t \in \mathbb{R}$ and every $x \in X_0$. Then there exists exactly one mild solution $u \in P_p(\mathbb{R}, X_0)$ of (SNP).*

Proof. For $f \in P_p(\mathbb{R}, X_0)$ set

$$Sf(t) = \lim_{\lambda\to\infty} (Id - U_B(t, t-p))^{-1} \int_{t-p}^{t} U(t,\sigma)\lambda R(\lambda, A)F(\sigma, f(\sigma))\, d\sigma, \quad t \in \mathbb{R}.$$

By Proposition 2.5 and Remark 3.11, S is well-defined and maps $P_p(\mathbb{R}, X_0)$ into itself. If $f, g \in P_p(\mathbb{R}, X_0)$, then

$$\begin{aligned}&\|Sf - Sg\|_\infty \\ &= \sup_{t\in\mathbb{R}} \left\| \lim_{\lambda\to\infty} (Id - U_B(t,t-p))^{-1} \int_{t-p}^{t} U_B(t,\sigma)\lambda R(\lambda, A)\Big(F(\sigma, f(\sigma)) - F(\sigma, g(\sigma))\Big)\, d\sigma \right\| \\ &\le \tilde{C}pCl\|f-g\|_\infty .\end{aligned}$$

Since $\tilde{C}pCl < 1$, the map S is contractive and there is a unique function $v \in P_p(\mathbb{R}, X_0)$ such that

$$(4.1) \quad v(t) = \lim_{\lambda\to\infty} (Id - U_B(t, t-p))^{-1} \int_{t-p}^{t} U_B(t,\sigma)\lambda R(\lambda, A)F(\sigma, v(\sigma))\, d\sigma, \quad t \in \mathbb{R}.$$

By Corollary 3.10, there is a unique mild solution $u \in P_p(\mathbb{R}, X_0)$ of $(INP)_{F(\cdot,v(\cdot))}$. The representation of u obtained in Remark 3.11 shows that $u = v$, and hence v is a mild solution of (SNP). On the other hand, it follows from (3.9) and Remark 3.11 that each p-periodic mild solution of (SNP) satisfies (4.1). Hence v is the only p-periodic mild solution of (SNP). □

5. Nonautonomous retarded differential equations

In this section we apply the results obtained for (INP) to retarded differential equations. Throughout the whole section Y is a fixed Banach space. We consider the inhomogeneous nonautonomous retarded differential equation

$$(RINP)_h \quad \frac{d}{dt}w(t) = Cw(t) + K(t)w_t + h(t), \quad t \in \mathbb{R},$$

where $(C, D(C))$ is a Hille-Yosida operator on Y and $h \in L^1_{loc}(\mathbb{R}, Y)$. The part C_0 of C on $Y_0 = \overline{D(C)}$ generates a C_0-semigroup $(S_0(t))_{t\geq 0}$ on Y_0, and by $(S_{-1}(t))_{t\geq 0}$ we denote the corresponding extrapolated C_0-semigroup on the extrapolation space Y_{-1}. We set $E = C([-p, 0], Y_0)$, $p > 0$, and for a function $w \in C(\mathbb{R}, Y_0)$ we define $w_t \in E$ by $w_t(r) = w(t + r)$, $r \in [-p, 0]$. Finally, we assume that $K(t)$, $t \in \mathbb{R}$, is a family of operators in $\mathcal{L}(E, Y)$ such that $t \mapsto K(t)\phi$ is strongly measurable for every $\phi \in E$, and $\|K(\cdot)\| \leq d(\cdot)$ for a function $d \in L^1_{loc,u}(\mathbb{R})$. We define mild solutions of $(RINP)_h$ as follows (cf. [4], [15], [16], [28], [34], and [41]).

Definition 5.1. *If $h \in L^1_{loc}(\mathbb{R}, Y)$, then $w = w(\cdot, h) \in C(\mathbb{R}, Y_0)$ is called a* mild solution *of $(RINP)_h$ if*

$$w(t) = S_0(t-s)w(s) + \int_s^t S_{-1}(t-\sigma)\Big(K(\sigma)w_\sigma + h(\sigma)\Big)\, d\sigma \quad \text{for } t \geq s.$$

Remark 5.2. *If C is the generator of a C_0-semigroup on Y, then the above definition of a mild solution coincides with that given in* [15], [28], [34], [41].

