FEYNMAN-KAC SEMIGROUPS, MARTINGALES AND WAVE OPERATORS

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ABSTRACT. In this paper we intend to discuss the following topics:

(1) Notation, generalities, Markov processes. The close relationship between (generators of) Markov processes and the martingale problem is exhibited. A link between the Korovkin property and generators of Feller semigroups is established.
(2) Feynman-Kac semigroups: 0-order regular perturbations, pinned Markov measures. A basic representation via distributions of Markov processes is depicted.
(3) Dirichlet semigroups: 0-order singular perturbations, harmonic functions, multiplicative functionals. Here a representation theorem of solutions to the heat equation is depicted in terms of the distributions of the underlying Markov process and a suitable stopping time.
(4) Sets of finite capacity, wave operators, and related results. In this section a number of results are presented concerning the completeness of scattering systems (and its spectral consequences).
(5) Some (abstract) problems related to Neumann semigroups: 1st order perturbations. In this section some rather abstract problems are presented, which lie on the borderline between first order perturbations together with their boundary limits (Neumann type boundary conditions and) and reflected Markov processes.
(6) Operator-valued Cauchy semigroups: some formulae, KMS-function. Some relevant general formulae are presented in which among other things the KMS-condition shows up.

1. NOTATION, GENERALITIES, MARKOV PROCESSES

The following notation will de used throughout in this paper. By E we mean a locally compact second countable Hausdorff space; $E^{\triangle} = E \cup \{\triangle\}$ is the one-point compactification of E, or, if E is itself compact, then $\triangle$ is an isolated point of $E^{\triangle}$. The *state space* E $(E^{\triangle})$ is supplied with its Borel field $\mathcal{E}$ $(\mathcal{E}^{\triangle})$. The sample space $\Omega := D\left([0,\infty], E^{\triangle}\right)$, called *Skorohod space*, which is a polish for a suitable metric,

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is given by

$$\Omega = \{\omega : [0,\infty] \to E^{\triangle}, \quad \omega \text{ cadlag on } \quad [0,\zeta), \omega(s) = \triangle, t \geq s \Rightarrow \omega(t) = \triangle\};$$

The symbol ζ stands for the life time $\zeta(\omega) = \inf\{s > 0 : \omega(s) = \triangle\}$ of the sample ω. The *state variables* $X(t) : \Omega \to E^{\triangle}$ are defined by $X(t)(\omega) = \omega(t)$.

The σ-fields $\mathcal{F}_t$, $t \geq 0$, constitute a *filtration*; in principle they are given by $\mathcal{F}_t = \sigma(X(s) : 0 \leq s \leq t)$. Their interpretation is that of *information from the past*. The σ-field generated by $\{X(s) : s \geq t\}$ is interpreted as the *information from the future*. The σ-field $\{X(t) \in B : \ B \in \mathcal{E}\}$ is interpreted as the *present information*.

The family $(\mathcal{F}_t)_{t\geq 0}$ is called a *filtration*, or *history* of the process. By $\mathcal{F}$ we mean $\mathcal{F} = \sigma(X(s) : s \geq 0)$. The (time) *translation* operators are defined by $[\vartheta_t(\omega)](s) = \omega(s+t)$.

1. Definition. A family $\{S(t) : t \geq 0\}$ of operators defined on $L^\infty(E)$ is a *Feller semigroup* on $C_0(E)$ if it possesses the following properties:

(i) It leaves $C_0(E)$ invariant: $S(t)C_0(E) \subseteq C_0(E)$ for $t \geq 0$;
(ii) It is a semigroup: $S(s+t) = S(s) \circ S(t)$ for all s, $t \geq 0$, and $S(0) = I$;
(iii) It consists of contraction operators: $\|S(t)f\|_\infty \leq \|f\|_\infty$ for all $t \geq 0$ and for all $f \in C_0(E)$;
(iv) It is positivity preserving: $f \geq 0$, $f \in C_0(E)$, implies $S(t)f \geq 0$;
(v) It is continuous for $t = 0$: $\lim_{t\downarrow 0} [S(t)f](x) = f(x)$, for all $f \in C_0(E)$ and for all $x \in E$.

In the presence of (iii) and (ii), property (v) is equivalent to:
(v′) $\lim_{t\downarrow 0} \|S(t)f - f\|_\infty = 0$ for all $f \in C_0(E)$. So that a Feller semigroup is in fact *strongly continuous* in the sense that, for every $f \in C_0(E)$,

$$\lim_{s\to t} \|S(s)f - S(t)f\|_\infty = 0.$$

A strongly continuous semigroup $\{S(t) : t \geq 0\}$ is called a *Feller semigroup* if it possesses the following positivity property: for all $f \in C_0(E)$, for which $0 \leq f \leq 1$, and for all $t \geq 0$, the inequality $0 \leq S(t)f \leq 1$ is true.

Parts of Theorem 2 are proved in [55].

2. Theorem. (a) *(Blumenthal and Getoor* [10]*) Let* $\{S(t) : t \geq\}$ *be a Feller semigroup in* $C_0(E)$. *Then there exists a strong Markov process (in fact a Hunt process)*

$$\{(\Omega, \mathcal{F}, \mathbf{P}_x), (X(t) : t \geq 0), (\vartheta_t : t \geq 0), (E, \mathcal{E})\},$$

such that

$$[S(t)f](x) = \mathbf{E}_x [f(X(t))], \ f \in C_0(E), \ t \geq 0.$$

Moreover this Markov process is normal (i.e. $\mathbf{P}_x [X(0) = x] = 1$*), is right continuous (i.e.* $\lim_{t\downarrow s} X(t) = X(s)$, $\mathbf{P}_x$*-almost surely), possesses left limits in* E *on its life time (i.e.* $\lim_{t\uparrow s} X(t)$ *exists in* E, *whenever* $\zeta > s$*), and is quasi-left-continuous (i.e. if* $(T_n : n \in \mathbf{N})$ *is an increasing sequence of* $(\mathcal{F}_t)$*-stopping times,* $X(T_n)$ *converges* $\mathbf{P}_x$*-almost surely to* $X(T)$ *on the event* $\{T < \infty\}$, *where* $T = \sup_{n\in\mathbf{N}} T_n$*).*

(b) *Conversely, let*

$$\{(\Omega, \mathcal{F}, \mathbf{P}_x), (X(t) : t \geq 0), (\vartheta_t : t \geq 0), (E, \mathcal{E})\}$$

be a strong Markov process which is normal, right continuous, and possesses left limits in E on its life time. Put $[S(t)f](x) = \mathbf{E}_x[f(X(t))]$, for f a bounded Borel function, $t \geq 0$, $x \in E$. Suppose that $S(t)f$ belongs to $C_0(E)$ for f belonging to $\in C_0(E)$, $t \geq 0$. Then $\{S(t) : t \geq 0\}$ is a Feller semigroup.

(c) *Let L be the generator of a Feller semigroup in $C_0(E)$ and let*

$$\{(\Omega, \mathcal{F}, \mathbf{P}_x), (X(t) : t \geq 0), (\vartheta_t : t \geq 0), (E, \mathcal{E})\}$$

be the corresponding Markov process. For every $f \in D(L)$ and for every $x \in E$, the process

$$t \mapsto f(X(t)) - f(X(0)) - \int_0^t Lf(X(s))ds$$

is a $\mathbf{P}_x$-martingale for the filtration $(\mathcal{F}_t)_{t\geq 0}$, where each σ-field $\mathcal{F}_t$, $t \geq 0$, is (some closure of) $\sigma(X(u) : u \leq t)$. In fact the σ-field $\mathcal{F}_t$ may taken to be $\mathcal{F}_t = \bigcap_{s>t} \sigma(X(u) : u \leq s)$. It is also possible to complete $\mathcal{F}_t$ with respect to $\mathbf{P}_\mu$, given by $\mathbf{P}_\mu(A) = \int \mathbf{P}_x(A)d\mu(x)$. For $\mathcal{F}_t$ the following σ-field may be chosen:

$$\mathcal{F}_t = \bigcap_{\mu \in P(E)} \bigcap_{s>t} \{\mathbf{P}_\mu\text{-completion of } \sigma(X(u) : u \leq s)\}.$$

(d) *Conversely, let L be a densely defined linear operator with domain $D(L)$ and range $R(L)$ in $C_0(E)$. Let $(\mathbf{P}_x : x \in E)$ be a unique family of probability measures, on an appropriate measure space $(\Omega, \mathcal{F})$ with an appropriate filtration $(\mathcal{F}_t)_{t\geq 0}$, such that, for all $x \in E$, $\mathbf{P}_x[X(0) = x] = 1$, and such that for all $f \in D(L)$ the process*

$$t \mapsto f(X(t)) - f(X(0)) - \int_0^t Lf(X(s))ds$$

is a $\mathbf{P}_x$-martingale with respect to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Then the operator L possesses a unique extension L_0, which generates a Feller semigroup in $C_0(E)$.

(e) *(Unique Markov extensions) Suppose that the densely defined linear operator L (with domain and range in $C_0(E)$) possesses the Korovkin property as well as the following one. For every $\lambda > 0$ (large) and for every $h \in D(L)$, the inequality*

$$\lambda \sup_{x \in E} h(x) \leq \sup_{x \in E} (\lambda I - L) h(x) \text{ or, equivalently} \tag{1.1}$$

$$\lambda \inf_{x \in E} h(x) \geq \inf_{x \in E} (\lambda I - L) h(x) \tag{1.2}$$

is valid. Then L extends to a unique generator L_0 of a Feller semigroup, and the martingale problem is well posed for the operator L. Moreover the Markov process associated with L_0 solves the martingale problem uniquely for L.

3. Definition. The operator L possesses the *Korovkin property* in the sense that there exists a strictly positive real number $t_0 > 0$ such that for every $x_0 \in E \cup \{\triangle\}$ the equality

$$\inf_{h\in D(L)} \sup_{x\in E} \left\{h(x_0) + \left[g - (I - t_0 L)\, h\right](x)\right\} \tag{1.3}$$

$$= \sup_{h\in D(L)} \inf_{x\in E} \left\{h(x_0) + \left[g - (I - t_0 L)\, h\right](x)\right\} \tag{1.4}$$

is valid for all $g \in C_0(E)$.

The proof of assertion (e) is based on the following result.

4. Proposition. *Let L be a linear operator with range $R(L)$ and domain $D(L)$ in $C_0(E)$. Fix $t_0 = \dfrac{1}{\lambda_0} > 0$. Suppose that for every $x_0 \in E \cup \{\triangle\}$, and for every $g \in C_0(E)$ the equality*

$$\begin{aligned}&\inf_{h\in D(L)} \sup_{x\in E} \left\{h(x_0) + \left[g - (I - t_0 L)\, h\right](x)\right\}\\ &= \sup_{h\in D(L)} \inf_{x\in E} \left\{h(x_0) + \left[g - (I - t_0 L)\, h\right](x)\right\}\end{aligned} \tag{1.5}$$

is valid. Also suppose that for every $\lambda > 0$ and for every $h \in D(L)$ the inequality

$$\lambda \sup_{x\in E} h(x) \le \sup_{x\in E} (\lambda I - L)\, h(x) \tag{1.6}$$

is valid. Then, for $0 < \lambda < 2\lambda_0$, the following identities are true:

$$\begin{aligned}
&\lambda R(\lambda) g(x_0) = \Lambda^+(g, x_0, \lambda)\\
&:= \inf_{h_0 \in D(L), h_1 \in D(L), h_2 \in D(L), \dots} \ \max_{x_1 \in E, x_2 \in E, \dots}\\
&\quad \left\{ \sum_{j=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_0}\right)^j h_j(x_j) + \frac{\lambda}{\lambda_0} \sum_{j=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_0}\right)^j g(x_{j+1}) \right.\\
&\qquad \left. - \sum_{j=0}^{\infty} \left(1 - \frac{\lambda}{\lambda_0}\right)^j \left(I - \frac{1}{\lambda_0} L\right) h_j\left(x_{j+1}\right) \right\}
\end{aligned} \tag{1.7}$$

