SEMIGROUPS AND LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH DELAY

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Abstract. We prove the equivalence of the well-posedness of a partial differential equation with delay and an associated abstract Cauchy problem. This is used to derive sufficient conditions for well-posedness, exponential stability and norm continuity of the solutions. Applications to a reaction-diffusion equation with delay are given.

1. Introduction

Partial differential equations with delay have been studied for many years. In an abstract way and using the standard notation (see [13] and [26]) they can be written as

$$\text{(DE)} \qquad \begin{cases} u'(t) = Au(t) + \Phi u_t, \quad t \geq 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

on a Banach space X, where $(A, D(A))$ is a (unbounded) linear operator on X and the delay operator Φ is supposed to belong to $\mathcal{L}(W^{1,p}([-1,0],X),X)$. J. Hale [10] and G. Webb [24] were among the first who applied semigroup theory to the study of such equations, and we refer to [21] and [26] for more recent references.

As a first step one has to choose an appropriate state space. One of the possibilities is to work in the space of continuous X-valued functions. In this case, the relationship between solutions of (DE) and a corresponding semigroup has been studied widely (see for example [7, Section VI.6]) and is well understood.

On the other hand, the state space $\mathcal{E} := X \times L^p([-1,0],X)$ turns out to be a better choice with regards to certain applications (to control theory, see [16]) and will be used in this paper.

We will show in Section 2 that the linear partial differential equation with delay is equivalent to an abstract Cauchy problem

$$\text{(ACP)} \qquad \begin{cases} \mathcal{U}'(t) = \mathcal{A}\mathcal{U}(t), \qquad t \geq 0, \\ \mathcal{U}(0) = \binom{x}{f}, \end{cases}$$

on the space $\mathcal{E}$. Similar results for neutral differential equations on finite dimensional spaces X were proved by F. Kappel and K. Zhang [11] (see also [2]).

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In the third section we give sufficient conditions such that the operator $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup on the space $\mathcal{E}$. Assuming that $(A, D(A))$ is the generator of a strongly continuous semigroup on X, we show that the operator $\mathcal{A}$ is given by the sum $\mathcal{A}_0 + \mathcal{B}$, where $(\mathcal{A}_0, D(\mathcal{A}_0))$ generates a strongly continuous semigroup on $\mathcal{E}$ and $\mathcal{B}$ is $\mathcal{A}_0$-bounded. Moreover, we give a condition on the operators A and Φ such that $\mathcal{B}$ is a Miyadera perturbation of $\mathcal{A}_0$ and thus $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{E}$.

Finally, for $1 < p < \infty$, we show that if the operator Φ is given by the Riemann-Stieltjes integral of a function of bounded variation $\eta : [-1, 0] \longrightarrow \mathcal{L}(X)$, then $\mathcal{B}$ is a Miyadera perturbation of $\mathcal{A}_0$ for every generator $(A, D(A))$ and $\mathcal{A} = \mathcal{A}_0 + \mathcal{B}$ generates a strongly continuous semigroup on $\mathcal{E}$. For similar equations the well-posedness was proved, using a completely different setting, by J. Prüß in [20, Theorem I.1.2].

This part was inspired, on the one side, by a paper of K. Kunisch and W. Schappacher [13] containing necessary conditions such that $\mathcal{A}$ generates a strongly continuous semigroup on $\mathcal{E}$, on the other side, by a work of K.-J. Engel [6], who used the theory of unbounded operator matrices to treat the case of bounded A.

In the fourth section, we prove that the stability of the equation without delay persists in the equation with delay. We characterize the resolvent and use the theorem of Gearhart. This part is a generalization of a recent result of A. Fischer and J. van Neerven [8]. In [4, Theorem 5.1.7] one finds a similar stability result for the finite dimensional case.

In the last section we give a sufficient condition for the eventual norm continuity of the semigroup generated by the operator $(\mathcal{A}, D(\mathcal{A}))$. We show that if $(A, D(A))$ generates an immediately norm continuous semigroup on X and Φ satisfies a technical condition, then $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup which is norm continuous for $t > 1$. We use these results to obtain exponential stability of the solutions.

Finally, we illustrate our results on a reaction-diffusion equation with delay.

2. The semigroup approach

Consider the equation

$$\text{(DE)} \qquad \begin{cases} u'(t) = Au(t) + \Phi u_t, & t \geq 0, \\ u(0) = x, \\ u_0 = f, \end{cases}$$

where

- $x \in X$, X is a Banach space,
- $A : D(A) \subseteq X \longrightarrow X$ is a closed and densely defined linear operator,
- $f \in L^p([-1, 0], X)$, $1 \leq p < \infty$,
- $\Phi : W^{1,p}([-1, 0], X) \longrightarrow X$ is a bounded linear operator,
- $u : [-1, \infty) \longrightarrow X$ and $u_t : [-1, 0] \longrightarrow X$ is defined by $u_t(\sigma) := u(t + \sigma)$ for $\sigma \in [-1, 0]$.

We say that a function $u : [-1, \infty) \longrightarrow X$ is a *(classical) solution* of (DE) if

(i) $u \in C([-1, \infty), X) \cap C^1([0, \infty), X)$,
(ii) $u(t) \in D(A)$ and $u_t \in W^{1,p}([-1, 0], X)$ for all $t \geq 0$,

(iii) u satisfies (DE) for all $t \geq 0$.

It is now our purpose to investigate existence and uniqueness of the solutions of (DE). To do this we introduce the Banach space

$$\mathcal{E} := X \times L^p([-1,0],X)$$

and the operator

$$\mathcal{A} := \begin{pmatrix} A & \Phi \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \tag{2.1}$$

with domain

$$D(\mathcal{A}) := \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(A) \times W^{1,p}([-1,0],X) \ : \ f(0) = x \right\}. \tag{2.2}$$

Lemma 2.1. *The operator $(\mathcal{A}, D(\mathcal{A}))$ is closed and densely defined.*

The proof of Lemma 2.1 is straightforward and is omitted.

Remark 2.2. A necessary condition for (DE) to have a solution is that $u_0 = f \in W^{1,p}([-1,0],X)$ and $u(0) = x \in D(A)$, i.e. $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A})$.

In the following, we will see that equation (DE) and the abstract Cauchy problem associated to the operator $(\mathcal{A}, D(\mathcal{A}))$

$$\text{(ACP)} \qquad \begin{cases} \dot{\mathcal{U}}(t) = \mathcal{A}\mathcal{U}(t), & t \geq 0, \\ \mathcal{U}(0) = \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right), \end{cases}$$

are "equivalent", i.e., (DE) has a unique solution for every $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A})$ and the solutions depend continuously on the initial values if and only if (ACP) is well-posed (in the sense of [7, Definition II.6.8]).