In [16] (see also [33], [40]) it is shown how $(RINP)$ can be transformed into an equation of the form (INP). For this we set $X = Y \times E$ and consider the equation

$$(INP)_f \quad \frac{d}{dt}u(t) = Au(t) + B(t)u(t) + f(t), \quad t \in \mathbb{R},$$

where $A : D(A) \to X$ is the linear operator on X given by

$$A\begin{pmatrix}0\\ \phi\end{pmatrix} = \begin{pmatrix}-\phi'(0) + C\phi(0)\\ \phi'\end{pmatrix},$$

$$D(A) = \{\begin{pmatrix}0\\ \phi\end{pmatrix} \in \{0\} \times E : \phi \in C^1([-p, 0], Y_0),\ \phi(0) \in D(C)\},$$

$B(t) \in \mathcal{L}(\{0\} \times E, X)$, $t \in \mathbb{R}$, is defined by

$$B(t)\begin{pmatrix}0\\ \phi\end{pmatrix} = \begin{pmatrix}K(t)\phi\\ 0\end{pmatrix},$$

and $f(\cdot) = \binom{h(\cdot)}{0}$. It is shown in [33] that A is a Hille-Yosida operator on X, and the C_0-semigroup $(T_0(t))_{t\geq 0}$ generated by the part A_0 of A in $X_0 = \overline{D(A)} = \{0\} \times E$ is given by

$$(T_0(t)_{|E}\phi)(r) = \begin{cases} \phi(t+r) & \text{if } t+r \leq 0, \\ S_0(t+r)\phi(0) & \text{if } t+r > 0. \end{cases}$$

We recall the following results obtained in [16, Theorem 5.3 and Proposition 5.4].

Proposition 5.3.

(a) *If for $u \in C(\mathbb{R}, E)$ the map $t \mapsto \binom{0}{u(t)}$ is a mild solution of $(INP)_f$, then $t \mapsto u(t)(0)$ is a mild solution of $(RINP)_h$ and $u(t)(\xi) = u(t+\xi)(0)$ for $t \in \mathbb{R}$ and $\xi \in [-p, 0]$.*

(b) *If $w \in C(\mathbb{R}, Y_0)$ is a mild solution of $(RINP)_h$, then $t \mapsto \binom{0}{w_t}$ is a mild solution of $(INP)_f$.*

Proposition 5.4. *If $(U_B(t,s))_{t\geq s}$ is the evolution family on E determined by the variation-of-parameters formula*

$$\begin{pmatrix} 0 \\ U_B(t,s)\phi \end{pmatrix} = T_0(t-s)\begin{pmatrix} 0 \\ \phi \end{pmatrix} + \int_s^t T_{-1}(t-\sigma)B(\sigma)\begin{pmatrix} 0 \\ U_B(\sigma,s)\phi \end{pmatrix} d\sigma, \tag{5.1}$$

$t \geq s$, $\phi \in E$, then each mild solution $w \in C(\mathbb{R}, Y_0)$ of $(RINP)_0$ satisfies

$$w_t = U_B(t,s)w_s \quad \textit{for } t \geq s.$$

Furthermore, if $\phi \in E$ and $t \geq s$, then

$$(U_B(t,s)\phi)(\xi) = \begin{cases} S_0(t+\xi-s)\phi(0) & \\ \quad + \int_s^{t+\xi} S_{-1}(t+\xi-\sigma)K(\sigma)U_B(\sigma,s)\phi\, d\sigma, & t+\xi \geq s, \\ \phi(t+\xi-s), & t+\xi \leq s. \end{cases} \tag{5.2}$$

Asymptotic properties of the mild solutions of $(RINP)_h$ are connected with properties of the evolution family $(U_B(t,s))_{t\geq s}$ on E in the following way.