$$\begin{aligned}
&= \liminf_{n\to\infty} \ \inf_{h_0 \in D(L), \dots, h_n \in D(L)} \ \max_{x_1 \in E, x_2 \in E, \dots, x_{n+1} \in E}\\
&\quad \left[\frac{\lambda}{\lambda_0} \sum_{j=1}^{n} \left(1 - \frac{\lambda}{\lambda_0}\right)^{j-1} \left\{ h_0(x_0) + \frac{\lambda}{\lambda_0} g(x_1) - \left(I - \frac{1}{\lambda_0} L\right) h_0(x_1) \right\} \right.\\
&\quad \left. + \left(1 - \frac{\lambda}{\lambda_0}\right) \sum_{j=1}^{n} \left(1 - \frac{\lambda}{\lambda_0}\right)^{j-1} \left\{ h_j(x_j) + \frac{\lambda}{\lambda_0} g\left(x_{j+1}\right) - \left(I - \frac{1}{\lambda_0} L\right) h_j\left(x_{j+1}\right) \right\} \right]
\end{aligned} \tag{1.8}$$

$$= \inf_{h_1 \in D(L), h_2 \in D(L), \dots, h_0 = \frac{\lambda}{\lambda_0} \sum_{j=1}^{\infty} \left(1 - \frac{\lambda}{\lambda_0}\right)^{j-1} h_j} \ \max_{x_1 \in E, x_2 \in E, \dots}$$

$$\left\{\sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^j h_j(x_j)+\frac{\lambda}{\lambda_0}\sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^j g(x_{j+1})\right.$$
$$\left.-\sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^j\left(I-\frac{1}{\lambda_0}L\right)h_j\left(x_{j+1}\right)\right\} \tag{1.9}$$
$$=\inf_{h\in D(L)}\max_{x\in E}\left\{h(x_0)+\left[g-\left(I-\frac{1}{\lambda}L\right)h\right](x)\right\} \tag{1.10}$$
$$=\sup_{h\in D(L)}\min_{x\in E}\left\{h(x_0)+\left[g-\left(I-\frac{1}{\lambda}L\right)h\right](x)\right\} \tag{1.11}$$
$$=\sup_{h_1\in D(L),h_2\in D(L),\dots,h_0=\frac{\lambda}{\lambda_0}\sum_{j=1}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^{j-1}h_j}\ \min_{x_1\in E,x_2\in E,\dots}$$
$$\left\{\sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^j h_j(x_j)+\frac{\lambda}{\lambda_0}\sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^j g(x_{j+1})\right.$$
$$\left.-\sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^j\left(I-\frac{1}{\lambda_0}L\right)h_j\left(x_{j+1}\right)\right\} \tag{1.12}$$
$$=\sup_{h_0\in D(L),h_1\in D(L),h_2\in D(L),\dots}\ \min_{x_1\in E,x_2\in E,\dots}$$
$$\left\{\sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^j h_j(x_j)+\frac{\lambda}{\lambda_0}\sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^j g(x_{j+1})\right.$$
$$\left.-\sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^j\left(I-\frac{1}{\lambda_0}L\right)h_j\left(x_{j+1}\right)\right\} \tag{1.13}$$
$$=\liminf_{n\to\infty}\ \sup_{h_0\in D(L),\dots,h_n\in D(L)}\ \min_{x_1\in E,x_2\in E,\dots,x_{n+1}\in E}$$
$$\left[\frac{\lambda}{\lambda_0}\sum_{j=1}^{n}\left(1-\frac{\lambda}{\lambda_0}\right)^{j-1}\left\{h_0(x_0)+\frac{\lambda}{\lambda_0}g(x_1)-\left(I-\frac{1}{\lambda_0}L\right)h_0(x_1)\right\}\right. \tag{1.14}$$
$$\left.+\left(1-\frac{\lambda}{\lambda_0}\right)\sum_{j=1}^{n}\left(1-\frac{\lambda}{\lambda_0}\right)^{j-1}\left\{h_j(x_j)+\frac{\lambda}{\lambda_0}g\left(x_{j+1}\right)-\left(I-\frac{1}{\lambda_0}L\right)h_j\left(x_{j+1}\right)\right\}\right]$$
$$=\frac{\lambda}{\lambda_0}\sum_{j=0}^{\infty}\left(1-\frac{\lambda}{\lambda_0}\right)^j\left(\lambda_0R\left(\lambda_0\right)\right)^{j+1}g(x_0). \tag{1.15}$$

A discussion of the proof of Theorem 2 can be found in [55].

For a concise formulation of our results we need another definition.

5. Definition. The operator $L^{\triangle}$ with domain $D\left(L^{\triangle}\right)$ in $C\left(E^{\triangle}\right)=C\left(E^{\triangle},\mathbf{R}\right)$ given by $D\left(L^{\triangle}\right)=\left\{h\in C\left(E^{\triangle}\right):h-h(\triangle)\in D(L)\right\}$, and $L^{\triangle}h=L(h-h(\triangle))$,

$h \in D\left(L^{\triangle}\right)$. Here we wrote $h(\triangle) = \lim_{x\to\triangle} h(x)$. It is noticed that $L^{\triangle}$ satisfies the maximum principle in the sense that if $h \in D\left(L^{\triangle}\right)$ satisfies $\sup_{x\in E} h(x) > h(\triangle)$, then there exists $x_0 \in E$ for which $h(x_0) = \sup_{x\in E} h(x)$, and for which $L^{\triangle}h(x_0) \le 0$. Fix $x_0 \in E$ and $\lambda > 0$. The functional $g \mapsto \Lambda^+(g, x_0, \lambda)$ is defined as follows:

$$\begin{aligned}
&\Lambda^+(g, x_0, \lambda)\\
&= \inf\left\{ h(x_0) : h \in D\left(L^{\triangle}\right),\ \left(I - \frac{1}{\lambda}L^{\triangle}\right) h \ge g \right\}\\
&= \inf_{h\in D(L)} \sup_{x\in E} \left(h(x_0) + \left[g - \left(I - \frac{1}{\lambda} L\right) h \right](x) \right)\\
&= \inf_{\substack{\Gamma\subset D(L)\\ \#\Gamma<\infty}} \sup_{\substack{\Phi\subset E\\ \#\Phi<\infty}} \min_{h\in\Gamma} \max_{x\in\Phi\cup\{\triangle\}} \left\{ h(x_0) + \left[g - \left(I - \frac{1}{\lambda} L\right) h \right](x) \right\}, \quad g \in C_0(E).
\end{aligned}$$

The functional $g \mapsto \Lambda^-(g, x_0, \lambda)$ is defined as follows:

$$\begin{aligned}
&\Lambda^-(g, x_0, \lambda)\\
&= \sup\left\{ h(x_0) : h \in D\left(L^{\triangle}\right),\ \left(I - \frac{1}{\lambda}L^{\triangle}\right) h \le g \right\}\\
&= \sup_{h\in D(L)} \inf_{x\in E} \left(h(x_0) + \left[g - \left(I - \frac{1}{\lambda} L\right) h \right](x) \right)\\
&= \sup_{\substack{\Gamma\subset D(L)\\ \#\Gamma<\infty}} \inf_{\substack{\Phi\subset E\\ \#\Phi<\infty}} \max_{h\in\Gamma} \min_{x\in\Phi\cup\{\triangle\}} \left\{ h(x_0) + \left[g - \left(I - \frac{1}{\lambda} L\right) h \right](x) \right\}, \quad g \in C_0(E).
\end{aligned}$$

Then, with $t\lambda = 1$,

$$-\Lambda^+(-g, x_0, \lambda) = \sup_{h\in D(L)} \inf_{x\in E} \left(h(x_0) + \left[g - (I - tL) h \right](x) \right) = \Lambda^-(g, x_0, \lambda),$$

with $g \in C_0(E)$. Since, as is readily proved, $\Lambda^+(g_1 + g_2, x_0, \lambda) \le \Lambda^+(g_1, x_0, \lambda) + \Lambda^+(g_2, x_0, \lambda)$, for g_1, $g_2 \in C_0(E)$, we see (again with $t\lambda = 1$)

$$\begin{aligned}
&\inf_{h\in D(L)} \sup_{x\in E} \left(h(x_0) + \left[g - (I - tL) h \right](x) \right)\\
&\quad - \sup_{h\in D(L)} \inf_{x\in E} \left(h(x_0) + \left[g - (I - tL) h \right](x) \right)\\
&= \Lambda^+(g, x_0, \lambda) - \left(-\Lambda^+(-g, x_0, \lambda)\right) = \Lambda^+(g, x_0, \lambda) + \Lambda^+(-g, x_0, \lambda)\\
&\ge \Lambda^+(0, x_0, \lambda) = \inf_{h\in D(L)} \left\{ \sup_{x\in E} (I - tL) h(x) - h(x_0) \right\} = 0.
\end{aligned}$$

The seminorm p_{x_0} is defined by ($t_0 = \lambda_0^{-1}$ is fixed throughout the remainder of this section) $p_{x_0}(g) = \Lambda^+(|g|, x_0, \lambda_0)$, $g \in C_0(E)$.

6. Proposition. *The following assertions hold true.*

(i) *The 3 defining expressions for* $\Lambda^{\pm}(g, x_0, \lambda)$ *are equal, and hence these functionals are well-defined.*

(ii) *The functionals* $\Lambda^+(\cdot, x_0, \lambda)$, $x_0 \in E$, *are sub-additive and positive homogeneous. The functionals* $\Lambda^-(\cdot, x_0, \lambda)$ *are super-additive and positive homogeneous. Moreover the inequalities*

$$\inf_{x\in E} g(x) \le \Lambda^-(g, x_0, \lambda) \le \Lambda^+(g, x_0, \lambda) \le \sup_{x\in E} g(x), \quad g \in C_0(E)$$

are valid. In addition $g_2 \ge g_1$, g_2, $g_1 \in C_0(E)$, *implies* $\Lambda^\pm(g_2, x_0, \lambda) \ge \Lambda^\pm(g_1, x_0, \lambda)$. *In addition,* $\Lambda^\pm(\alpha, x_0, \lambda) = \alpha$, $\alpha \in \mathbf{R}$.

(iii) *The functionals* p_{x_0}, $x_0 \in E$, *are seminorms indeed.*

(iv) *Let* $g_1 = (I - t_0L)\, h_1$ *belong to the range of the operator* $I - t_0L$. *For* g *belonging to* $C_0(E)$ *the equalities* ($t_0\lambda_0 = 1$)

$$\Lambda^\pm(g + g_1, x_0, \lambda_0) = \Lambda^\pm(g, x_0, \lambda_0) + \Lambda^\pm(g_1, x_0, \lambda_0) = \Lambda^\pm(g, x_0, \lambda_0) + h_1(x_0)$$

hold true.

(v) *The subset* $M(x_0)$ *of* $C_0(E)$ *defined by*

$$M(x_0) = \Big\{ g \in C_0(E) : \inf_{h\in D(L)} \sup_{x\in E} \left(h(x_0) + \left[g - (I - t_0L)\, h\right](x)\right) \tag{1.16}$$

$$= \sup_{h\in D(L)} \inf_{x\in E} \left(h(x_0 + \left[g - (I - t_0L)\, h\right](x)\right) \Big\} \tag{1.17}$$

is a closed linear subspace of $C_0(E)$ *and* $g \mapsto \Lambda^+(g, x_0, \lambda_0) = \Lambda^-(g, x_0, \lambda_0)$, $g \in M(x_0)$, *is a linear functional on* $M(x_0)$.

(vi) *Let* μ *be a sub-probability measure on* E *with the property that*

$$\int (I - t_0L)\, h\, d\mu = h(x_0) \text{ for all } h \in D(L).$$

Then $\Lambda^-(g, x_0) \le \int g d\mu \le \Lambda^+(g, x_0)$ *for all* $g \in C_0(E)$.

The existence of a probability measure μ *with the property described in* (vi) *is guaranteed by the argument in* Remark 2 *following the proof of Theorem 2.*

For a proof the reader is referred to [55].

Outline of a proof of Theorem 2. (a) Assertion (a) is proved in Blumenthal and Getoor [10]: Theorem 4.9, page 46.

(b) The semigroup property is an easy consequence of the Markov property:

$$\begin{aligned}
[S(s)S(t)f](x) &= \mathbf{E}_x\left[[S(t)f]\left(X(s)\right)\right] \\
&= \mathbf{E}_x\left[\mathbf{E}_{X(s)}\left[f\left(X(t)\right)\right]\right] = \mathbf{E}_x\left[\mathbf{E}_x\left[f\left(X(t+s)\right) \mid \mathcal{F}_s\right]\right] \\
&= \mathbf{E}_x\left[f\left(X(t+s)\right)\right] = S(t+s)f(x),
\end{aligned}$$

where f is a bounded Borel measurable function. The other assertions are automatically true.