Proposition 2.3. *Let $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A})$ and let $u : [-1,\infty) \longrightarrow X$ be a solution of (DE). Then the map*

$$\mathbb{R}_+ \ni t \mapsto \left(\begin{smallmatrix} u(t) \\ u_t \end{smallmatrix}\right) \in \mathcal{E}$$

is a classical solution of the abstract Cauchy problem (ACP) associated to the operator $(\mathcal{A}, D(\mathcal{A}))$ with initial value $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)$.

Proof. Since the function u is a solution of (DE), we have $u \in C^1([0,\infty),X)$, $u(t) \in D(A)$, $u_t \in W^{1,p}([-1,0],X)$ and $\dot{u}(t) = Au(t) + \Phi u_t$ for all $t \geq 0$, $u(0) = x$, and finally $u_0 = f$.

It remains to show that the map $t \mapsto u_t$ is continuously differentiable for $t \geq 0$ and that $\frac{d}{dt}u_t = \frac{d}{d\sigma}u_t$ for all $t \geq 0$. So let $T > t \geq 0$. Then the function $u_{|[-1,T]}$ can be extended to a function $v \in W^{1,p}(\mathbb{R},X)$, i.e., v is in the domain of the first derivative, which is the generator of the left shift semigroup on $L^p(\mathbb{R},X)$. From the definition of generator we have

$$\frac{d}{dt}v(t+\cdot) = \frac{d}{d\sigma}v(t+\cdot)$$

in $L^p(\mathbb{R},X)$ for $t \geq 0$. This implies that

$$\frac{d}{dt}u_t = \frac{d}{dt}u(t+\cdot) = \frac{d}{d\sigma}u(t+\cdot) = \frac{d}{d\sigma}u_t$$

in $L^p([-1,0],X)$ and that the map $\mathbb{R}_+ \ni t \mapsto \frac{d}{d\sigma}u_t \in L^p([-1,0],X)$ is continuous. $\square$

We now prove the converse to Proposition 2.3. Similar results with different conditions on A and Φ were proved by G. Webb in [25].

Proposition 2.4. *Let* $\binom{x}{f} \in D(\mathcal{A})$ *and* $\mathcal{U} : [0,\infty) \longrightarrow \mathcal{E}$, $\mathcal{U}(t) = \binom{z(t)}{v(t)}$, *be a classical solution of the abstract Cauchy problem (ACP) with initial value* $\binom{x}{f}$ *associated to the operator* $(\mathcal{A}, D(\mathcal{A}))$. *Let* $u : [-1,\infty) \longrightarrow X$ *be the function defined by*

$$u(t) := \begin{cases} z(t), & t \geq 0, \\ f(t), & t \in [-1,0). \end{cases} \tag{2.3}$$

Then $u_t = v(t)$ *for every* $t \geq 0$ *and* u *is a solution of (DE).*

Proof. Since $\mathcal{U}$ is a classical solution of (ACP), $v \in C^1(\mathbb{R}_+, L^p([-1,0],X))$ solves the Cauchy problem

$$\begin{cases} \frac{d}{dt}v(t) = \frac{d}{d\sigma}v(t), & t \geq 0, \\ v(t)(0) = z(t), & t \geq 0, \\ v(0) = f \end{cases} \tag{2.4}$$

in the space $L^p([-1,0],X)$. We now observe that the map $\mathbb{R}_+ \ni t \mapsto u_t \in L^p([-1,0],X)$ solves (2.4) in the space $L^p([-1,0],X)$. So let $w(t) := u_t - v(t)$ for $t \geq 0$. Then w is a classical solution of the Cauchy problem

$$\begin{cases} \frac{d}{dt}w(t) = \frac{d}{d\sigma}w(t), & t \geq 0, \\ w(t)(0) = 0, & t \geq 0, \\ w(0) = 0. \end{cases} \tag{2.5}$$

Since (2.5) is the abstract Cauchy problem associated to the generator of the nilpotent left shift semigroup on $L^p([-1,0],X)$ with initial value 0, we have that $w(t) = 0$ for all $t \geq 0$. Therefore $\mathcal{U}(t) = \binom{u(t)}{u_t}$ for all $t \geq 0$ and u is a solution of (DE). ☐

Let $\pi_1 : \mathcal{E} \longrightarrow X$ be the projection onto the first component of $\mathcal{E}$, i.e., $\pi_1\binom{x}{f} := x$ for all $\binom{x}{f} \in \mathcal{E}$.

Corollary 2.5. *If* $(\mathcal{A}, D(\mathcal{A}))$ *is the generator of a strongly continuous semigroup* $(\mathcal{T}(t))_{t\geq 0}$ *on* $\mathcal{E}$, *then equation (DE) has a unique solution* u *for every* $\binom{x}{f} \in D(\mathcal{A})$, *which is given by*

$$u(t) = \begin{cases} \pi_1\big(\mathcal{T}(t)\binom{x}{f}\big), & t \geq 0, \\ f(t), & t \in [-1,0). \end{cases} \tag{2.6}$$

Proof. Since $(\mathcal{A}, D(\mathcal{A}))$ is the generator of the strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$, the Cauchy problem (ACP) has a unique classical solution $\mathcal{U}$ for all $\binom{x}{f} \in D(\mathcal{A})$ and

$$\mathcal{U}(t) = \mathcal{T}(t)\tbinom{x}{f}, \qquad t \geq 0.$$

Then the function u defined in (2.6) is a solution of (DE) by Proposition 2.4. Uniqueness follows from Proposition 2.3. ☐

Corollary 2.6. *If* $(\mathcal{A}, D(\mathcal{A}))$ *is the generator of a strongly continuous semigroup* $(\mathcal{T}(t))_{t\geq 0}$*, the function* $u : [-1,\infty) \longrightarrow X$ *defined by (2.6) for a given* $\binom{x}{f} \in \mathcal{E}$ *satisfies the integral equation*

$$u(t) = \begin{cases} x + A\int_0^t u(s)\,ds + \Phi\int_0^t u_s\,ds, & t \geq 0, \\ f(t), & a.e.\ t \in [-1,0). \end{cases} \tag{2.7}$$

Proof. Let $\pi_2 : \mathcal{E} \longrightarrow L^p([-1,0],X)$ be the projection onto the second component of $\mathcal{E}$, i.e., $\pi_2\binom{x}{f} := f$ for all $\binom{x}{f} \in \mathcal{E}$.
Then, by Proposition 2.4 and using the density of $D(\mathcal{A})$ in $\mathcal{E}$, we obtain $u_t = \pi_2\big(\mathcal{T}(t)\binom{x}{f}\big)$ for all $\binom{x}{f} \in \mathcal{E}$ and $t \geq 0$. Take now the first component of the identity

$$\mathcal{T}(t)\tbinom{x}{f} - \tbinom{x}{f} = \mathcal{A}\int_0^t \mathcal{T}(s)\tbinom{x}{f}\,ds, \qquad t \geq 0,$$

to obtain (2.7). □

A solution of the integral equation (2.7) is called *mild solution* in the literature, see for example [17, Section 2.2].