Theorem 5.5. *Assume that the evolution family $(U_B(t,s))_{t\geq s}$ defined by (5.1) has an exponential dichotomy. Then:*

(a) *For every $h \in L^1_{loc,u}(\mathbb{R}, Y)$ (in particular, for every $h \in C_b(\mathbb{R}, Y)$) there exists a unique mild solution $w \in C_b(\mathbb{R}, Y_0)$ of $(RINP)_h$.*

(b) *For every $h \in C_0(\mathbb{R}, Y)$ there exists a unique mild solution $w \in C_0(\mathbb{R}, Y_0)$ of $(RINP)_h$.*

Proof. If h is in $L^1_{loc,u}(\mathbb{R}, Y)$ resp. $C_0(\mathbb{R}, Y)$, then $f : \mathbb{R} \to X$ defined by $f(t) = \binom{h(t)}{0}$ is in $C_b(\mathbb{R}, X)$ resp. $C_0(\mathbb{R}, X)$. Proposition 5.3 shows that there is a one-to-one correspondence between the mild solutions $w \in C(\mathbb{R}, Y_0)$ of $(RINP)_h$ and the mild solutions $u \in C(\mathbb{R}, X_0)$ of $(INP)_f$, and w is in $C_b(\mathbb{R}, Y_0)$ resp. $C_0(\mathbb{R}, Y_0)$ if and only if u is in $C_b(\mathbb{R}, X_0)$ resp. $C_0(\mathbb{R}, X_0)$. An application of Theorem 3.5 and proves the theorem. □

In the same way the following results can be derived from Corollary 3.9 and Corollary 3.10.

Theorem 5.6. *If the evolution family $(U_B(t,s))_{t\geq s}$ defined by (5.1) is p-periodic and $S^1 \subseteq \rho(U_B(p,0))$, then for every $h \in AP(\mathbb{R}, Y)$ there is a unique mild solution $w \in AP(\mathbb{R}, Y_0)$ of $(RINP)_h$.*

Theorem 5.7. *If the evolution family $(U_B(t,s))_{t\geq s}$ defined by (5.1) is p-periodic and $1 \in \rho(U_B(p,0))$, then for every $h \in P_p(\mathbb{R},Y)$ there is a unique mild solution $w \in P_p(\mathbb{R},Y_0)$ of $(RINP)_h$.*

To give more concrete results we impose the following additional condition on the family $K(t)$, $t \in \mathbb{R}$.

(K) Each operator $K(t)$, $t \in \mathbb{R}$, is of the form

$$K(t)\phi = \hat{K}(t)\phi(-p), \quad \phi \in E,$$

where $\hat{K}(t)$, $t \in \mathbb{R}$, is a p-periodic family in $\mathcal{L}(Y_0,Y)$ such that $t \mapsto \hat{K}(t)y$ is strongly measurable for all $y \in Y_0$, and $\|\hat{K}(\cdot)\| \leq d(\cdot)$ for some $d \in L^1_{loc}(\mathbb{R})$.

If condition (K) holds, then the evolution family $(U_B(t,s))_{t\geq s}$ is p-periodic. Now, we want to determine the spectrum $\sigma(U_B(p,0))$ of the monodromy operator $U_B(p,0)$. To that purpose we consider for each $\lambda \in \mathbb{C}$ the evolution family $(V_K^\lambda(t,s))_{t\geq s}$ on Y_0 determined by the integral equation

$$V_K^\lambda(t,s)y = S_0(t-s)y + \int_s^t S_{-1}(t-\sigma)e^{-\lambda p}\hat{K}(\sigma)V_K^\lambda(\sigma,s)y\,d\sigma \tag{5.3}$$

for $y \in Y_0$ and $t \geq s$. The existence of $(V_K^\lambda(t,s))_{t\geq s}$ is guaranteed by the same reasons as for the evolution family $(U_B(t,s))_{t\geq s}$ in (2.3). One can easily see that $(V_K^\lambda(t,s))_{t\geq s}$ is p-periodic for every $\lambda \in \mathbb{C}$.