(c) Let f be a member of $D(L)$ and put $M_f(t) = f(X(t)) - f(X(0)) - \int_0^t Lf(X(s))ds$. Then, for $t_2 > t_1$ we have

$$\mathbf{E}_x\left[M_f(t_2) \mid \mathcal{F}_{t_1}\right] - M_f(t_1) = \mathbf{E}_x\left[M_f(t_2 - t_1) \circ \vartheta_{t_1} \mid \mathcal{F}_{t_1}\right]$$

$$\text{(Markov property)} = \mathbf{E}_{X(t_1)}\left[M_f(t_2 - t_1)\right]. \tag{1.18}$$

Since, in addition, by virtue of the fact that L generates the semigroup $\{S(t) : t \geq 0\}$,

$$\begin{aligned}\mathbf{E}_z\left[M_f(t)\right] &= S(t)f(z) - f(z) - \int_0^t S(u)Lf(z)du \\ &= S(t)f(z) - f(z) - \int_0^t \frac{\partial}{\partial u}\left(S(u)f(z)\right)du \\ &= S(t)f(z) - f(z) - \left(S(t)f(z) - S(0)f(z)\right) = 0,\end{aligned}$$

the asserion in (c) follows from 1.18.

(d) The proof of (d) is to be found in [51]. An outline of the proof can also be found in [40].There are two issues involved. One is related to the fact that the functions $x \mapsto \mathbf{E}_x\left[f(X(t)\right]$ belongs to $C_0(E)$, whenever f does. The other one is related to the proof of the fact that the complement of Ω in the product space $\left(E^{\triangle}\right)^{[0,\infty]}$ is of $\mathbf{P}_x$-measure zero. The proof of this fact follows more or less the same pattern as the proof of (a) in [10]. The invariance of the space $C_0(E)$ under the action of the semigroup $\{S(t) : t \geq 0\}$, where $S(t)f(x) = \mathbf{E}_x\left[f(X(t))\right]$, f bounded measurable may be based on the following facts.

Fact 1: Let $P(\Omega)$ denote the set of probability measures on $\mathcal{F}$. The set

$$P'(\Omega) = \bigcup_{x\in E^{\triangle}} \Big\{ \mathbf{P} \in P(\Omega) : \mathbf{P}(X(0) = x) = 1 \text{ and for every } f \in D(L) \text{ the process } f(X(t)) - f(X(0)) - \int_0^t Lf(X(s))ds, t \geq 0, \text{ is a } \mathbf{P}\text{-martingale} \Big\}.$$

is a compact metrizable Hausdorff space for an appropriate metric. In other words, the collections of solutions to the martingale problem is a compact Hausdorff space.

Fact 2: Let E' be the largest subset of $E^{\triangle}$ on which the martingale problem is well-posed. So for every $x \in E'$ the martingale problem is uniquely solvable. Put

$$P'(E', \Omega) = \bigcup_{x\in E'} \left\{\mathbf{P} \in P'(\Omega) : \mathbf{P}\left[X(0) = x\right] = 1\right\}.$$

Define the map $F : P'(E', \Omega) \to E^{\triangle}$ by $F(\mathbf{P}) = x$, where $\mathbf{P} \in P'(\Omega)$ is such that $\mathbf{P}(X(0) = x) = 1$. Also notice that $F(P_{\triangle}) = \triangle$. Then F is a homeomorphism from $P'(E', \Omega)$ onto E'. Consequently for every function $u \in C_0(E)$, the function $(s, x) \mapsto \mathbf{E}_x(u(X(s))$ is continuous on $[0, \infty) \times E'$. In particular it follows that $\lim_{x\to\triangle, x\in E'} \mathbf{E}_x\left[u(X(s))\right] = 0$.

A combination of these two facts shows that $S(t)$ leaves the space $C_0(E)$ invariant.

(e) Define the operator $\lambda_0 R(\lambda_0)$ as follows ($\lambda_0 t_0 = 1$):

$$\begin{aligned}\lambda_0 R(\lambda_0)g(x_0) &= \inf_{h\in D(L)} \sup_{x\in E} \left\{h(x_0) + \left[g - (I - t_0L)\,h\right](x)\right\} = \Lambda^+(g, x_0, \lambda) \\ &= \Lambda^-(g, x_0, \lambda) = \sup_{h\in D(L)} \inf_{x\in E} \left\{h(x_0) + \left[g - (I - t_0L)\,h\right](x)\right\}.\end{aligned}$$

The function $x_0 \mapsto \Lambda^+ (g, x_0, \lambda_0)$ is lower semi-continuous, and the function $x_0 \mapsto \Lambda^- (g, x_0, \lambda_0)$ is upper semi-continuous. Consequently the function $\lambda_0 R(\lambda_0) g$ is continuous. Fix $\alpha > 0$ and choose $h_0 \in D(L)$ in such a way that $g - (I - t_0 L) h_0 \leq \frac{1}{2}\alpha$. Then the inclusions

$$\begin{aligned}&\{x_0 \in E : \lambda_0 R(\lambda_0) g(x_0) \geq \alpha\}\\ &\subseteq \left\{x_0 \in E : h_0(x_0) + \sup_{x\in E} \left[g - (I - t_0 L) h_0\right](x_0) \geq \alpha\right\} \subseteq \left\{x_0 \in E : h_0(x_0) \geq \frac{1}{2}\alpha\right\}\end{aligned}$$

are valid. These observations prove that $\lambda_0 R(\lambda_0) g$ belongs to $C_0(E)$ whenever g does so. Then prove that

$$\inf_{x\in E} g(x) \leq \lambda_0 R(\lambda_0) g(x_0) \leq \sup_{x\in E} g(x), \quad g \in C_0(E).$$

Define the operator L_0 on $D(L_0) = R(R(\lambda_0))$ by the equality

$$L_0 (R(\lambda_0) g) = \lambda_0 R(\lambda_0) g - g, \quad g \in C_0(E).$$

Then L_0 verifies the maximum principle (which will be a consequence of Proposition 4: see Proof of Theorem 2 part (e) conclusion) and the range of $\lambda_0 I - L_0$ coincides with $C_0(E)$. Moreover L_0 extends L. Since $D(L)$ is dense, the domain of L_0 is dense as well. Consequently L_0 generates a Feller semigroup. If L_1 and L_2 are two generators of Feller semigroups which extend L_0 with respective resolvent families $\{R_1(\lambda) : \lambda > 0\}$ and $\{R_2(\lambda) : \lambda > 0\}$, then

$$\begin{aligned}&\sup_{h\in D(L)} \inf_{x\in E} \left\{h(x_0) + \left[g - (I - t_0 L) h\right](x)\right\} \leq \lambda_0 R_1(\lambda_0) g(x_0)\\ &\leq \inf_{h\in D(L)} \sup_{x\in E} \left\{h(x_0) + \left[g - (I - t_0 L) h\right](x)\right\}\\ &\leq \sup_{h\in D(L)} \inf_{x\in E} \left\{h(x_0) + \left[g - (I - t_0 L) h\right](x)\right\}.\end{aligned}$$

The same is true for $\lambda_0 R_2(\lambda_0) g(x_0)$. Consequently $R_1(\lambda_0) = R_2(\lambda_0)$, and thus $L_1 = L_2$. First we give an alternative description of the operator L_0. Again the equality in (1.5) is available. The quantities in (1.10) and (1.11) of Proposition 4 are equal. As a consequence, we may repeat the construction in Proposition 4 for any $0 < \lambda_1 < 2\lambda_0$ instead of λ_0. In this way we obtain a resolvent family $\{R(\lambda) : 0 < \lambda < 4\lambda_0\}$, for which Proposition 4 is applicable. By induction we find a resolvent family $\{R(\lambda) : \lambda > 0\}$ with the property

$$\begin{aligned}\lambda R(\lambda) g(x_0) &= \inf_{\substack{\Gamma\subset D(L)\\ \#\Gamma<\infty}} \sup_{\substack{\Phi\subset E\\ \#\Phi<\infty}} \min_{h\in\Gamma} \max_{x\in\Phi\cup\{\triangle\}} \left\{h(x_0) + \left[g - \left(I - \frac{1}{\lambda} L\right) h\right](x)\right\}\\ &= \sup_{\substack{\Gamma\subset D(L)\\ \#\Gamma<\infty}} \inf_{\substack{\Phi\subset E\\ \#\Phi<\infty}} \max_{h\in\Gamma} \min_{x\in\Phi\cup\{\triangle\}} \left\{h(x_0) + \left[g - \left(I - \frac{1}{\lambda} L\right) h\right](x)\right\}. \quad (1.19)\end{aligned}$$

From the first part of the proof of (e) of Theorem 2 we see that the operator $R(\lambda_0)$ leaves the space $C_0(E)$ invariant. Hence, so do the operators $R(\lambda)$, $\lambda > 0$. Put $L_0 R(\lambda) g = \lambda R(\lambda) g - g$, $g \in C_0(E)$. First we show that L_0 is well-defined and that $L_0 = L_1$, where $L_1 = \text{s-}\lim_{\alpha\to\infty} \alpha (\alpha R(\alpha) - I)$. Therefore we consider, for

$h \in D(L)$, $\lambda R(\lambda)h - h = R(\lambda)(Lh)$. Since $\inf_{x\in E} g(x) \leq \lambda R(\lambda)g \leq \sup_{x\in E} g(x)$, we see that $\lim_{\lambda\to\infty} \lambda R(\lambda)h = h$, for $h \in D(L)$. Since $D(L)$ is dense in $C_0(E)$, and since $\|\lambda R(\lambda)\| \leq 1$, for $\lambda > 0$, we infer $\lim_{\lambda\to\infty} \lambda R(\lambda)g = g$, $g \in C_0(E)$. Next we prove that L_0 is well-defined. Suppose that $R(\lambda_1) g_1 = R(\lambda_2) g_2$, g_1, $g_2 \in C_0(E)$, λ_1, $\lambda_2 > 0$. Then

$$\begin{aligned}
&R(\lambda_2)\left(\lambda_2 R(\lambda_2) g_2 - \lambda_1 R(\lambda_1) g_1 - g_2 + g_1\right) \\
&= R(\lambda_2)\left(\left(\lambda_2 - \lambda_1\right) R(\lambda_1) g_1 - g_2 + g_1\right) \\
&= R(\lambda_1) g_1 - R(\lambda_2) g_1 - R(\lambda_2) g_2 + R(\lambda_2) g_1 = 0.
\end{aligned}$$

Put $g = \lambda_2 R(\lambda_2) g_2 - \lambda_1 R(\lambda_1) g_1 - (g_2 - g_1)$. Then $R(\lambda_2) g = 0$. Consequently, $R(\lambda)g = 0$ for all $\lambda > 0$, and hence $g = 0$. This proves that L_0 is well-defined. Since

$$\begin{aligned}
\alpha\left(\alpha R(\alpha) - I\right) R(\beta) &= \alpha\left(\frac{\alpha}{\alpha-\beta}\left(R(\beta) - R(\alpha)\right) - R(\beta)\right) \\
&= \alpha\left(\frac{\beta}{\alpha-\beta} R(\beta) - \frac{\alpha}{\alpha-\beta} R(\alpha)\right) \to \beta R(\beta) - I, \qquad (1.20)
\end{aligned}$$

if α tends to ∞, it follows that $L_1 R(\beta) = \beta R(\beta) - I$. As a consequence we see that L_1 extends L_0. Next suppose that g_1 belongs to $D(L_1)$. Then

$$(\lambda I - L_1) g_1 = (\lambda I - L_0) R(\lambda) (\lambda I - L_1) g_1 = (\lambda I - L_1) R(\lambda) (\lambda I - L_1) g_1. \qquad (1.21)$$

Since, for $g \in D(L_1)$,

$$\begin{aligned}
R(\beta)\left(\lambda I - L_1\right) g &= \lambda R(\beta) g - \lim_{\alpha\to\infty} \alpha R(\beta)\left(\alpha R(\alpha) - I\right) g \\
&= \lambda R(\beta) g + \lim_{\alpha\to\infty} \frac{\alpha}{\alpha-\beta}\left(\alpha R(\alpha) g - \beta R(\beta)\right) g \\
&= \lambda R(\beta) g + g - \beta R(\beta) g = \left(\lambda I - L_0\right) R(\beta) g. \qquad (1.22)
\end{aligned}$$

From (1.21) and (1.22) we obtain

$$\begin{aligned}
0 &= \beta R(\beta)\left(\lambda I - L_1\right)\left(g_1 - R(\lambda)\left(\lambda I - L_1\right) g_1\right) \\
&= \beta\left(\lambda I - L_0\right) R(\beta)\left(g_1 - R(\lambda)\left(\lambda I - L_1\right) g_1\right). \qquad (1.23)
\end{aligned}$$

In (1.23) we let β tend to ∞. Since L_0 is a closed linear operator, we obtain that the function $g_1 - R(\lambda)(\lambda I - L_1) g_1$ belongs to the domain of L_0, and consequently g_1 is a member of $D(L_0)$. Next we show that the operator L_0 verifies the maximum principle. Fix $g \in D(L_0)$, and let $x_0 \in E$ be such that $g(x_0) = \sup_{x\in E} g(x)$. Then

$$\begin{aligned}
L_0 g(x_0) = L_1 g(x_0) &= \lim_{\alpha\to\infty} \alpha\left(\alpha R(\alpha) g(x_0) - g(x_0)\right) \\
&\leq \lim_{\alpha\to\infty}\left(\sup_{x\in E} g(x) - g(x_0)\right) \leq 0. \qquad (1.24)
\end{aligned}$$

In addition, we show that L_0 extends L. If h_0 belongs to $D(L)$, then

$$\begin{aligned}
&L_0 \Lambda\left((I - t_0 L) h_0, \cdot\right)(x_0) = \lambda_0 \Lambda\left((I - t_0 L) h_0, x_0\right) - \lambda_0\left(I - t_0 L\right) h_0(x_0) \\
&= \lambda_0 h_0(x_0) - \lambda_0 h_0(x_0) + L h_0(x_0) = L h_0(x_0).
\end{aligned}$$

Since $D(L)$ is dense, it follows that the domain of L_0 is dense. Since

$$(\lambda I - L_0)\, R(\lambda) = I$$

we see that the range of $\lambda I - L_0$ coincides with $C_0(E)$. From the Lumer-Phillips theorem, we may conclude that the operator L_0 generates a Feller semigroup: see e.g. [50].