At this point we need the appropriate terminology.

Definition 2.7. *We call (DE)* **well-posed** *if*

(i) for every $\binom{x}{f} \in D(\mathcal{A})$ *there is a unique solution* $u(x,f,\cdot)$ *of (DE) and*
(ii) the solutions depend continuously on the initial values, i.e., if a sequence $\binom{x_n}{f_n}$ *in* $D(\mathcal{A})$ *converges to* $\binom{x}{f} \in D(\mathcal{A})$*, then* $u(x_n,f_n,t)$ *converges to* $u(x,f,t)$ *uniformly for* t *in compact intervals.*

This well-posedness of (DE) can now be characterized by the well-posedness of the abstract Cauchy problem (ACP) for the operator $(\mathcal{A}, D(\mathcal{A}))$.

Theorem 2.8. *Let* $(\mathcal{A}, D(\mathcal{A}))$ *be the operator defined by (2.1) and (2.2). Then the following assertions are equivalent.*

(i) (DE) is well-posed.
(ii) $(\mathcal{A}, D(\mathcal{A}))$ *is the generator of a strongly continuous semigroup on* $\mathcal{E}$.

Proof. We first show $(i) \Rightarrow (ii)$. Assume that for every $\binom{x}{f} \in D(\mathcal{A})$ equation (DE) has a unique solution u. Then Proposition 2.3 yields that for every $\binom{x}{f} \in D(\mathcal{A})$ the abstract Cauchy problem (ACP) has a classical solution which is unique by Proposition 2.4. It is easy to see that these solutions depend continuously on the initial values. Finally, by Lemma 2.1, $(\mathcal{A}, D(\mathcal{A}))$ is a closed and densely defined operator. So $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup on $\mathcal{E}$ by [7, Theorem II.6.7].

Conversely, if $\mathcal{A}$ is a generator, we have by Corollary 2.5 that for every initial value $\binom{x}{f} \in D(\mathcal{A})$ there is a unique solution u of (DE) which is given by (2.6). This implies that the solutions depend continuously on the initial values. □

3. The generator property

In the previous section we transformed the problem of solving the partial differential equation with delay (DE) into the functional analytical problem: When does $\mathcal{A}$ generate a strongly continuous semigroup on $\mathcal{E}$?

In the following, we will give sufficient conditions on A and Φ such that this is true.

First, we observe that we can write $\mathcal{A}$ as the sum $\mathcal{A}_0 + \mathcal{B}$, where

$$\mathcal{A}_0 := \begin{pmatrix} A & 0 \\ 0 & \frac{d}{d\sigma} \end{pmatrix} \tag{3.1}$$

with domain

$$D(\mathcal{A}_0) := D(\mathcal{A}) = \left\{ \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(A) \times W^{1,p}([-1,0],X) \ : \ f(0) = x \right\} \tag{3.2}$$

and

$$\mathcal{B} := \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \in \mathcal{L}(D(\mathcal{A}_0), \mathcal{E}).$$

The idea now is to show first that under appropriate conditions $\mathcal{A}_0$ becomes a generator and then apply perturbation results to show that the sum $\mathcal{A}_0 + \mathcal{B}$ is a generator as well. The first step is quite easy.

Proposition 3.1. *Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X. Then $(\mathcal{A}_0, D(\mathcal{A}_0))$ generates the strongly continuous semigroup $(\mathcal{T}_0(t))_{t\geq 0}$ on $\mathcal{E}$ given by*

$$\mathcal{T}_0(t) := \begin{pmatrix} S(t) & 0 \\ S_t & T_0(t) \end{pmatrix}, \tag{3.3}$$

where $(T_0(t))_{t\geq 0}$ is the nilpotent left shift semigroup on $L^p([-1,0],X)$ and $S_t : X \to L^p([-1,0],X)$ is defined by

$$(S_t\, x)(\tau) := \begin{cases} S(t+\tau)x, & -t < \tau \leq 0, \\ 0, & -1 \leq \tau \leq -t. \end{cases}$$

Therefore, we will always assume that

$(A, D(A))$ *generates a strongly continuous semigroup* $(S(t))_{t\geq 0}$ *on* X.

We will see later in Remark 3.5 that this condition is necessary for the well-posedness in case of many applications .

To the perturbation $\mathcal{B}$ we will apply the theorem of Miyadera-Voigt (see [14] and [22]), which we quote from [7, Corollary III.3.16].

Theorem 3.2. *Let $(G, D(G))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X and let $C \in \mathcal{L}((D(G), \|\cdot\|_G), X)$ satisfy*

$$\int_0^{t_0} \|CT(r)x\|\, dr \leq q\|x\| \qquad \text{for all } x \in D(G) \tag{3.4}$$

and some $t_0 > 0$, $0 \leq q < 1$. Then $(G + C, D(G))$ generates a strongly continuous semigroup $(U(t))_{t\geq 0}$ on X which satisfies

$$U(t)x = T(t)x + \int_0^t T(t-s)CU(s)x\, ds \qquad \text{and}$$

$$\int_0^{t_0} \|CU(t)x\|\, dt \leq \frac{q}{1-q}\|x\| \qquad \text{for } x \in D(G) \text{ and } t \geq 0.$$

In our situation, $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup on $\mathcal{E}$ if there exist $t_0 > 0$ and $0 \le q < 1$ such that

$$\int_0^{t_0} \left\| \mathcal{B}\,\mathcal{T}_0(r) \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\| \, dr \le q \left\| \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\| \tag{3.5}$$

for all $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A}_0)$. However, since

$$\begin{aligned}\int_0^{t_0} \left\| \mathcal{B}\,\mathcal{T}_0(r) \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\| \, dr &= \int_0^{t_0} \left\| \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S(r) & 0 \\ S_r & T_0(r) \end{pmatrix} \begin{pmatrix} x \\ f \end{pmatrix} \right\| \, dr \\ &= \int_0^{t_0} \| \Phi(S_r x + T_0(r) f) \| \, dr \,,\end{aligned}$$

we conclude that (3.5) holds if and only if there exist $t_0 > 0$ and $0 \le q < 1$ such that

$$\int_0^{t_0} \| \Phi(S_r\, x + T_0(r) f) \| \, dr \le q \, \left\| \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\| \tag{M}$$

for all $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A}_0)$. We therefore obtain the following result.