Remark 5.8. *If $\lambda \in \mathbb{C}$ and $y \in Y_0$, then $t \mapsto V_K^\lambda(t,s)y$, $t \geq s$, is the unique mild solution of the initial value problem*

$$(\lambda P)_{s,y} \quad \begin{cases} \frac{d}{dt}w(t) = & Cw(t) + e^{-\lambda p}\hat{K}(t)w(t), \quad t \geq s, \\ \quad w(s) = & y \in Y_0. \end{cases}$$

Next we derive a generalized *characteristic equation* for $(RINP)_0$ under the additional condition (K). In the finite dimensional case, i.e., $Y = \mathbb{C}^n$, $C = 0$, and $\hat{K}(t)y = k(t)y$, where k is a p-periodic matrix-valued function, this leads to the classical characteristic equation as it can be found e.g. in [17, Theorem 8.3.1]. In the autonomous case, i.e., $\hat{K}(t) = \hat{K} \in \mathcal{L}(Y_0)$, a related result is shown in [14, Chapter VI, Proposition 6.7].

Theorem 5.9. *Assume that condition (K) holds and let $\lambda \in \mathbb{C}$. If $(U_B(t,s))_{t\geq s}$ is the evolution family defined in (5.1) and $(V_K^\lambda(t,s))_{t\geq s}$ is the evolution family defined in (5.3), then*

$$e^{\lambda p} \in \rho(U_B(p,0)) \ \text{ if and only if } \ e^{\lambda p} \in \rho(V_K^\lambda(p,0)). \tag{5.4}$$

Proof. First we show the"only if" part. Let $e^{\lambda p} \in \rho(U_B(p,0))$. Then for every $\phi \in E$ there exists $\psi_\phi \in E$ such that $\phi = e^{\lambda p}\psi_\phi - U_B(p,0)\psi_\phi$. By (5.2) we have

$$\phi(-p) = e^{\lambda p}\psi_\phi(-p) - \psi_\phi(0), \tag{5.5}$$

and

$$\begin{aligned} \phi(\xi) &= e^{\lambda p}\psi_\phi(\xi) - S_0(p+\xi)\psi_\phi(0) - \int_0^{p+\xi} S_{-1}(p+\xi-\sigma)\hat{K}(\sigma)\psi_\phi(\sigma-p)\,d\sigma \\ &= e^{\lambda p}\psi_\phi(\xi) + S_0(p+\xi)\phi(-p) - e^{\lambda p}S_0(p+\xi)\psi_\phi(-p) \\ &\quad - \int_{-p}^{\xi} S_{-1}(\xi-\sigma)\hat{K}(\sigma)\psi_\phi(\sigma)\,d\sigma \end{aligned} \tag{5.6}$$

for $\xi \in [-p,0]$. In order to show surjectivity of $e^{\lambda p}Id - V_K^\lambda(p,0)$ fix $y \in Y_0$ and set

$$\phi(\xi) = S_0(p+\xi)y, \quad \xi \in [-p,0].$$

Then (5.6) leads to

$$\psi_\phi(\xi) = S_0(p+\xi)\psi_\phi(-p) + \int_{-p}^{\xi} S_{-1}(\xi-\sigma)e^{-\lambda p}\hat{K}(\sigma)\psi_\phi(\sigma)\,d\sigma, \quad \xi \in [-p,0].$$

By Remark 5.8, $\xi \mapsto V_K^\lambda(\xi,-p)\psi_\phi(-p)$ is the unique mild solution of $(\lambda P)_{-p,\psi_\phi(-p)}$. Hence $\psi_\phi(\xi) = V_K^\lambda(\xi,-p)\psi_\phi(-p)$ for $\xi \in [-p,0]$. Since $\phi(-p) = y$ we obtain from (5.5)

$$y = e^{\lambda p}\psi_\phi(-p) - V_K^\lambda(0,-p)\psi_\phi(-p)$$

which proves the surjectivity of $e^{\lambda p}Id - V_K^\lambda(p,0)$.
In order to prove injectivity assume that $V_K^\lambda(0,-p)x = e^{\lambda p}x$ for some $x \in Y_0$. Let $\psi_x(\xi) = V_K^\lambda(\xi,-p)x$, $\xi \in [-p,0]$. Using (5.3), (5.2), and condition (K), a straightforward computation shows that $U_B(0,-p)\psi_x = e^{\lambda p}\psi_x$. Thus $\psi_x = 0$. In particular, $x = \psi_x(-p) = 0$.
Now we prove the "if" part. Let $\phi \in E$. An application of the generalized form of Banach's Fixed Point Theorem shows that for every $y \in Y_0$ there is $\psi_y \in E$ such that