Next we prove the *uniqueness*. Let L_1 and L_2 be two linear extensions of L which generate Feller semigroups with respective resolvent families $\{R_1(\lambda) : \lambda > 0\}$ and $\{R_2(\lambda) : \lambda > 0\}$. Then there exists a probability measures $\mu^1_{x_0}$ and $\mu^2_{x_0}$ on the Borel field of $E^{\triangle}$ such that $\lambda_0 R_j(\lambda_0) g(x_0) = \int g(y) d\mu^j_{x_0}$, $g \in C_0(E)$, $j = 1, 2$. Fix $\varepsilon > 0$. Then we obtain, for some finite subset $\Gamma = \Gamma_\varepsilon$ of $D(L)$,

$$\begin{aligned}
-\frac{\varepsilon}{2} + \Lambda^-(g, x_0) &\leq \inf_{\substack{\Phi \subset E \\ \#\Phi < \infty}} \max_{h \in \Gamma} \min_{x \in \Phi \cup \{\triangle\}} \left\{ h(x_0) + \left[g - (I - t_0 L)\, h \right](x) \right\} - \frac{\varepsilon}{4} \\
&\leq \max_{h \in \Gamma} \left\{ h(x_0) + \lambda_0 R_1(\lambda_0)\, g(x_0) - \lambda_0 R_1(\lambda_0)\,(I - t_0 L)\, h(x_0) \right\} \\
&= \lambda_0 R_1(\lambda_0)\, g(x_0) \leq \Lambda^+(g, x_0) + \frac{\varepsilon}{2}.
\end{aligned} \tag{1.25}$$

By the same token we have

$$-\frac{\varepsilon}{2} + \Lambda^-(g, x_0) \leq \lambda_0 R_2(\lambda_0)\, g(x_0) \leq \Lambda^+(g, x_0) + \frac{\varepsilon}{2}. \tag{1.26}$$

Consequently, from (1.25) and (1.26) together with the equality of the expressions in (1.3) and (1.4) we obtain

$$\begin{aligned}
-\varepsilon = \Lambda^-(g, x_0) - \Lambda^+(g, x_0) - \varepsilon &\leq \lambda_0 R_1(\lambda_0)\, g(x_0) - \lambda_0 R_2(\lambda_0)\, g(x_0) \\
&\leq \Lambda^+(g, x_0) - \Lambda^-(g, x_0) + \varepsilon = \varepsilon.
\end{aligned} \tag{1.27}$$

Since $\varepsilon > 0$ is arbitrary we conclude $R_1(\lambda_0) g(x_0) = R_2(\lambda_0) g(x_0)$, $g \in C_0(E)$, $x_0 \in E$, and hence $R_1(\lambda_0) = R_2(\lambda_0)$. Thus $L_1 = L_2$.

Let L_0 be the (unique) extension of L, which generates a Feller semigroup, and let

$$\left\{ (\Omega, \mathcal{F}, \mathbf{P}_x), (X(t), t \geq 0), (\vartheta_t, t \geq 0), (E, \mathcal{E}) \right\}$$

be the corresponding Markov process with $\mathbf{E}_x\left[g(X(t)\right] = \exp\left(tL_0\right) g(x)$, $g \in C_0(E)$, $x \in E$, $t \geq 0$. Then the family $\{\mathbf{P}_x : x \in E\}$ is a solution to the martingale problem associated to L. The proof of the uniqueness part follows a pattern similar to the proof of the uniqueness part (e) of Theorem 2. Let $\left\{\mathbf{P}^{(1)}_x : x \in E\right\}$ and $\left\{\mathbf{P}^{(2)}_x : x \in E\right\}$ be two solutions to the martingale problem for L. Fix $x_0 \in E$, $g \in C_0(E)$, and $s > 0$. Then, as in the proof of the first part of (e) of Theorem 2

$$\Lambda^-(g, X(s), \lambda) \leq \lambda \int_0^\infty \exp(-\lambda t) \mathbf{E}^{(j)}_{x_0} \left[g\left(X(t+s)\right) \,\middle|\, \mathcal{F}_s \right] dt \leq \Lambda^+(g, X(s), \lambda), \tag{1.28}$$

for $j = 1, 2$, where

$$\Lambda^{+}\left(g, x_0, \lambda\right) = \inf_{\substack{\Gamma\subset D(L)\\ \#\Gamma<\infty}} \sup_{\substack{\Phi\subset E\\ \#\Phi<\infty}} \min_{h\in\Gamma} \max_{x\in\Phi\cup\{\triangle\}} \left\{ h(x_0) + \left[g - \left(I - \frac{1}{\lambda} L\right) h\right](x)\right\}; \tag{1.29}$$

$$\Lambda^{-}\left(g, x_0, \lambda\right) = \sup_{\substack{\Gamma\subset D(L)\\ \#\Gamma<\infty}} \inf_{\substack{\Phi\subset E\\ \#\Phi<\infty}} \max_{h\in\Gamma} \min_{x\in\Phi\cup\{\triangle\}} \left\{ h(x_0) + \left[g - \left(I - \frac{1}{\lambda} L\right) h\right](x)\right\}. \tag{1.30}$$

In the same spirit we get

$$\Lambda^{-}\left(g, X(s), \lambda\right) \leq \lambda \int_0^\infty \exp(-\lambda t) \mathbf{E}^{(j)}_{X(s)}\left[g\left(X(t)\right)\right] dt \leq \Lambda^{+}\left(g, X(s), \lambda\right), \tag{1.31}$$

for $j = 1, 2$. Since, by Proposition 4 (formula (1.10) and (1.11)) $\Lambda^{+}\left(g, x, \lambda\right) = \Lambda^{-}\left(g, x, \lambda\right)$, $g \in C_0(E)$, $x \in E$, $\lambda > 0$, we obtain, by putting $s = 0$, $\mathbf{E}^{(1)}_x\left[g(X(t))\right] = \mathbf{E}^{(2)}_x\left[g(X(t))\right]$, $t \geq 0$, $g \in C_0(E)$. We also obtain, $\mathbf{P}^{(1)}_x$-almost surely,

$$\mathbf{E}^{(1)}_x\left[g(X(t+s)) \bigm| \mathcal{F}_s\right] = \mathbf{E}^{(1)}_{X(s)}\left[g(X(t))\right],$$

and, $\mathbf{P}^{(2)}_x$-almost surely,

$$\mathbf{E}^{(2)}_x\left[g(X(t+s)) \bigm| \mathcal{F}_s\right] = \mathbf{E}^{(2)}_{X(s)}\left[g(X(t))\right], \text{ for } t,\ s \geq 0, \text{ and } g \in C_0(E).$$

It necessarily follows that $\mathbf{P}^{(1)}_x = \mathbf{P}^{(2)}_x$, $x \in E$. This proves the uniqueness of the solutions to the martingale problem for the operator L. $\square$

Remark 1. For Ω we may take the Skorohod space $\Omega = D\left([0,\infty], E^{\triangle}\right)$. So a sample ω belongs to Ω if it possesses the following properties:

(i) ω is a mapping from $[0,\infty]$ to $E^{\triangle} = E \cup \{\triangle\}$; $\omega(0) \in E$.

(ii) ω is right continuous and possesses left limits in E on the stochastic interval $[0, \zeta(\omega))$, in the sense that $\lim_{t\uparrow s}\omega(s)$ exists in E for
$$s < \zeta(\omega) := \inf\left\{t > 0 : \omega(t) = \triangle\right\}.$$
Moreover, if $\omega(s) = \triangle$ and if $t \geq s$, then $\omega(t) = \triangle$.

(iii) The set $E^{\triangle}$ is the one-point compactification of E, or, if E is compact, $\triangle$ is an isolated point of $E^{\triangle} = E \cup \{\triangle\}$.

Remark 2. The collection $\{\mathcal{F}_t : t \geq 0\}$ is a *filtration*: if $s < t$, then $\mathcal{F}_s \subset \mathcal{F}_t$. Every σ-field $\mathcal{F}_t$ is an appropriate completion (extension) of the σ-field $\sigma\left(X(u) : u \leq t\right)$. The family $\{\mathcal{F}_t : t \geq 0\}$ is continuous from the right: $\mathcal{F}_t = \bigcap_{s>t}\mathcal{F}_s$. Since we consider more or less the internal history $\{\mathcal{F}_t : t \geq 0\}$, $t \geq 0$, we suppress the notation $\mathcal{F}_t$, $t \geq 0$, in our symbolism of our Markov process:

$$\left\{\left(\Omega, \mathcal{F}, \mathbf{P}_x\right), \left(X(t) : t \geq 0\right), \left(\vartheta_t : t \geq 0\right), \left(E, \mathcal{E}\right)\right\}.$$

Authors often write things like $\left\{\left(\mathbf{P}_x\right)_{x\in E},\ \left(X(t)\right)_{t\geq 0}\right\}$, when the other items are clear from the context.

Remark 3. The mappings $X(t) : \Omega \to E^{\triangle}$ are called *state variables*; E is referred to as the *state space* (sometimes *stochastic state space*). Put

$$\zeta = \inf \{s > 0 : X(s) = \triangle\}.$$

Then ζ is called the *life time* of the process $\{X(t) : t \geq 0\}$.
The motion $\{X(t) : t \geq 0\}$ is $\mathbf{P}_x$-almost surely right continuous and possesses left limits in E on its life time:

(i) $\lim_{s \downarrow t} X(s) = X(t)$, (right continuity);
(ii) $s \geq t$, $X(t) = \triangle$, implies $X(s) = \triangle$, ($\triangle$ is cemetery);
(iii) $\lim_{s \uparrow t} X(s) = X(t-) \in E$, $t < \zeta$, (left limits in E on its life time).

These assertions hold $\mathbf{P}_x$-almost surely for all $x \in E$. The probability $\mathbf{P}_{\triangle}$ may be defined by $\mathbf{P}_{\triangle}(A) = \delta_{\omega_{\triangle}}(A)$, where $\omega_{\triangle}(s) = \triangle$, $s > 0$.

Remark 4. The *shift* or *translation* operators $\vartheta_s : \Omega \to \Omega$, $s \geq 0$, possess the property that $X(t) \circ \vartheta_s = X(t+s)$, $\mathbf{P}_x$-almost surely, for all $x \in E$ and for all s and $t \geq 0$. This is an extremely important property. For example $f(X(t)) \circ \vartheta_s = f(X(t+s))$, $f \in C_0(E)$, s, $t \geq 0$. If Ω is the Skorohod space $\Omega = D\left([0, \infty], E^{\triangle}\right)$, then $X(t)(\omega) = \omega(t) = X(t, \omega) = \omega(t)$, $\vartheta_t(\omega)(s) = \omega(s+t)$, $\omega \in \Omega$.