Theorem 3.3. *Let $(A, D(A))$ be the generator of a strongly continuous semigroup on X and let condition (M) be satisfied. Then the operator $(\mathcal{A}, D(\mathcal{A}))$ is the generator of a strongly continuous semigroup on $\mathcal{E}$. Thus, (DE) is well-posed.*

We now give some important examles of Φ to satisfy this condition (M).

Examples 3.4. (a) Let Φ be bounded from $L^p([-1,0], X)$ to X. Then the perturbation $\mathcal{B}$ is bounded, and $(\mathcal{A}, D(\mathcal{A}))$ is a generator on $\mathcal{E}$.

(b) Let $1 < p < \infty$ and let $\eta : [-1, 0] \to \mathcal{L}(X)$ be of bounded variation. Let $\Phi : C([-1,0], X) \to X$ be the bounded linear operator given by the Riemann-Stieltjes integral

$$\Phi(f) := \int_{-1}^{0} d\eta\, f \qquad \text{for all } f \in C([-1,0], X). \tag{3.6}$$

Since $W^{1,p}([-1,0],X)$ is continuously embedded in $C([-1,0],X)$, Φ defines a bounded operator from $W^{1,p}([-1,0],X)$ to X. For $0<t<1$ we obtain that

$$\begin{aligned}
\int_0^t \|\Phi(S_r\, x + T_0(r)f)\|\, dr &= \int_0^t \left\| \int_{-1}^{-r} d\eta(\sigma)\, f(r+\sigma) + \int_{-r}^{0} d\eta(\sigma)\, S(r+\sigma)x \right\| dr \\
&\le \int_0^t \int_{-1}^{-r} \|f(r+\sigma)\|\, d|\eta|(\sigma)\, dr + \int_0^t \int_{-r}^{0} \|S(r+\sigma)x\|\, d|\eta|(\sigma)\, dr \\
&\le \int_{-t}^{0} \int_{\sigma}^{0} \|f(s)\|\, ds\, d|\eta|(\sigma) + \int_{-1}^{-t} \int_{\sigma}^{t+\sigma} \|f(s)\|\, ds\, d|\eta|(\sigma) \\
&\quad + \int_0^t M\|x\||\eta|([-1,0])\, dr \\
&\le \int_{-t}^{0} (-\sigma)^{1/p'} \|f\|_p\, d|\eta|(\sigma) + \int_{-1}^{-t} t^{1/p'} \|f\|_p\, d|\eta|(\sigma) \\
&\quad + tM\|x\||\eta|([-1,0]) \\
&\le \int_{-1}^{0} t^{1/p'} \|f\|_p\, d|\eta|(\sigma) + tM\|x\||\eta|([-1,0]) \\
&= (t^{1/p'}\|f\|_p + tM\|x\|)\, |\eta|([-1,0]),
\end{aligned}$$

where $\frac{1}{p}+\frac{1}{p'}=1$, $M := \sup_{r\in[0,1]} \|S(r)\|$ and $|\eta|$ is the positive Borel measure on $[-1,0]$ defined by the total variation of η.

Finally we conclude that

$$\int_0^t \|\Phi(S_r x + T_0(r)f)\|\, dr \le t^{1/p'} M\, |\eta|([-1,0])(\|f\|_p + \|x\|) \tag{3.7}$$

for all $0<t<1$. Choose now t_0 small enough such that $t_0^{1/p'} M\, |\eta|([-1,0]) < 1$. Then condition (M) is satisfied with $q := t_0^{1/p'} M\, |\eta|([-1,0])$.

(c) An important special case of (b) are the operators Φ defined by $\Phi(f) := \sum_{k=0}^{n} B_k f(h_k)$, $f \in W^{1,p}([-1,0],X)$, where $B_k \in \mathcal{L}(X)$ and $h_k \in [-1,0]$ for $k=0,\dots,n$.

Remark 3.5. We note that from the perturbation theorem of Miyadera-Voigt (see Theorem 3.2) it follows that the generator property of $(A,(D(A))$ is necessary and sufficient for the well-posedness of (DE) if Φ is defined as in (3.6) because we can choose q small enough. Similar results for more special cases were proved by Kunisch and Schappacher (see [13, Proposition 4.2]) and for a slightly different equation by Prüß (see [20, Corollary I.1.4]).

4. Stability

In this section, we prove a stability result for the delay equation (DE) extending a recent result of Fischer and van Neerven [8]. First, we recall some notations and definitions.

Let $\mathcal{T} := (T(t))_{t\ge 0}$ be a C_0-semigroup of bounded linear operators on the Banach space X with generator $(G,D(G))$. The *spectral bound* of G is given by

$$s(G) := \{\Re\lambda \,:\, \lambda \in \sigma(G)\},$$

the *abscissa of uniform boundedness* of the resolvent of G is defined by

$$s_0(G) := \inf\left\{\omega \in \mathbb{R} \,:\, \{\Re\lambda > \omega\} \subset \varrho(G) \text{ and } \sup_{\Re\lambda>\omega} \|R(\lambda, G)\| < \infty\right\},$$

and *the uniform growth bound* or *type* of the semigroup

$$\omega_0(G) := \inf\left\{\omega \in \mathbb{R} \,:\, \exists M > 0 \text{ such that } \|T(t)\| \leq Me^{\omega t}\ \forall t \geq 0\right\}.$$

We say that $\mathcal{T}$ is *uniformly exponentially stable* if $\omega_0(G) < 0$. It is known (see [18, Sections 1.2, 4.1]) that

$$-\infty \leq s(G) \leq s_0(G) \leq \omega_0(G) < \infty.$$

The theorem of Gearhart (see [7, Theorem V.1.11]) says that in a Hilbert space X

$$s_0(G) = \omega_0(G). \tag{4.1}$$

In order to find estimates for the above quantities, we first calculate the resolvent $R(\lambda, \mathcal{A})$ and the resolvent set $\rho(\mathcal{A})$ of the operator $\mathcal{A}$. Given $\lambda \in \mathbb{C}$ and $\binom{y}{g} \in \mathcal{E}$ we are looking for $\binom{x}{f} \in D(\mathcal{A})$ such that

$$(\lambda - \mathcal{A})\binom{x}{f} = \begin{pmatrix} (\lambda - A)x - \Phi f \\ \lambda f - f' \end{pmatrix} = \begin{pmatrix} y \\ g \end{pmatrix}.$$

Since $f(0) = x$, the second component of this identity is equivalent to

$$f = \epsilon_\lambda \otimes x + R(\lambda, A_0)g, \tag{4.2}$$

where $\epsilon_\lambda(s) := e^{\lambda s}$ for $s \in [-1, 0]$ and $(A_0, D(A_0))$ is the infinitesimal generator of the nilpotent left shift semigroup $(T_0(t))_{t\geq 0}$ on $L^p([-1, 0], X)$ whose spectrum is empty. Hence, x has to satisfy the equation

$$(\lambda - A - \Phi(\epsilon_\lambda \otimes Id))x = \Phi R(\lambda, A_0)g + y. \tag{4.3}$$

This leads to the following lemma (see also [6] and [17]).