$$e^{\lambda p}\psi_y(\xi) = \phi(\xi) + S_0(p+\xi)y + \int_{-p}^{\xi} S_{-1}(\xi-\sigma)\hat{K}(\sigma)\psi_y(\sigma)\,d\sigma \tag{5.7}$$

for $\xi \in [-p,0]$. By subtracting (5.3) we obtain

$$\begin{aligned} &e^{\lambda p}\psi_y(\xi) - V_K^\lambda(\xi,-p)y \\ &= \phi(\xi) + \int_{-p}^{\xi} S_{-1}(\xi-\sigma)e^{-\lambda p}\hat{K}(\sigma)(e^{\lambda p}\psi_y(\sigma) - V_K^\lambda(\sigma,-p)y)\,d\sigma \end{aligned} \tag{5.8}$$

For y_1, $y_2 \in Y_0$ this leads to

$$\begin{aligned} &e^{\lambda p}\psi_{y_1}(\xi) - V_K^\lambda(\xi,-p)y_1 - e^{\lambda p}\psi_{y_2}(\xi) + V_K^\lambda(\xi,-p)y_2 \\ &= \int_{-p}^{\xi} S_{-1}(\xi-\sigma)e^{-\lambda p}\hat{K}(\sigma)\Big(e^{\lambda p}\psi_{y_1}(\sigma) - V_K^\lambda(\sigma,-p)y_1 - e^{-\lambda p}\psi_{y_2}(\sigma) + V_K^\lambda(\sigma,-p)y_2\Big)\,d\sigma. \end{aligned}$$

An application of Gronwall's inequality yields

$$e^{\lambda p}\psi_{y_1}(\xi) - V_K^\lambda(\xi,-p)y_1 = e^{\lambda p}\psi_{y_2}(\xi) - V_K^\lambda(\xi,-p)y_2 \quad \text{for } \xi \in [-p,0].$$

By the assumption and the y-independence of $e^{\lambda p}\psi_y - V_K^\lambda(\cdot,-p)y$, we can choose $\tilde{y} \in Y_0$ such that

$$e^{\lambda p}\tilde{y} - V_K^\lambda(0,-p)\tilde{y} = \phi(0) + \int_{-p}^{0} S_{-1}(-\sigma)e^{-\lambda p}\hat{K}(\sigma)\Big(e^{\lambda p}\psi_{\tilde{y}}(\sigma) - V_K^\lambda(\sigma,-p)\tilde{y}\Big)\,d\sigma.$$

Evaluation of (5.8) at $\xi = 0$ then leads to

$$\tilde{y} = \psi_{\phi,\tilde{y}}(0).$$

By using (5.2) and (5.7) a direct computation yields $U_B(0,-p)\psi_{\tilde{y}} = e^{\lambda p}\psi_{\tilde{y}} - \phi$, which shows the surjectivity of $e^{\lambda p} - U_B(p,0)$.
To prove injectivity assume that $e^{\lambda p}\psi - U_B(p,0)\psi = 0$ for some $\psi \in E$. Then (5.2) leads to

$$e^{\lambda p}\psi(\xi) - S_0(p+\xi)\psi(0) - \int_{-p}^{\xi} S_{-1}(\xi-\sigma)\hat{K}(\sigma)\psi(\sigma)\,d\sigma = 0 \quad \text{for } \xi \in [-p,0].$$

Since $\xi \mapsto V_K^\lambda(\xi,-p)\psi(0)$ is the unique mild solution of $(\lambda P)_{-p,\psi(0)}$, we obtain $e^{\lambda p}\psi(\xi) = V_K^\lambda(\xi,-p)\psi(0)$, $\xi \in [-p,0]$. In particular $e^{\lambda p}\psi(0) = V_K^\lambda(0,-p)\psi(0)$, and the invertibility of $e^{\lambda p}Id - V_K^\lambda(0,-p)$ implies $\psi(0) = 0$. Hence $\psi = e^{-\lambda p}V_K^\lambda(\cdot,-p)\psi(0) = 0$. □