Remark 5. For every $x \in E$, the measure $\mathbf{P}_x$ is a probability measure on $\mathcal{F}$ with the property that $\mathbf{P}_x[X(0) = x] = 1$. So the process starts at $X(0) = x$, $\mathbf{P}_x$-almost surely, at $t = 0$. This is the *normality property*.

Remark 6. The Markov property can be expressed as follows:

$$\begin{aligned} \mathbf{E}_x\left[f(X(s+t)) \,\middle|\, \mathcal{F}_s\right] &= \mathbf{E}_x\left[f(X(s+t)) \,\middle|\, \sigma(X(s))\right] \\ &= \mathbf{E}_{X(s)}\left[f(X(t))\right], \end{aligned} \tag{1.32}$$

$\mathbf{P}_x$-almost surely for all $f \in C_0(E)$ and for all s and $t \geq 0$. Of course, the expression $\mathbf{E}\left[Y \,\middle|\, \mathcal{F}\right]$ denotes conditional expectation. The meaning of $\mathcal{F}_t$ is explained in Remark 1. Let $Y : \Omega \to \mathbf{C}$ be a bounded random variable. This means that Y is measurable with respect to the field generated by $\{X(u) : u \geq 0\}$. The Markov property is then equivalent to

$$\mathbf{E}_x\left[Y \circ \vartheta_s \,\middle|\, \mathcal{F}_s\right] = \mathbf{E}_{X(s)}[Y], \tag{1.33}$$

$\mathbf{P}_x$-almost surely for all bounded random variables Y and for all $s \geq 0$. Notice that, intuitively speaking, $\mathcal{F}_s$ is the information from the past, $\sigma(X(s))$ is the information at the present, and $Y \circ \vartheta_s$ is measurable with respect t o some completion of $\sigma\{X(u) : u \geq s\}$, the information from the future. Put $P(t, x, B) = \mathbf{P}_x[X(t) \in B]$. Then $\mathbf{E}_x[f(X(t))] = \int f(y) P(t, x, dy)$, $f \in C_0(E)$. Moreover (1.32) is equivalent to (1.33) and to

$$\mathbf{E}_x\left[\prod_{j=1}^n f_j(X(t_j))\right] = \int\int \dots \int \prod_{j=1}^n \left(f_j(x_j) P(t_j - t_{j-1}, x_{j-1}, dx_j)\right), \tag{1.34}$$

for all $0 = t_0 \leq t_1 < t_2 < \cdots < t_n < \infty$ and for all $f_1, \dots, f_n$ in $C_0(E)$.

Remark 7. Next we explain the strong Markov property. Since the paths $\{X(t) : t \geq 0\}$ are right continuous $\mathbf{P}_x$-almost surely our Markov process is a strong Markov process. Let $S : \Omega \to \infty$ be a *stopping* meaning that for every $t \geq 0$ the event $\{S \leq t\}$ belongs to $\mathcal{F}_t$. This is the same as saying that the process $t \mapsto 1_{[S \leq t]}$ is adapted. Let $\mathcal{F}_S$ be the natural σ-field associated with the stopping time S, i.e.

$$\mathcal{F}_S = \bigcap_{t \geq 0} \{A \in \mathcal{F} : A \cap \{S \leq t\} \in \mathcal{F}_t\}.$$

Define $\vartheta_S(\omega)$ by $\vartheta_S(\omega) = \vartheta_{S(\omega)}(\omega)$. Consider $\mathcal{F}_S$ as the information from the past, $\sigma\big(X(S)\big)$ as information from the present, and

$$\sigma\{X(t) \circ \vartheta_S : t \geq 0\} = \sigma\{X(t+S) : t \geq 0\}$$

as the information from the future. The strong Markov property can be expressed as follows:

$$\mathbf{E}_x\left[Y \circ \vartheta_S | \mathcal{F}_S\right] = \mathbf{E}_{X(S)}\left[Y\right], \ \mathbf{P}_x\text{-almost surely} \tag{1.35}$$

on the event $\{S < \infty\}$, for all bounded random variables Y, for all stopping times S, and for all $x \in E$. One can prove that under the "cadlag" property events like $\{X(S) \in B,\ S < \infty\}$, B Borel, are $\mathcal{F}_S$-measurable. The passage from (1.35) to (1.32) is easy: put $Y = f(X(t))$ and $S(\omega) = s$, $\omega \in \Omega$. The other way around is much more intricate and uses the cadlag property of the process $\{X(t) : t \geq 0\}$. In this procedure the stopping time S is approximated by a decreasing sequence of discrete stopping times $(S_n = 2^{-n}\lceil 2^n S\rceil : n \in \mathbf{N})$. The equality

$$\mathbf{E}_x\left[Y \circ \vartheta_{S_n} | \mathcal{F}_{S_n}\right] = \mathbf{E}_{X(S_n)}\left[Y\right], \ \mathbf{P}_x\text{-almost surely},$$

is a consequence of 1.32 for a fixed time. Let n tend to infinity in (1) to obtain (1.35). The "strong Markov property" can be extended to the "strong time dependent Markov property":

$$\begin{aligned} &\mathbf{E}_x\left[Y\left(S + T \circ \vartheta_S, \vartheta_S\right) \bigm| \mathcal{F}_S\right](\omega) \\ &= \mathbf{E}_{X\left(S(\omega)\right)}\left[\omega' \mapsto Y\left(S(\omega) + T\left(\omega'\right), \omega'\right)\right], \end{aligned} \tag{1.36}$$

$\mathbf{P}_x$-almost surely on the event $\{S < \infty\}$. Here $Y : [0, \infty) \times \Omega \to \mathbf{C}$ is a bounded random variable. The Cartesian product $[0, \infty) \times \Omega$ is supplied with the product field $\mathcal{B} \otimes \mathcal{F}$; $\mathcal{B}$ is the Borel field of $[0, \infty)$ and $\mathcal{F}$ is (some extension of) $\sigma\left(X(u) : u \geq 0\right)$. Important stopping times are "hitting times", or times related to hitting times:

$$T = \inf\left\{s > 0 : X(s) \in E^{\triangle} \setminus U\right\},$$

$$\text{and } S = \inf\left\{s > 0 : \int_0^s 1_{E \setminus U}(X(u))du > 0\right\},$$

where U is some open (or Borel) subset of $E^{\triangle}$. This kind of stopping times have the extra advantage of being *terminal* stopping times, i.e. $t + S \circ \vartheta_t = S$ $\mathbf{P}_x$-almost surely on the event $\{S > t\}$. A similar statement holds for the *hitting time* T. The time S is called the *penetration time* of $E \setminus U$. Let $p : E \to [0,\infty)$ be a Borel measurable function. Stopping times of the form

$$S_\xi = \inf\left\{ s > 0 : \int_0^s p\big(X(u)\big)du > \xi \right\}$$

serve as a *stochastic time change*, because they enjoy the equality: $S_\xi + S_\eta \circ \vartheta_{S_\xi} = S_{\xi+\eta}$, $\mathbf{P}_x$-almost surely on the event $\{S_\xi < \infty\}$. As a consequence operators of the form $\mathcal{S}(\xi)f(x) := \mathbf{E}_x\left[f\left(X\left(S_\xi\right)\right)\right]$, f a bounded Borel function, possess the semigroup property. Also notice that $S_0 = 0$, provided that the function p is strictly positive.

Remark 8. Since a Feller semigroup possesses a generator, L say, one also says that L generates the associated strong Markov process. For example $\frac{1}{2}\Delta$ generates Brownian motion. This concept yields a direct relation between certain (lower order) pseudo-differential operators and probability theory: see Jacob [30]. The order has to be less than or equal to 2. This follows from the theory of Lévy processes and the Lévy-Khinchin formula, which decomposes a continuous negative-definite function into a linear term (probabilistically this corresponds to a deterministic drift), a quadratic term (this corresponds to a diffusion: a continuous Brownian motion-like process), and a term that corresponds to the jumps of the process (compound Poisson process, Lévy measure). Quite a number of problems in classical analysis can be reformulated in probabilistic terms. For more work on the connection between the martingale problem and (pseudo-)differential operators, the reader may consult e.g. papers by Hoh [28, 29], and papers by Mikulyavichyus and Pragarauskas [38, 39]. For a connection between the martingale problem and quadratic forms, see e.g. Albeverio and Röckner [3]

For instance for certain Dirichlet boundary value problems hitting times are appropriate, for certain initial value problems Markov process theory is relevant. For other problems the martingale approach is more to the point. For example there exists a one-to-one correspondence between the following concepts:

- **(i):** Unique (weak) solutions of stochastic differential equations in $\mathbf{R}^\nu$:
- **(ii):** Unique solutions to the corresponding martingale problem;
- **(iii):** Markovian diffusion semigroups in $\mathbf{R}^\nu$;
- **(iv):** Feller semigroups generated by certain second order differential operators of elliptic type.

(Regular) first order perturbations of second order elliptic differential operators can be studied using the Cameron-Martin-Girsanov transformation. Perturbations of order zero are treated via the Feynman-Kac formula.

Remark 9. In our discussion we started with (generators of) Feller semigroups. Another approach would be to begin with symmetric Dirichlet forms (quadratic form theory) in $L^2(E,m)$, where m is a Radon measure on the Borel field $\mathcal{E}$ of E. (By definition a Radon measure assigns finite values to compact subsets and it is inner and outer regular.) The reader may consult the books by Bouleau and Hirsch [11], by Fukushima, Oshima and Takeda, [25], or by Z. Ma and M. Röckner [35]. In the latter reference Ma and Röckner treat somewhat more general Dirichlet forms. These Dirichlet need not be symmetric, but they obey a certain *cone type* inequality:

$$|\mathcal{E}(f,g)|^2 \le K\mathcal{E}(f,f)\mathcal{E}(g,g), \quad f, \quad g \in D(\mathcal{E}).$$

Again one says that the Markov process is generated by (or associated to the Dirichlet form $\mathcal{E}$ or to the corresponding closed linear operator: $\mathcal{E}(f,g) = -\langle Lf, g\rangle$, $f \in D(L)$, $g \in D(\mathcal{E})$. (Note that only regular Dirichlet forms correspond to Markov processes.)

Problem 1. Is a result like Theorem 2 true if the locally compact space E is replaced with a Polish space, and if $C_b(E)$ (space of all bounded continuous functions on E) replaces $C_0(E)$? Instead of the topology of uniform convergence we consider the *strict topology*. This topology is generated by seminorms of the form: $f \mapsto \sup_{x\in E} |u(x)f(x)|$, $f \in C_b(E)$. The functions $u \ge 0$ have the property that for every $\alpha > 0$ the set $\{u \ge \alpha\}$ is compact (or is contained in a compact subset of E. The functions u need not be continuous. What about Markov uniqueness? Is there a relationship with work done by Eberle [20, 21]?

Is it possible to rephrase Theorem 2 for reciprocal Markov processes and diffusions? Martingales should then replaced with differences of forward and backward martingales. A stochastic process $(M(t) : t \ge 0)$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is called a *backward martingale* if $\mathbf{E}\left[M(t) \mid \mathcal{F}^s\right] = M(s)$, $\mathbf{P}$-almost surely, where $t < s$, and $\mathcal{F}^s$ is the σ-field generated by the information from the future: $\mathcal{F}^s = \sigma(X(u) : u \ge s\}$. Of course we assume that $M(t)$ belongs to $L^1(\Omega, \mathcal{F}, \mathbf{P})$, $t \ge 0$.

Let $(\Omega, \mathcal{F}, \mathbf{P})$ be a probability space. An E-valued process $(X(t) : 0 \le t \le 1)$ is called a *reciprocal Markov* process if for any $0 \le s < t \le 1$ and every pair of events $A \in \sigma(X(\tau) : \tau \in (s,t))$, $B \in \sigma(X(\tau) : \tau \in [0,s] \cup [t,1])$ the equality

$$\mathbf{P}\left[A \cap B \mid X(s), X(t)\right] = \mathbf{P}\left[A \mid X(s), X(t)\right] \mathbf{P}\left[B \mid X(s), X(t)\right] \tag{1.37}$$

is valid. By $\mathcal{D}$ we denote the set

$$\mathcal{D} = \{(s,x,t,B,u,z) : (x,z) \in E \times E,\ 0 \le s < t < u \le 1,\ B \in \mathcal{E}\}. \tag{1.38}$$

A function $P : \mathcal{D} \to [0,\infty)$ is called a *reciprocal probability distribution* or a *Bernstein probability* if the following conditions are satisfied:

(i) the mapping $B \mapsto P(s,x,t,B,u,z)$ is probability measure on $Ecal$ for any $(x,z) \in E \times E$ and for any $0 \le s < t < u \le 1$;
(ii) the function $(x,z) \mapsto P(s,x,t,B,u,z)$ is $\mathcal{E} \otimes Ecal$-measurable for any $0 \le s < t < u \le 1$;
(iii) For every pair $(C,D) \in \mathcal{E} \otimes E$, $(x,y) \in E \times E$, and for all $0 \le s < t < u \le 1$ the equality

$$\int_D P(s,x,u,d\xi,v,y)\, P(s,x,t,C,u,\xi) = \int_C P(s,x,t,d\eta,v,y)\, P(t,\eta,u,C,v,y)$$

is valid.