Lemma 4.1. *For $\lambda \in \mathbb{C}$ we have $\lambda \in \rho(\mathcal{A})$ if and only if $\lambda \in \rho(A+\Phi(\epsilon_\lambda \otimes Id))$. Moreover, for $\lambda \in \rho(\mathcal{A})$ the resolvent $R(\lambda, \mathcal{A})$ is given by*

$$\begin{pmatrix} R(\lambda, A + \Phi(\epsilon_\lambda \otimes Id)) & R(\lambda, A + \Phi(\epsilon_\lambda \otimes Id))\Phi R(\lambda, A_0) \\ \epsilon_\lambda \otimes R(\lambda, A + \Phi(\epsilon_\lambda \otimes Id)) & [\epsilon_\lambda \otimes R(\lambda, A + \Phi(\epsilon_\lambda \otimes Id))\Phi + Id]R(\lambda, A_0) \end{pmatrix}. \tag{4.4}$$

Proof. Let $\lambda \in \rho(A + \Phi(\epsilon_\lambda \otimes Id))$. Then the matrix in (4.4) is a bounded operator from $\mathcal{E}$ to $D(\mathcal{A})$ defining the inverse of $(\lambda - \mathcal{A})$.

Conversely, if $\lambda \in \rho(\mathcal{A})$, then for every $\binom{y}{g} \in \mathcal{E}$ there exists a unique $\binom{x}{f} \in D(\mathcal{A})$ such that (4.2) and (4.3) hold. In particular, for $g = 0$ and for every $y \in X$, there exists a unique $x \in D(A)$ such that

$$(\lambda - A - \Phi(\epsilon_\lambda \otimes Id))x = y.$$

This means that $(\lambda - A - \Phi(\epsilon_\lambda \otimes Id))$ is invertible, i.e. $\lambda \in \rho(A - \Phi(\epsilon_\lambda \otimes Id))$. □

We now assume the well-posedness conditions from the previous section. In particular, we will assume that $(A, D(A))$ generates a strongly continuous semigroup on X, that $p > 1$ and that Φ satisfies the assumption (3.6) of Example 3.4 (b), i.e.

$$\Phi f = \int_{-1}^{0} d\eta\, f,$$

where $\eta : [-1,0] \longrightarrow \mathcal{L}(X)$ is a function of bounded variation. Hence, our matrix $(\mathcal{A}, D(\mathcal{A}))$, defined in (2.1) and (2.2), is the generator of a strongly continuous semigroup $(\mathcal{T}(t))_{t\geq 0}$ on the Banach space $\mathcal{E}$ and (DE) is well-posed. For this generator we can now estimate $s_0(\mathcal{A})$, generalizing [8, Theorem 3.3] with a similar proof.

Theorem 4.2. *Assume that $s_0(A) < 0$ and let $\alpha \in (s_0(A), 0]$. If*

$$\sup_{\omega\in\mathbb{R}} \|\Phi(\epsilon_{\alpha+i\omega} \otimes Id)\| < \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(\alpha+i\omega, A)\|}, \tag{4.5}$$

then $s_0(\mathcal{A}) < \alpha \leq 0$.

To prove this statement we recall from [8, Proposition 1.1] the following lemma in a modified form (compare also [12, Theorem IV.1.16]).

Lemma 4.3. *Let A be a closed linear operator on a Banach space X, and suppose $\lambda \in \rho(A)$. If $\Delta \in \mathcal{L}(X)$ satisfies*

$$\|\Delta\| \leq (1-\delta)\frac{1}{\|R(\lambda, A)\|}$$

for some $\delta \in (0,1)$, then $\lambda \in \rho(A+\Delta)$ and

$$\|R(\lambda, A+\Delta)\| \leq \frac{1}{\delta}\|R(\lambda, A)\|.$$

Proof of Theorem 4.2. Choose $\delta \in (0,1)$ such that

$$\sup_{\omega\in\mathbb{R}} \|\Phi(\epsilon_{\alpha+i\omega} \otimes Id)\| \leq (1-\delta)\frac{1}{\sup_{\omega\in\mathbb{R}} \|R(\alpha+i\omega, A)\|}.$$

We observe that $\Phi(\epsilon_\lambda \otimes Id)$ is an analytic function and that the suprema of bounded analytic functions along vertical lines, $\Re\lambda = c$, decrease as c increases (see [3, Chapter 6.4] for the details). For all $\lambda \in \mathbb{C}$ with $\Re\lambda > \alpha$ we obtain

$$\|\Phi(\epsilon_\lambda \otimes Id)\| \leq \sup_{\omega\in\mathbb{R}} \|\Phi(\epsilon_{\alpha+i\omega} \otimes Id)\| \leq \frac{(1-\delta)}{\sup_{\omega\in\mathbb{R}} \|R(\alpha+i\omega, A)\|} \leq \frac{(1-\delta)}{\|R(\lambda, A)\|}.$$

Therefore, by Lemma 4.3, $\{\Re\lambda > \alpha\} \subset \rho\left(A + \Phi(\epsilon_\lambda \otimes Id)\right)$, and for all $\lambda \in \mathbb{C}$ with $\Re\lambda > \alpha$ we have

$$\|R\left(\lambda, A + \Phi(\epsilon_\lambda \otimes Id)\right)\| \leq \frac{1}{\delta}\|R(\lambda, A)\|.$$

Hence, by Lemma 4.1 ,

$$\{\Re\lambda > \alpha\} \subset \rho(\mathcal{A}).$$

We show that $R(\lambda, \mathcal{A})$ is bounded on this halfplane. We have shown above that $R(\lambda, A + \Phi(\epsilon_\lambda \otimes Id))$ is bounded. The operator A_0 generates the nilpotent left shift, so $R(\lambda, A_0)$ is bounded on this right halfplane. The function $\Phi(\epsilon_\lambda \otimes Id)$ is bounded, analytic and