As a concrete example we discuss the retarded differential equation

$$(E)\left\{\begin{array}{rl}\frac{\partial}{\partial t}w(t,x) &= \frac{\partial^2}{\partial x^2}w(t,x)-aw(t,x)-b(t)w(t-1,x)+f(t,x),\ 0\le x\le 2\pi,\ t\ge s,\\ w(t,x) &= \varphi(t-s,x),\ 0\le x\le 2\pi,\ s-1\le t\le s,\end{array}\right.$$

with initial value $\varphi\in C([-1,0]\times[0,2\pi])$. We assume that $a\in\mathbb{R}$, $b:\mathbb{R}\to\mathbb{R}$ is 1-periodic and locally integrable, and $f:\mathbb{R}\times[0,2\pi]\to\mathbb{R}$ is continuous. It is known (see [12]) that on the Banach space $Y=C[0,2\pi]$ the operator $(C,D(C))$ given by

$$(C\psi)(x)=\frac{\partial^2}{\partial x^2}\psi(x)-a\psi(x),\ x\in[0,2\pi],$$
$$D(C)=\{\psi\in C^2([0,2\pi]):\psi(0)=\psi(2\pi)=0\},$$

is a Hille-Yosida operator. The spectrum of the part C_0 of C in $Y_0=\overline{D(C)}=\{\psi\in C[0,2\pi]:\psi(0)=\psi(2\pi)=0\}$ is the set $\{-n^2-a:n=1,2,3,...\}$. For $E=C([-1,0],Y_0)$ and $t\in\mathbb{R}$ we define

$$K(t):E\to Y_0:\varphi\mapsto -b(t)\varphi(t-1,\cdot).$$

Clearly, the operator family $(K(t))_{t\in\mathbb{R}}$ satisfies condition (K) for $p=1$. The evolution family $(U_B(t,s))_{t\ge s}$ on E given by equation (5.1) is 1-periodic. Hence we can apply Theorem 5.9 to determine the spectrum of $U_B(1,0)$. We set $\bar b=\int_0^1 b(\tau)\,d\tau$.

Proposition 5.10. *$e^\lambda\in\sigma(U_B(1,0))$ if and only if there exists $k\in\mathbb{Z}$ and $n\in\{1,2,3,...\}$ such that*

$$\lambda+2\pi ik=-e^{-\lambda}\bar b-n^2-a. \tag{5.9}$$

Proof. The evolution family $(V_K^\lambda(t,s))_{t\ge s}$ defined in (5.3) is given by

$$V_K^\lambda(t,s)=e^{-\int_s^t e^{-\lambda}b(\tau)\,d\tau}S_0(t-s),\ t\ge s,$$

where $(S_0(t))_{t\ge0}$ is the C_0-semigroup on $Y_0=\overline{D(C)}$ generated by C_0. From Theorem 5.9 it follows that $e^\lambda\in\sigma(U_B(1,0))$ if and only if $e^\lambda\in\sigma(e^{-\int_0^1 e^{-\lambda}b(\tau)\,d\tau}S(1))$. Since for C_0 the spectral mapping theorem holds we have

$$\sigma(S(1))=\{e^{-n^2-a}:n=1,2,3,...\}.$$

Thus $e^\lambda\in\sigma(U_B(1,0))$ if and only if

$$e^\lambda=e^{(-\int_0^1 e^{-\lambda}b(\tau)\,d\tau)}e^{(-n^2-a)}=e^{-e^{-\lambda}\bar b-n^2-a}\ \text{ for some } n\in\{1,2,3,...\}.$$

However, this is the case if and only if

$$\lambda+2\pi ik=-e^{-\lambda}\bar b-n^2-a\ \text{ for some }\ n\in\{1,2,3,...\}\text{ and some }k\in\mathbb{Z}.$$

□

While it appears difficult to determine the set of all $\lambda\in\mathbb{C}$ satisfying equation (5.9), there are results saying for which values of n^2+a and $\bar b$ all solutions $\mu\in\mathbb{C}$ of

$$\mu=-e^{-\mu}\bar b-n^2-a \tag{5.10}$$

have negative real part (see e.g. [17, p.135]). This is the case if $(n^2+a,\bar b)$ belongs to the shaded region shown in Figure 1.
Since $n^2\ge1$, we obtain $Re\mu<0$ for each μ satisfying (5.10) for some $n\in\{1,2,3,...\}$ if $(a,\bar b)$ is in the shaded region in Figure 2.