Then the following theorem is valid for $E = \mathbf{R}^\nu$ (see Jamison [31]).

7. Theorem. *Let $P(s, x, t, B, u, y)$ be a reciprocal transition probability function and let μ be a probability measure on $\mathcal{E} \otimes \mathcal{E}$. Then there exists a unique probability measure $\mathbf{P}_\mu$ on $\mathcal{F}$ with the following properties:*

(1) *With respect to $\mathbf{P}_\mu$ the process $(X(t) : 0 \le t \le 1)$ is reciprocal;*
(2) *For all $(A, B) \in \mathcal{E} \otimes \mathcal{E}$ the equality $\mathbf{P}_\mu [X_0 \in A,\ X_1 \in B] = \mu (A \times B)$ is valid;*
(3) *For every $0 \le s < t < u \le 1$ and for every $A \in \mathcal{E}$ the equality*

$$\mathbf{P}_\mu \left[X(t) \in A \;\middle|\; X(s), X(u) \right] = P\left(s, X(s), t, A, u, X(u)\right) \quad \textit{is valid.}$$

For more details see Thieullen [45] and [46].

What is the equivalent of all this in the non-commutative setting? Notice that it is possible to define a strict topology on C^*-algebras: see *Problem 15*.

We notice that a positive solution to Problem 1 would make a link with work by Dorroh and Neuberger [19] in as much that the Lie generator will also be the generator of a Markov process.

Problem 2. How useful is the martingale result on operators with the Korovkin property? We don't have a good example or application. Is there a non-commutative version of the Korovkin property?

2. Feynman-Kac semigroups: 0-order regular perturbations

In the present section we will quote one central theorem from [18]. We assume that $V : E \to [-\infty, \infty]$ is a Kato-Feller potential function in the following sense:

$$\lim_{t \downarrow 0} \sup_{x \in E} \int_0^t d\tau \mathbf{E}_x \left(V_-\big(X(\tau)\big) \right) = 0; \tag{2.1}$$

$$\lim_{t \downarrow 0} \sup_{x \in E} \int_0^t d\tau \mathbf{E}_x \left(1_K\big(X(\tau)\big) V_+\big(X(\tau)\big) \right) = 0$$

for all compact subsets K of E, We note that a Kato-Feller potential function belongs to $L^1_{\mathrm{loc}}(E, m)$. Furthermore we consider Σ: (large) open subset of E, with complement Γ. The symbol S denotes the *penetration time* of Γ:

$$S = \inf \left\{ s > 0 : \int_0^s 1_\Gamma(X(u)) du > 0 \right\} \tag{2.2}$$

It is a blanket assumption that $p_0(t, x, y) = p_0(t, y, x)$ (symmetry), and that the function $(t, x, y) \mapsto p_0(t, x, y)$ is continuous on $(0, \infty) \times E \times E$. In addition $dx = dm(x)$ is a *reference measure* on E.

The operator $-K_0$ is supposed to generate a symmetric strong Markov process

$$\left\{ (\Omega, \mathcal{F}, \mathbf{P}_x), (X(t) : t \ge 0), (\vartheta_t : t \ge 0), (E, \mathcal{E}) \right\}.$$

So the law $\mathbf{P}_x (X(t) \in B)$ is given by

$$\mathbf{P}_x (X(t) \in B) = \textstyle\int_B p_0(t, x, y) dm(y).$$

8. Definition. The *pinned measure* $\mu_{0,x}^{t,y}$ on $\mathcal{F}_{t-}$ is defined as follows:

$$\mu_{0,x}^{t,y}(A) = \mathbf{E}_x\left[p_0(t-s, X(s), y)1_A\right],$$

where $A \in \mathcal{F}_s$, $s < t$. Extend this pre-measure as a genuine measure on $\mathcal{F}_{t-}$ and notice that the process $s \mapsto p_0(t-s, X(s), y)$ is a martingale. The measure $\mu_{0,x}^{t,y}$ lives on the event $\{X(0) = x,\ X(t-) = y\}$.

9. Theorem. (a): *There exist a closed densely defined linear operator $K_0 \dot{+} V$ extending $K_0 + V$, which generates a positivity preserving (self-adjoint) semigroup in $L^2(E, m)$, denoted by $\left\{\exp\left(-t\left(K_0 \dot{+} V\right)\right) : t \geq 0\right\}$. This semigroup is given by the Feynman-Kac formula ($f \in L^2(E, m)$):*

$$\left[\exp\left(-t\left(K_0 \dot{+} V\right)\right) f\right](x) = \mathbf{E}_x\left[\exp\left(-\int_0^t V(X(u))du\right) f(X(t))\right]. \tag{2.3}$$

(b): *Every operator* $\exp\left(-t\left(K_0 \dot{+} V\right)\right)$ *is an integral operator with a continuous, symmetric integral kernel* $\exp\left(-t\left(K_0 + V\right)\right)(x, y)$ *given by*

$$\begin{aligned}
&\exp\left(-t\left(K_0 + V\right)\right)(x, y) \\
&= \lim_{s \uparrow t} \mathbf{E}_x\left[\exp\left(-\int_0^s V(X(u))du\right) p_0(t-s, X(s), y)\right] &(2.4)\\
&= \int \exp\left(-\int_0^t V(X(u))du\right) d\mu_{0,x}^{t,y}. &(2.5)
\end{aligned}$$

The measure $\mu_{0,x}^{t,y}$ is defined on the σ-field $\sigma\left(X(u) : u < t\right)$, and as usual can be extended on some completion of this σ-field. It is determined by

$$\mu_{0,x}^{t,y}(A) = \mathbf{E}_x\left[1_A p_0(t-s, X(s), y)\right],$$

where the event A belongs to $\mathcal{F}_s = \sigma\left(X(u) : u \leq s\right)$, for $s < t$. Since the process $s \mapsto p_0(t-s, X(s), y)$ is a $\mathbf{P}_x$-martingale on the interval $0 \leq s < t$, it follows that the quantity $\mu_{0,x}^{t,y}(A)$ is well-defined: its value does not depend on s, as long as A belongs to $\mathcal{F}_s$ and $s < t$.

(c): *The quadratic form (generalized Schrödinger form) $\mathcal{E}^V$ associated with the above Feynman-Kac semigroup is given by*

$$\mathcal{E}^V(f, g) = \left\langle \sqrt{K_0} f, \sqrt{K_0} g \right\rangle + \left\langle \sqrt{V_+} f, \sqrt{V_+} g \right\rangle - \left\langle \sqrt{V_-} f, \sqrt{V_-} g \right\rangle, \tag{2.6}$$

for f, g members of

$$D\left(\sqrt{K_0}\right) \cap \left\{ f \in L^2(E, m),\ \int V_+(x) \left|f(x)\right|^2 dm(x) < \infty \right\}. \tag{2.7}$$

For a proof we refer the interested reader to Chapter 2 in [18].

3. Dirichlet semigroups: 0-order singular perturbations, harmonic functions

In this section we present some results concerning Feynman-Kac semigroups, but with infintely high potentials in certain regions (obstacles, potential barriers): the repulsive part of the potentials takes the value ∞ in such regions.

10. Definition. The *Dirichlet semigroup* $\left\{\exp\left(-\left(K_0\dot{+}V\right)_\Sigma\right) : t \geq 0\right\}$ is defined by

$$\exp\left(-t\left(K_0\dot{+}V\right)_\Sigma\right) f(x) = \mathbf{E}_x\left[\exp\left(-\int_0^t V(X(u))du\right) f(X(t)) : S > t\right]. \tag{3.1}$$

The V-*harmonic extension* operator H_Σ^V is defined by

$$H_\Sigma^V f(x) = \mathbf{E}_x\left(\exp\left(-\int_0^S V(X(u))du\right) f(X(S)) : S < \infty\right). \tag{3.2}$$

We have $H_\Sigma^{a+V} = a\int_0^\infty e^{-as} T_\Sigma^V(s)ds$, where

$$T_\Sigma^V(s)f(x) = \mathbf{E}_x\left[\exp\left(-\int_0^S V(X(u))du\right) f(X(S)) : S < s\right]. \tag{3.3}$$

11. Theorem. *The operator*

$$\left(aI + K_0\dot{+}V\right)^{1/2} H_\Sigma^{a+V}\left(aI + K_0\dot{+}V\right)^{-1/2} \tag{3.4}$$

extends to an orthogonal projection and Dynkin's formula

$$\left(aI + K_0\dot{+}V\right)^{-1} = J^*\left(aI + \left(K_0\dot{+}V\right)_\Sigma\right)^{-1} J + H_\Sigma^{a+V}\left(aI + K_0\dot{+}V\right)^{-1} \tag{3.5}$$

is valid.

At the level of integral kernels, the latter formula reads:
Newton potential = Green function + Harmonic correction. We also have

$$D\left(\mathcal{E}^{a+V}\right) = \overline{D\left(\left(aI + K_0\dot{+}V\right)_\Sigma\right) \oplus_{\mathcal{E}^{a+V}} R\left(H_\Sigma^{a+V}\right)}^{\mathcal{E}^{a+V}} : \tag{3.6}$$

H_Σ^{a+V} is an orthogonal projection in $D\left(\mathcal{E}^{a+V}\right)$, endowed with the canonical quadratic form $\mathcal{E}^{a+V}$ corresponding to the Feynman-Kac semigroup corresponds, and which is given by

$$\mathcal{E}^{a+V}(f,g) = \left\langle K_0^{1/2}f, K_0^{1/2}g\right\rangle + \left\langle V_+^{1/2}f, V_+^{1/2}g\right\rangle + a\langle f,g\rangle - \left\langle V_-^{1/2}f, V_-^{1/2}g\right\rangle,$$

where f and g belong to $D(K_0^{1/2}) \cap D(V_+^{1/2})$. A proof of Dynkin's formula can be found in Chapter 2 of [18], and the assertion about the projection H_Σ^{a+V} is proved in detail in [53].

Put

$$h_\Sigma^{a+V}(x) = H_\Sigma^{a+V}1(x) = \mathbf{E}_x\left[\exp\left(-\int_0^S (a + V(X(u)))\,du\right) : S < \infty\right].$$

12. Definition. The set $E \setminus \Sigma$ has *finite* $a+V$-*capacity* if $\int h_\Sigma^{a+V}(x)dx$ is finite. Note:

$$\int h_\Sigma^{a+V}(x)dx = \inf\left\{\mathcal{E}^{a+V}(u,u) : u \in D\left(\mathcal{E}^{a+V}\right),\ u \geq 1_\Gamma\right\}. \tag{3.7}$$

The following theorem appears in [54] as Theorem 3.1.

13. Theorem. *Let $\{E_0(\xi):\xi\in\mathbf{R}\}$ and $\{E_1(\xi):\xi\in\mathbf{R}\}$ be the spectral decompositions corresponding to $K_0\dot{+}V$ and to $K_0\dot{+}W$, respectively. Let*

$$\{(\Omega,\mathcal{F},\mathbf{P}_x),(X(t):t\geq 0),(\vartheta_t:t\geq 0),(E,\mathcal{E})\}$$

be the strong Markov process generated by $-K_0$. Suppose that, for some $t_0>0$, the function $\exp(-t_0K_0)|W-V|$ is bounded, or suppose that

$$\lim_{t\downarrow 0}\sup_{x\in E}\mathbf{E}_x\left[\left(\int_0^t (W(X(u))-V(x(u)))\,du\right)^2\right]=0. \tag{3.8}$$

The following assertions are equivalent:

- **(i):** *For every bounded interval A the operator $E_0(A)(W-V)E_1(A)$ is compact;*
- **(ii):** *For some $t>0$ (for all $t>0$) the operator*

  $$\exp\left(-t\left(K_0\dot{+}V\right)\right)(W-V)\exp\left(-t\left(K_0\dot{+}W\right)\right)$$

  is compact;
- **(iii):** *For some $t>0$ (for all $t>0$), the operator $D(t)$ is compact.*

Remark 1. If $\lim_{t\downarrow 0}\sup_{x\in E}\int_0^t\left[\exp(-sK_0)|W-V|\right](x)ds=0$, then

$$\lim_{t\downarrow 0}\sup_{x\in E}\mathbf{E}_x\left[\left(\int_0^t (W(X(u))-V(x(u)))\,du\right)^2\right]=0. \tag{3.9}$$

This is a consequence of the Markov property.