$\Phi R(\lambda, A_0)$ is continuous and bounded, since

$$\begin{aligned}
\|\Phi R(\lambda, A_0)f\| &= \left\| \int_{-1}^{0} d\eta(\sigma) \int_{\sigma}^{0} e^{\lambda(\sigma-\tau)} f(\tau)\, d\tau \right\| \\
&\leq \int_{-1}^{0} \int_{\sigma}^{0} \|e^{\lambda(\sigma-\tau)} f(\tau)\|\, d\tau\, d|\eta|(\sigma) \\
&= \int_{-1}^{0} \int_{\sigma}^{0} e^{\Re\lambda(\sigma-\tau)} \|f(\tau)\|\, d\tau\, d|\eta|(\sigma) \\
&\leq \int_{-1}^{0} \int_{\sigma}^{0} e^{-\alpha} \|f(\tau)\|\, d\tau\, d|\eta|(\sigma) \\
&\leq \int_{-1}^{0} \int_{-1}^{0} e^{-\alpha} \|f(\tau)\|\, d\tau\, d|\eta|(\sigma) \\
&\leq e^{-\alpha}\, |\eta|([-1,0])\, \|f\|_p
\end{aligned}$$

for every λ with $\Re\lambda > \alpha$ and every $f \in L^p([-1,0], X)$. Therefore $\sup_{\Re\lambda>\alpha} \|R(\lambda, \mathcal{A})\| < \infty$, and we conclude that $s_0(\mathcal{A}) < \alpha \leq 0$. □

Using Gearhart's theorem quoted above, this result can be improved for Hilbert spaces.

Corollary 4.4. *Take $p = 2$ and X a Hilbert space. If $\omega_0(A) < \alpha \leq 0$ and*

$$\sup_{\omega\in\mathbb{R}} \|\Phi(\epsilon_{\alpha+i\omega} \otimes Id)\| < \frac{1}{\sup_{\omega\in\mathbb{R}} \|R(\alpha + i\omega, A)\|},$$

then $\omega_0(\mathcal{A}) < \alpha \leq 0$.

Example 4.5. We consider the reaction diffusion equation with delay (see [26, Section 2.1])

$$\begin{cases}
\partial_t w(x,t) = \Delta w(x,t) + c \int_{-1}^{0} w(x, t+\tau) dg(\tau), \quad x \in \Omega,\ t \geq 0, \\
w(x,t) = 0, \quad x \in \partial\Omega,\ t \geq 0, \\
w(x,t) = f(x,t), \quad (x,t) \in \Omega \times [-1, 0],
\end{cases}$$

where c is a constant, $\Omega \subset \mathbb{R}^n$ a bounded domain, $f(\cdot, t) \in L^2(\Omega)$ for all $t \geq 0$, $f(\cdot, 0) \in W_0^{1,2}(\Omega) \cap W^{2,2}(\Omega)$ and the map $[-1,0] \ni t \mapsto f(\cdot, t) \in L^2(\Omega)$ belongs to $W^{1,2}([-1,0], L^2(\Omega))$. The function $g : [-1,0] \to [0,1]$ is the Cantor function (see [9, Example I.8.15]), which is singular and has total variation 1.

We consider $X := L^2(\Omega)$, $A := \Delta_D$ the Dirichlet-Laplacian with usual domain and $\eta := c \cdot g \cdot Id$. The well-posedness follows from the calculations in Examples 3.4(b).

We have to verify the stability estimate. First, the expression with Φ satisfies

$$\|\Phi(\epsilon_\lambda \otimes Id)y\| = |c|\|y\| \left\| \int_{-1}^{0} e^{\lambda\tau} dg(\tau) \right\| \leq |c|\|y\|.$$

The other expression, using that A is a normal operator on a Hilbert space (see [12, Section V.3.8], can be computed as

$$\sup_{\omega\in\mathbb{R}} \|R(i\omega, A)\| = \sup_{\omega\in\mathbb{R}} \frac{1}{d\,(i\omega, \sigma(A))} = \frac{1}{d\,(0, \sigma(A))} = \frac{1}{|\lambda_1|},$$

where λ_1 is the first eigenvalue of the Laplacian. Thus the solutions decay exponentially if

$$|c| < |\lambda_1|.$$

We refer for example to [5, Chapter 6] for estimates on λ_1 and for further references. The same result holds for more general elliptic operators as considered in [5, Section 6.3].

5. Norm continuity

In this section, we show that if the operator $(A, D(A))$ in (DE) generates an immediately norm continuous semigroup on X, then, under an appropriate assumption, the operator matrix associated to the delay equation generates an eventually norm continuous semigroup on $\mathcal{E}$. This fact is important for the study of the asymptotic behaviour of the solutions.

For convenience, we repeat here the definitions from [7, Definition II.4.17]. A strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space Y is called *eventually norm continuous*, if there exists $t_0 \geq 0$ such that the function $t \mapsto T(t)$ is norm continuous from (t_0, ∞) to $\mathcal{L}(Y)$. The semigroup is called *immediately norm continuous* if t_0 can be chosen to be $t_0 = 0$.

Let $(A, D(A))$ be the generator of a strongly continuous semigroup $(T(t))_{t\geq 0}$ on a Banach space X and let $C \in \mathcal{L}((D(A), \|\cdot\|_A), X)$. Moreover, let us assume that there exist $\varepsilon > 0$ and a function $q : (0, \varepsilon) \to \mathbb{R}_+$ such that $\lim\limits_{t\searrow 0} q(t) = 0$ and

$$\int_0^t \|C\,T(s)x\|\,ds \leq q(t)\,\|x\| \tag{5.1}$$

for every $x \in D(A)$ and every $0 < t < \varepsilon$. Then we know from the perturbation theorem of Miyadera-Voigt that $(A + C, D(A))$ generates a strongly continuous semigroup $(U(t))_{t\geq 0}$ on X given by the Dyson-Phillips series

$$U(t) = \sum_{n=0}^{\infty} (V^n T)(t), \qquad t \geq 0, \tag{5.2}$$

where V is the abstract Volterra operator defined in [7, Theorem III.3.14] converging uniformly on compact intervals of $\mathbb{R}_+$ (see [7, Corollary III.3.15]). For $x \in D(A)$ we have

$$(VT)(t)x := \int_0^t T(t-s)CT(s)x\,ds.$$

Theorem 5.1. *If $(T(t))_{t\geq 0}$ is norm continuous for $t > \alpha$ and there exists $n \in \mathbb{N}$ such that $V^n T$ is norm continuous for $t > 0$, then the perturbed semigroup $(U(t))_{t\geq 0}$ is norm continuous for $t > n\alpha$.*

The proof is a straightforward generalization of [15, Theorem 6.1]. We refer to [19, Corollary 2.7] for the details.