For example, if $a=-1$ and $\bar b=\frac{\pi}{3}$, then all $\lambda\in\mathbb{C}$ such that $e^\lambda\in\sigma(U_B(1,0))$ have negative real part. Hence $(U_B(t,s))_{t\ge s}$ is exponentially stable and, in particular, has an exponential dichotomy. Note that in this case the semigroup $(S_0(t))_{t\ge0}$ does not

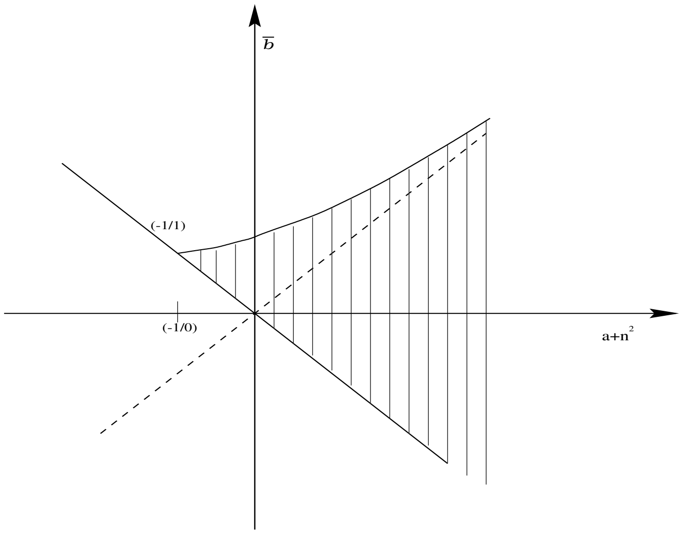

FIGURE 1

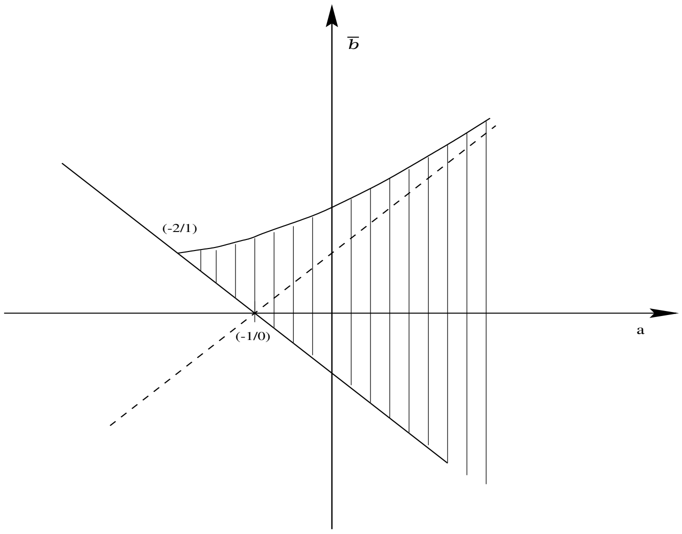

FIGURE 2

have an exponential dichotomy. The proof our next theorem follows from Theorem 5.5 and Theorem 5.6.

Theorem 5.11. *Assume that $(a, \bar{b})$ belongs to the shaded region shown in Figure 2. Then the following holds.*

(i) *If the function f is bounded, then there exists exactly one bounded mild solution w of equation (E).*
(ii) *If $f(\cdot, x) \in C_0(\mathbb{R})$ for every $x \in [0, 2\pi]$, then there exists exactly one mild solution w of (E) such that $\lim_{t\to\pm\infty} w(t, x) = 0$.*
(iii) *If $f(\cdot, x)$ is almost periodic uniformly for $x \in [0, 2\pi]$, then there exists exactly one mild solution w of (E) such that $w(\cdot, x)$ is almost periodic uniformly for $x \in [0, 2\pi]$.*

Proposition 5.10 and Theorem 5.7 lead to conditions for the existence of a unique periodic mild solution.

Theorem 5.12. *If $f(\cdot,x) \in P_p(\mathbb{R})$ for all $x \in [0,2\pi]$ and $\bar{b} + a \neq n^2$ for all $n \in \{1,2,3,...\}$, then there exists exactly one mild solution w of (E) such that $w(\cdot,x) \in P_p(\mathbb{R})$.*

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