Remark 2. An equality like (3.9) can probably be used for first order perturbations, where the Cameron-Martin formula is applicable. In such a case we probably have to deal with stochastic integrals instead of the process

$$t\mapsto\int_0^t (W(X(u))-V(X(u)))\,du.$$

Remark 3. Theorem 13 is probably not known, even in case we consider $K_0=H_0=-\frac{1}{2}\Delta$. So the corresponding process is Brownian motion.

Remark 4. We introduced Brownian motion as a Markov process with a certain transition function. It can also be introduced as a ν-dimensional Gaussian process $\{X(t):t\geq 0\}$ such that $\mathbf{E}\left[\langle X(t),X(s)\rangle\right]=\nu\min(s,t)$, or as a Lèvy process with negative definite function $\xi\mapsto\frac{1}{2}|\xi|^2$, or as a martingale with variation process $t\mapsto t$.

Remark 5. In the implication (iii) $\Rightarrow$ (ii) the following identity is relevant:

$$\begin{aligned} &t\exp\left(-\frac{t}{2}\left(K_0\dot{+}V\right)\right)(W-V)\exp\left(-\frac{t}{2}\left(K_0\dot{+}W\right)\right) \qquad (3.10)\\ &=\frac{\pi}{2}\int_{-\infty}^{\infty}\frac{1}{(\cosh(\pi\tau))^2}\exp\left(i\tau t\left(K_0\dot{+}V\right)\right)\mathcal{D}(t)(W-V)\exp\left(-i\tau t\left(K_0\dot{+}W\right)\right)d\tau,\end{aligned}$$

$$\text{where}\quad \mathcal{D}(t)T=\int_0^t\exp\left(-u\left(K_0\dot{+}V\right)\right)T\exp\left(-(t-u)\left(K_0\dot{+}W\right)\right)du. \tag{3.11}$$

The following result is applicable for

$$M(t)=\exp\left(-\int_0^t V(X(u))du\right) \text{ or } M(t)=\exp\left(-\int_0^t V(X(u))du\right)1_{\{S>t\}},$$

where V is a Kato-Feller potential, and where S is a terminal stopping times, i.e. $t+S\circ\vartheta_t=S$ $\mathbf{P}_x$-almost surely on the event $\{S>t\}$. Theorem 14 proves part of Theorem 9.

The proof of the following result can be found in [53].

14. Theorem. *Let $\{M(t):t\geq 0\}$ be a multiplicative process taking its values in $[0,\infty)$. This means that, for every $t\geq 0$, $M(t):\Omega\to[0,\infty)$ is $\mathcal{F}_t$-measurable and that $M(s+t)=M(s)M(t)\circ\vartheta_s$ for all s and $t\geq 0$. Assume*

$$\lim_{\epsilon\downarrow 0}\int M(t-\epsilon)d\mu_{0,x}^{t,y}=\int M(t)d\mu_{0,x}^{t,y}.$$

As above, the defining property of $\mu_{0,x}^{t,y}$ is the equality

$$\int F d\mu_{0,x}^{t,y}=\mathbf{E}_x\left[Fp_0(t-s,X(s),y)\right],$$

where $F:\Omega\to\mathbf{R}$ is bounded and $\mathcal{F}_s$-measurable ($s<t$). The following assertions are valid:

(1) *The process*

$$s\mapsto M(s)\int M(t-s)d\mu_{0,X(s)}^{t-s,y}$$

is a $\mathbf{P}_x$-martingale on the interval $[0,t)$.

(2) *The following equality is valid:*

$$\mathbf{E}_x\left[M(t)f(X(t))\right]=\int\int M(t)d\mu_{0,x}^{t,y}f(y)dy, \tag{3.12}$$

where f is greater than or equal to zero and Borel measurable.

(3) *The following Chapman-Kolmogorov identity is valid:*

$$\int\int M(t_1)d\mu_{0,x}^{t_1,z}\int M(t_2)d\mu_{0,z}^{t_2,y}dz=\int M\left(t_1+t_2\right)d\mu_{0,x}^{t_1+t_2,y}. \tag{3.13}$$

Problem 3. Let $M(t)$ be as in Theorem 14. Define the semigroup $\exp\left(-tK_M\right)$ by

$$\left[\exp\left(-tK_M\right)f\right](x)=\int\exp\left(-tK_M\right)(x,y)f(y)dy,$$

where $\exp\left(-tK_M\right)(x,y)=\int M(t)d\mu_{0,x}^{t,y}$. Suppose that the operators $\exp\left(-tK_M\right)$, $t>0$, are self-adjoint. Then, formally,

$$\begin{aligned}&\int\left[\exp\left(-isK_M\right)\exp\left(-tK_M\right)(\cdot,y)\right](x)f(y)dy\\ &=\left[\exp\left(-isK_M\right)\int\exp\left(-tK_M\right)(\cdot,y)f(y)dy\right](x)\\ &=\left[\exp\left(-isK_M\right)\exp\left(-tK_M\right)f\right](x)=\left[\exp\left(-(t+is)K_M\right)f\right](x).\end{aligned}$$

In what sense do we have convergence of $[\exp(-isK_M)\exp(-tK_M)(\cdot,y)](x)$ to $\exp(-isK_M)(x,y)$, if t tends to 0 downward?

Remark 1. In relation to the previous problem, we like to point out that Zambrini and coworkers [60, 2, 47, 48] have kind of a transition scheme to go from classical stochastic calculus (with non-reversible processes) to physical real time (reversibile) quantum mechanics and vice versa. An important tool in this connection is the so-called Noether theorem. In fact, in Zambrini's words, reference [60] contains the first concrete application of this theorem. In [60] the author formulates a theorem like Theorem 15 below, he also uses so-called "Bernstein diffusions" (see e.g. [13]) for the "Euclidean Born interpretation" of quantum mechanics. The Bernstein diffusions are related to solutions of $\left(\frac{\partial}{\partial t} - \left(K_0 \dot{+} V\right)\right)\eta = 0$, and of $\left(\frac{\partial}{\partial t} + K_0 \dot{+} V\right)\eta^* = 0$. It would be nice to formulate the stochastic Noether theorem (Theorem 2.4 in [60]) in terms of the carré du champ operator and ideas from stochastic control.

In relation to the previous remark we have the following result for generators of diffusions: see Remark 2 following Theorem 27. For the notion of the squared gradient operator (carré du champ operateur) see equality (5.10). The operator K_0 acts on the first variable and so does the squared gradient operator Γ_1.

15. Theorem. *Let $\chi : E \to [0,\infty]$ be a function such that $\mathbf{E}_{x,t}\left[\left|\log\chi\left(X(u),u\right)\right|\right]$ is finite. Here $u > 0$ is a fixed time and $\{(\Omega,\mathcal{F},\mathbf{P}_x), (X(t):t\geq 0), (\vartheta_t : t \geq 0), (E,\mathcal{E})\}$ is the strong Markov process generated by $-K_0$. Let S_L be a solution to the following Riccati type equation. (This equation is called the Hamilton-Jacobi-Bellman equation.) For $t \leq s \leq u$ and $x \in E$ the following identity is true:*

$$\begin{cases} -\dfrac{\partial S_L}{\partial s}(x,s) + \dfrac{1}{2}\Gamma_1\left(S_L,S_L\right)(x,s) + K_0 S_L(x,s) - V(x,s) = 0; & \\ S_L(x,u) = -\log\chi(x,u), & x \in E. \end{cases} \tag{3.14}$$

Then for any real valued $v \in D(K_0)$ the following inequality is valid:

$$S_L(x,t) \leq \mathbf{E}_{x,t}^{M_{v,t}}\left[\int_t^u \left(\frac{1}{2}\Gamma_1(v,v) + V\right)(X(\tau),\tau)d\tau\right] - \mathbf{E}_{x,t}^{M_{v,t}}\left[\chi\left(X(u),u\right)\right], \tag{3.15}$$

and equality is attained for the "action" $v = S_L$.

By definition $E_{x,t}[Y]$ is the expectation, conditoned at $X(t) = x$, of the random variable Y which is measurable with respect to the information from the future: i.e. with respect to $\sigma\{X(s) : s \geq t\}$. The measure $\mathbf{P}_{x,t}^{M_{v,t}}$ is defined in equality (3.17) below. Put $\eta_\chi = \exp(-S_L)$, where S_L satisfies (3.14). From (5.21) it follows that $\left(\frac{\partial}{\partial t} - \left(K_0 \dot{+} V\right)\right)\eta_\chi = 0$, provided that $K_0 1$ is interpreted as 0, i.e. $\int K_0 f\, dm = 0$ for all $f \in D(K_0)$.

Fix a function $v : E \times \mathbf{R} \to \mathbf{R}$ in $D(K_0 - D_1)$, where $D_1 = \frac{\partial}{\partial t}$ is differentiation with respect to t. Let the process

$$\{(\Omega,\mathcal{F},\mathbf{P}_{x,t}), ((q_v(t),t) : t \geq 0), (\vartheta_t : t \geq 0), (E \times \mathbf{R}, \mathcal{E} \otimes \mathcal{B})\}$$

be the Markov process generated by the operator $-K_v + D_1$, where K_v is defined by $K_v(f)(x,t) = K_0 f(x,t) + \Gamma_1(v,f)(x,t)$. Here, $\mathcal{B}$ denotes the Borel field of $\mathbf{R}$, and by $\Gamma_1(v,f)(x,t)$ we mean

$$\Gamma_1(v,f)(x,t) = \lim_{s\downarrow 0} \frac{1}{s} \mathbf{E}_x \left[\left(v(X(s),t) - v(X(0),t) \right) \left(f(X(s),t) - f(X(0),t) \right) \right]. \tag{3.16}$$

We also believe that the following version of the Cameron-Martin formula is valid. For all finite n-tuples $t_1, \dots, t_n$ in $(0,\infty)$ the identity (3.18 is valid:

$$\mathbf{E}_{x,t}^{M_{v,t}} \left[\prod_{j=1}^{n} f_j \left(X\left(t_j + t\right), t_j + t \right) \right] \tag{3.17}$$

$$= \mathbf{E}_{x,t} \left[\exp\left(-\frac{1}{2} \int_t^u \Gamma_1(v,v)(X(\tau),\tau)\, d\tau - M_{v,t}(u) \right) \prod_{j=1}^{n} f_j \left(X\left(t_j + t\right), t_j + t \right) \right]$$

$$= \mathbf{E}_{x,t} \left[\prod_{j=1}^{n} f_j \left(q_v\left(t_j + t\right), t_j + t \right) \right] \tag{3.18}$$

where the $\mathbf{E}_{x,t}$-martingale $M_{v,t}(s)$, $s \geq t$, is given by

$$M_{v,t}(s) = v\left(X(s),s\right) - v\left(X(t),t\right) + \int_t^s \left(-\frac{\partial}{\partial \tau} + K_0 \right) v\left(X(\tau),\tau\right) d\tau. \tag{3.19}$$

Its quadratic variation part $\langle M_{v,t} \rangle(s) := \langle M_{v,t}, M_{v,t} \rangle(s)$ is given by

$$\langle M_{v,t} \rangle(s) = \int_t^s \Gamma_1(v,v)\left(X(\tau),\tau\right) d\tau. \tag{3.20}$$

The equality in (3.17) serves as a definition of the measure $\mathbf{P}_{x,t}^{M_{v,t}}(\cdot)$, and the equality in (3.18) is a statement. We notice that the following processes are $\mathbf{P}_{x,t}$ martingales on the interval $[t,u]$:

$$\exp\left(-\frac{1}{2} \langle M_{v,t} \rangle(s) - M_{v,t}(s) \right) \quad \text{and} \tag{3.21}$$

$$\exp\left(-\frac{1}{2} \langle M_{v,t} \rangle(s) - M_{v,t}(s) \right) \left(\langle M_{v,t} \rangle(s) + M_{v,t}(s) \right). \tag{3.22}$$