We now apply this result to the operator $(\mathcal{A}, D(\mathcal{A}))$ associated to the delay equation (DE).

As before, we will assume that $(A, D(A))$ generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on X and that the perturbation $\mathcal{B}$ satisfies the following condition being slightly stronger than condition (M). There exists $q : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ with $\lim_{t\to 0^+} q(t) = 0$ and

$$\int_0^t \|\Phi(S_s x + T_0(s)f)\|\, ds \le q(t)\, \left\|\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right)\right\| \tag{K}$$

for all $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A}_0)$ and $t > 0$. We then have that $(\mathcal{A}, D(\mathcal{A}))$ is a generator by Theorem 3.2. Moreover, from (3.7) it follows that all the cases in Examples 3.4 satisfy this condition.

We recall that $(\mathcal{A}_0, D(\mathcal{A}_0))$ is the operator defined in (3.1) and (3.2) generating the strongly continuous semigroup $(\mathcal{T}_0(t))_{t\ge0}$ given by (3.3).

Proposition 5.2. *If $(S(t))_{t\ge0}$ is immediately norm continuous, then $(\mathcal{T}_0(t))_{t\ge0}$ is norm continuous for $t \ge 1$.*

Proof. Let $t \ge 1$. Then $T_0(t) = 0$ and $\mathcal{S}_0(t) = \left(\begin{smallmatrix} S(t) & 0 \\ S_t & 0 \end{smallmatrix}\right)$. So it suffices to show that the map $t \mapsto S_t$ from $[1, \infty)$ to $\mathcal{L}(X, L^p([-1, 0], X))$ is norm continuous.

For $s, t \ge 1$ we have

$$\begin{aligned}
\lim_{s\to t} \|S_s - S_t\| &= \lim_{s\to t} \sup_{\|x\|\le1} \left(\int_{-1}^0 \|S(s+\sigma)x - S(t+\sigma)x\|^p\, d\sigma \right)^{\frac{1}{p}} \\
&\le \lim_{s\to t} \sup_{\|x\|\le1} \big(\int_{-1}^0 \|S(s+\sigma) - S(t+\sigma)\|^p\, d\sigma \big)^{\frac{1}{p}} \|x\| \\
&= \lim_{s\to t} \big(\int_{-1}^0 \|S(s+\sigma) - S(t+\sigma)\|^p\, d\sigma \big)^{\frac{1}{p}},
\end{aligned}$$

which converges to 0 as s tends to t since $(S(t))_{t\ge0}$ is immediately norm continuous and therefore uniformly norm continuous on compact intervals. ☐

We are now ready to apply Theorem 5.1.

Proposition 5.3. *If the semigroup $(S(t))_{t\ge0}$ generated by $(A, D(A))$ is immediately norm continuous, then $(\mathcal{T}(t))_{t\ge0}$ is norm continuous for $t \ge 1$.*

The same result was already proved by a different technique, and for the special case $\Phi := \sum_{k=0}^n B_k \delta_{h_k}$, by Fischer and van Neerven [8, Proposition 3.5].

Proof of Proposition 5.3. From Proposition 5.2, we have that $(\mathcal{T}_0(t))_{t\ge0}$ is norm continuous for $t \ge 1$. We now show that $V\mathcal{T}_0$ is norm continuous for $t \ge 0$. In fact, for $t \ge 0$ and $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A}_0)$ we have

$$\begin{aligned}
V\mathcal{T}_0(t)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) &= \int_0^t \mathcal{T}_0(t-s)\mathcal{B}\mathcal{T}_0(s)\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) ds \\
&= \int_0^t \mathcal{T}_0(t-s) \begin{pmatrix} 0 & \Phi \\ 0 & 0 \end{pmatrix} \begin{pmatrix} S(s)x \\ S_s x + T_0(s)f \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} S(t-s) & 0 \\ S_{t-s} & T_0(t-s) \end{pmatrix} \begin{pmatrix} \Phi(S_s x + T_0(s)f) \\ 0 \end{pmatrix} ds \\
&= \int_0^t \begin{pmatrix} S(t-s)\Phi(S_s x + T_0(s)f) \\ S_{t-s}\Phi(S_s x + T_0(s)f) \end{pmatrix} ds.
\end{aligned}$$

We prove norm continuity of both components separately.

1. Let $t \geq 0$ and $1 > h > 0$. Then we have

$$\begin{aligned}
&\left\| \int_0^{t+h} S(t+h-s)\Phi(S_s x + T_0(s)f)\, ds - \int_0^t S(t-s)\Phi(S_s x + T_0(s)f)\, ds \right\| \\
&\quad \leq \left\| \int_t^{t+h} S(t+h-s)\Phi(S_s x + T_0(s)f)\, ds \right\| \\
&\qquad + \left\| \int_0^t (S(t+h-s) - S(t-s))\Phi(S_s x + T_0(s)f)\, ds \right\| \\
&\quad \leq \int_0^h \|S(h-s)\| \, \|\Phi(S_{s+t}x + T_0(s+t)f)\|\, ds \\
&\qquad + \int_0^t \|S(t+h-s) - S(t-s)\| \, \|\Phi(S_s x + T_0(s)f)\|\, ds. \\
&\quad \leq \sup_{0 \leq r \leq 1} \|S(r)\| q(h) \left\| \mathcal{T}_0(t) \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\| \\
&\qquad + \int_0^t \|S(t+h-s) - S(t-s)\| \, \|\Phi(S_s x + T_0(s)f)\|\, ds.
\end{aligned}$$

By condition (K), the Lebesgue dominated convergence theorem and by the immediate norm continuity of $(S(t))_{t \geq 0}$, we have that

$$\sup_{0 \leq r \leq 1} \|S(r)\| q(h) \left\| \mathcal{T}_0(t) \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\| + \int_0^t \|S(t+h-s) - S(t-s)\| \, \|\Phi(S_s x + T_0(s)f)\|\, ds$$

tends to 0 as $h \to 0^+$ uniformly in $\left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \in D(\mathcal{A}_0)$, $\left\| \left(\begin{smallmatrix} x \\ f \end{smallmatrix}\right) \right\| \leq 1$. The proof for $h \to 0^-$ is analogous.

Since $D(A)$ is dense in $\mathcal{E}$, the first component of $V\mathcal{T}_0$ is immediately norm continuous.

2. To prove immediate norm continuity of the second component of $V\mathcal{T}_0$ one proceeds in a similar way. We only have to use the norm continuity of the map $t \mapsto S_t$, which was proved in Proposition 5.2.

Hence, the map $t \mapsto V\mathcal{T}_0(t)$ is norm continuous on $\mathbb{R}_+$ and by Theorem 5.1 we have that $(\mathcal{T}(t))_{t \geq 0}$ is norm continuous for $t \geq 1$. □

Using the stability results obtained in the previous section and the spectral mapping theorem for eventually norm continuous semigroups (see [7, Theorem IV.3.9] for the details), we can prove the following stability result.

Corollary 5.4. *Assume that A generates an immediately norm continuous semigroup, $\omega_0(A) < 0$ and let $\alpha \in (\omega_0(A), 0]$. If*

$$\sup_{\omega \in \mathbb{R}} \|\Phi(\epsilon_{\alpha + i\omega} \otimes Id)\| < \frac{1}{\sup_{\omega \in \mathbb{R}} \|R(\alpha + i\omega, A)\|}, \tag{5.3}$$

then $\omega_0(\mathcal{A}) < \alpha \leq 0$.

Example 5.5. We consider the reaction-diffusion equation from Example 4.5 in the state space $\mathcal{E} := L^r(\Omega) \times L^p([-1,0], L^r(\Omega))$ for $1 \leq r < \infty$, $1 < p < \infty$.

The well-posedness follows again from the calculations in Examples 3.4(b). We show again that the solutions decay exponentially if

$$|c| < |\lambda_1|, \tag{5.4}$$

extending our result from the Hilbert space case.

Unfortunately, Corollary 5.4 is not optimal for this problem. If we estimate the resolvent for the stability condition in Corollary 5.4, we obtain that it is sufficient for the exponential stability if

$$|c| < c_r,$$

where $c_r \leq |\lambda_1|$ is a constant depending on r.

To obtain estimate (5.4), we extend the result obtained for the Hilbert space case in two steps.

Consider first the case $r = 2$. We know from Lemma 4.1 that $\sigma(\mathcal{A})$, the spectrum of $(\mathcal{A}, D(\mathcal{A}))$, does not depend on p, and that

$$s(\mathcal{A}) = \omega_0(\mathcal{A})$$

since by Proposition 5.3 $(\mathcal{A}, D(\mathcal{A}))$ generatates an eventually norm continuous semigroup on $\mathcal{E}$. For $p = 2$ we had a condition to obtain exponential stability, i.e., $\omega_0(\mathcal{A}) < 0$. Hence, as in the Hilbert space case, we obtain that the solutions decay exponentially if (5.4) holds.

For the general case $\mathcal{E} := L^r(\Omega) \times L^p([-1,0], L^r(\Omega))$, we observe first that the operator $(A, D(A))$ has compact resolvent. From [7, Proposition 1.12(ii)] it follows that the operator $(A + \Phi(\epsilon_\lambda \otimes Id), D(A))$ has compact resolvent in $X := L^r(\Omega)$ and thus its spectrum does not depend on r, see [1, Proposition 2.6] for the details. Using a spectral mapping argument as before, we obtain condition (5.4) for the exponential stability.

References

[1] Arendt, W., *Gaussian estimates and interpolation of the spectrum in L^p*, Diff. Int. Eq., **7**, 1153-1168 (1994).

[2] Burns, J.A., Herdman, T.L., Stech, H.W., *Linear functional differential equations as semigroups on product spaces*, SIAM J. Math. Anal., **14**, 98-116 (1983).

[3] Conway, J., "Functions of One Complex Variable", Springer-Verlag, 1986.

[4] Curtain, R.F., Zwart, H.J., "An Introduction to Infinite Dimensional Linear Systems Theory", Springer-Verlag, 1995.

[5] Davies, E.B., "Spectral Theory and Differential Operators", Cambridge UP, 1995.

[6] Engel, K.-J., *Spectral theory and generator property of one-sided coupled operator matrices*, Semigroup Forum, **58**, 267-295 (1999).

[7] Engel, K.-J., Nagel, R. "One-parameter Semigroups for Linear Evolution Equations", to appear in: Graduate Texts in Mathematics, Springer-Verlag.

[8] Fischer, A., van Neerven, J.M.A.M., *Robust stability of C_0-semigroups and an application to stability of delay equations*, J. Math. Anal. Appl. **226**, 82-100 (1998).

[9] Gelbaum, B.R., Olmsted, J.M.H., "Counterexamples in Analysis", Holden-Day, 1964.

[10] Hale, J.K., "Functional Differential Equations", Springer-Verlag, 1971.

[11] Kappel, F., Zhang, K., *Equivalence of functional-differential equations of neutral type and abstract Cauchy problems*, Mh. Math. **101**, 115-133 (1986).

[12] Kato, T., "Perturbation Theory for Linear Operators", Springer-Verlag, 1980.

[13] Kunisch, K., Schappacher, W., *Necessary conditions for partial differential equations with delay to generate C_0-semigroups*, J. Differ. Equations, **50**, 49-79 (1983).

[14] Miyadera, I., *On perturbation theory for semi-groups of operators*, Tôhoku Math. **18**, 299-310 (1966).

[15] Nagel, R., Piazzera, S., *On the regularity properties of perturbed semigroups*, Rend. Circ. Matem. Palermo, Serie II, Suppl. **56**, 99-110 (1998).

[16] Nakagiri, S., *Optimal control of linear retarded systems in Banach spaces*, J. Math. Anal. Appl. **120**, 169-210 (1986).

[17] Nakagiri, S., *Structural properties of functional differential equations in Banach spaces*, Osaka J. Math., **25**, 353-398 (1988).

[18] van Neerven, J. M. A. M., "The Asymptotic Behaviour of Semigroups of Linear Operators", Operator Theory: Advances and Applications, Vol. 88, Birkhäuser, 1996.

[19] Piazzera, S., "Qualitative Properties of Perturbed Semigroups", Ph.D. Thesis, 1999.

[20] Prüß, J., "Evolutionary Integral Equations and Applications", Birkhäuser, 1993.

[21] Rhandi, A., *Extrapolation methods to solve non-autonomuos retarded partial differential equations*, Studia Math. **126**, 219-233 (1997).

[22] Voigt, J., *On the perturbation theory for strongly continuous semigroups*, Math. Ann. **229**, 163-171 (1977).

[23] Voigt, J., *Absorption semigroups, Feller property, and Kato class*, Oper. Theory. Adv. Appl. **78**, 389-396 (1995).

[24] Webb, G. F., *Autonomous nonlinear functional differential equations and nonlinear semigroups*, J. Math. Anal. Appl. **46**, 1-12 (1974).

[25] Webb, G. F., *Functional differential equations and nonlinear semigroups in L^p-spaces*, J. Differ. Equations, **29**, 71-89 (1976).

[26] Wu, J., "Theory and Applications of Partial Functional Differential Equations", Springer-Verlag, 1996.

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