Proof. The proof is based on the following version of Jensen inquality and should be compared with the arguments in Zambrini [60], who used ideas from Fleming and Soner: see Chapter VI in [24]. The inequality we have in mind is the following one:

$$-\log \mathbf{E}_{x,t}^{M_{v,t}} \left[\exp(-\varphi) \right] \leq \mathbf{E}_{x,t}^{M_{v,t}} \left[\varphi \right], \tag{3.23}$$

with equality only if φ is constant $\mathbf{P}_{x,t}$-almost surely. We apply 3.23 for the stochastic variable $\varphi = \varphi_v$, given by

$$\varphi_v = -\int_t^u \left[\frac{1}{2}\Gamma_1(v,v) + V\right](X(\tau),\tau)\,d\tau - M_{v,t}(u) - \log\chi(X(u)). \tag{3.24}$$

By the Jensen inequality we have

$$\mathbf{E}_{x,t}^{M_{v,t}}\left[\frac{1}{2}\langle M_{v,t}\rangle(u) + \int_t^u V(X(\tau),\tau)\,d\tau - \log\chi(X(u),u)\right]$$

(the process in (3.22) is a $= \mathbf{P}_{x,t}^{M_{v,t}}$-martingale)

$$\mathbf{E}_{x,t}^{M_{v,t}}\left[-\frac{1}{2}\langle M_{v,t}\rangle(u) - M_{v,t}(u) + \int_t^u V(X(\tau),\tau)\,d\tau - \log\chi(X(u),u)\right] \tag{3.25}$$

(here we apply Jensen inequality)

$$\geq -\log\mathbf{E}_{x,t}^{M_{v,t}}\left[\exp\left(\frac{1}{2}\langle M_{v,t}\rangle(u) + M_{v,t}(u) - \int_t^u V(X(\tau),\tau)\,d\tau + \log\chi(X(u),u)\right)\right] \tag{3.26}$$

(definition of the probability measure $\mathbf{E}_{x,t}^{M_{v,t}}$)

$$= -\log\mathbf{E}_{x,t}\left[\exp\left(-\int_t^u V(X(\tau),\tau)\,d\tau + \log\chi(X(u),u)\right)\right]$$

(the function $S_L(y,s)$ obeys the Hamilton-Jacobi-Bellmann equation (3.14))

$$\begin{aligned} &= -\log\mathbf{E}_{x,t}\Big[\exp\left(-\frac{1}{2}\int_t^u \Gamma_1(S_L,S_L)(X(\tau),\tau)\,d\tau - M_{S_L,t}(u)\right) \\ &\qquad \times\exp\left(-S_L(X(t),t) + S_L(X(u),u) + \log\chi(X(u),u)\right)\Big] \end{aligned}$$

(the quadratic variation process $\langle M_{S_L,t}\rangle(s)$ is given by $\int_t^s \Gamma_1(S_L,S_L)(X(\tau),\tau)\,d\tau$)

$$\begin{aligned} &= -\log\mathbf{E}_{x,t}\Big[\exp\left(-\frac{1}{2}\langle M_{S_L,t}\rangle(u) - M_{S_L,t}(u)\right) \\ &\qquad \times\exp\left(-S_L(X(t),t) + S_L(X(u),u) + \log\chi(X(u),u)\right)\Big] \qquad (3.27)\\ &= S_L(x,t). \end{aligned}$$

Since we have $X(t) = x$, $\mathbf{P}_{x,t}$-almost surely, since $S_L(X(u),u) + \log\chi(X(u),u) = 0$, and since the process $\exp\left(-\frac{1}{2}\langle M_{v,t}\rangle(u) - M_{v,t}(u)\right)$ is a $\mathbf{P}_{x,t}$-martingale, we see that the expression in (3.27) is equal to $S_L(x,t)$. This proves the inequality part of the theorem. If $v = S_L$, then the Hamilton-Jacobi-Bellmann equation implies that the expression in (3.25) equals $S_L(x,t)$. Here we employ again the identity $\langle M_{S_L,t}\rangle(s) = \int_t^s \Gamma_1(S_L,S_L)(X(\tau),\tau)\,d\tau$. Altogether this proves Theorem 15. $\square$

Problem 4. Prove Theorem 15 for viscosity solutions of the equation

$$H\left(x, s_1, s_2, v(x,\cdot), \left(\Gamma_1\left(v,v\right)\left(x,\cdot\right)\right)^{1/2}, -K_0 v\left(x,\cdot\right)\right) = 0, \tag{3.28}$$

where the function $H\left(x, s_1, s_2, v(\cdot), p(\cdot), M(\cdot)\right)$ is defined by

$$\begin{aligned} &H\left(x, s_1, s_2, v(\cdot), p(\cdot), M(\cdot)\right) \\ &= v(s_1) - v(s_2) + \frac{1}{2}\int_{s_1}^{s_2} p(s)^2 ds - \int_{s_1}^{s_2} M(s) ds - \int_{s_1}^{s_2} V(x,s) ds. \end{aligned} \tag{3.29}$$

Here a *viscosity solution* is defined as a function S_L for which

$$H\left(x_0, s_1, s_2, \varphi\left(x_0,\cdot\right), \left(\Gamma_1\left(\varphi,\varphi\right)\left(x_0,\cdot\right)\right)^{1/2}, -K_0\varphi\left(x,\cdot\right)\right) \geq 0, \tag{3.30}$$

whenever φ belongs to $D\left(K_0\right)$ and possesses the property that

$$S_L\left(x_0,s\right) - \varphi\left(x_0,s\right) \leq S_L\left(x,s\right) - \varphi\left(x,s\right), \quad \text{for all} \quad s_1 \leq s \leq s_2, \tag{3.31}$$

and for all x in a neighborhood of x_0 (or for all $x \in E$). Moreover, if the inequality in (3.31) is reversed, then the one in (3.30) should also be reversed.

Formulate and prove a stochastic Noether theorem in terms of the squared gradient operator: see Zambrini [60]. It should read something like what follows ($(x,s) \in E \times [t,u]$, $\dot{h}$ denotes the time derivative of h).

16. Theorem. *Suppose that the functions h, $S_L : E \times \mathbf{R} \to \mathbf{C}$, $T : [0,\infty) \to [0,\infty)$ satisfy*

$$\begin{aligned} &\dot{h}(x.s) - \dot{h}(x,t) - K_0 h(x,s) + K_0 h(x,t) \\ &= \int_t^s \Gamma_1(h,V)(x,\sigma) d\sigma + V(x,s)T(s) - V(x,t)T(t); \end{aligned} \tag{3.32}$$

$$\dot{S}_L(x,s) = K_0 S_L(x,s) + \frac{1}{2}\Gamma_1\left(S_L,S_L\right)(x,s) - V(x,s).$$

Then the proccess

$$\left[\Gamma_1\left(S_L,h\right) + \dot{S}_L T + \dot{h}\right]\left(q_{S_L}(s), s\right), \quad t \leq s \leq u, \tag{3.33}$$

is a $\mathbf{E}_{x,t}$-martingale. Under suitable conditions on the function $v : E \times [t,u] \to \mathbf{C}$ the process

$$\left[\Gamma_1\left(v,h\right) + \left(K_0 v + \frac{1}{2}\Gamma_1(v,v) - V\right)T + \dot{h}\right]\left(q_v(s), s\right), \quad t \leq s \leq u, \tag{3.34}$$

is a $\mathbf{E}_{x,t}$-martingale as well.

Remark 2. There is also a connection with work by Albeverio, Johnson and Ma [1], and Lim [34] about Feynman operational calculus for Kato-Feller potentials. In her work Lim extends the Feynman operational calculus to so-called *smooth* Kato-Feller measures. In [32] the authors, G.W. Johnson and M.L. Lapidus treat the Feynman Calculus in great length.

17. Corollary. *Let $-K_M$ be the generator of the semigroup $(\exp(-tK_M))$ defined as in Theorem 14. The following processes are martingales:*

$$\tau \mapsto M(\tau)\exp\left(-(t-\tau)K_M\right)(X(\tau),y),\ 0 \le \tau < t; \tag{3.35}$$

$$M(\tau)f\big(X(\tau)\big) - f(X(0)) + \int_0^\tau M(v)K_M f\big(X(v)\big)dv. \tag{3.36}$$

18. Corollary. *Let S be a terminal stopping time, and put*

$$\Sigma = \{x \in E : \mathbf{P}_x[S=0]=0\}, \quad \Gamma = E \setminus \Sigma; \tag{3.37}$$

$$\exp\left(-t\left(K_M\right)_S\right)f(x) = \mathbf{E}_x\left[M(t)f(X(t)) : S > t\right]; \tag{3.38}$$

$$H_S^M f(x) = \mathbf{E}_x\left[M(S)f(X(S)) : S < \infty\right]; \tag{3.39}$$

$$D_S(t) = \exp\left(-tK_M\right) - J_S^* \exp\left(-t\left(K_M\right)_S\right)J_S; \tag{3.40}$$

$$\mathcal{D}_S(t)T = \int_0^t \exp\left(-u\left(K_M\right)_S\right)T\exp\left(-(t-u)K_M\right)du. \tag{3.41}$$

Here $J_S f = f\big|_\Sigma$ and J_S^* is its adjoint. Then (singular Duhamel's formula)

$$\begin{aligned} D_S(t) - 1_\Gamma \exp\left(-tK_M\right) &= J_S^*\mathcal{D}_S(t)\left(K_M\right)_S J_S H_S^M \\ &= 1_\Sigma H_S^M \exp\left(-tK_M\right) - J_S^*\exp\left(-t\left(K_M\right)_S\right)J_S H_S^M + J_S^*\mathcal{D}_S(t)H_S^M K_M. \end{aligned} \tag{3.42}$$

Khas'minskii's lemma is available for multiplicative processes.

19. Theorem. *(Khas'minskii's Lemma) Let $W : E \to [0,\infty]$ be a Borel measurable function. Put $\gamma = \lim_{t\downarrow 0}\sup_{x\in E}\mathbf{E}_x\left[\int_0^t W(X(s))ds\right]$, and suppose $\gamma < 1$. The following assertions are true:*

(1) $\gamma = \lim_{a\to\infty}\sup_{x\in E}\left(aI + K_0\right)^{-1}W(x)$.

(2) *Choose $t_0 > 0$ in such a way that $\alpha := \sup_{x\in E}\mathbf{E}_x\left[\int_0^{t_0} W(X(s))ds\right] < 1$. Then*

$$\sup_{x\in E}\mathbf{E}_x\left[\exp\left(\int_0^{t_0} W(X(s))ds\right)\right] \le \frac{1}{1-\alpha}.$$

(3) *Let t_0 and α be as in (2). Put $M = \dfrac{1}{1-\alpha}$ and $e^b = \left(\dfrac{1}{1-\alpha}\right)^{1/t_0}$. Then, for $x \in E$ and $t \ge 0$,*

$$\mathbf{E}_x\left[\exp\left(\int_0^t W(X(s))ds\right)\right] \le M\exp(bt).$$

(4) *Let $\{M(t) : t \ge 0\}$ be a multiplicative functional attaining values in $[0,\infty]$. Suppose that*

$$\inf_{\eta>0}\inf_{t>0}\eta\sup_{y\in E}\mathbf{P}_y\left[\sup_{0\le s\le t}M(s) \ge \eta\right] < 1.$$

Then there exist constants M and b such that

$$\mathbf{E}_x\left[M(t)\right] \le Me^{bt}, \quad x \in E, \quad t \ge 0.$$

The proof of (4) can be based on stopping times of the form

$$T_\xi = \inf\left\{t > 0 : \sup_{0\le s\le t} M(s) > e^{\xi}\right\}.$$

Then $T_\xi + T_\eta \circ \vartheta_{T_\xi} = T_{\xi+\eta}$.

4. Sets of finite capacity, wave operators, and related results

In the present section we collect without proof some of the results obtained in [18], where the proofs can be found as well. The operator $J = J_\Sigma$ denotes the restriction operator: $Jf = f\restriction_\Sigma$. Its adjoint J^* extends a function f, defined on Σ with 0 on $E \setminus \Sigma$.

20. Theorem. *If $\int h_\Sigma^{a+V}(x)^2 dm(x) < \infty$ for some $a > 0$, then the semigroup difference*

$$D_\Sigma(t) := \exp\left(-t\left(K_0 \dot{+} V\right)\right) - J^*\exp\left(-t\left(K_0\dot{+}V\right)_\Sigma\right) J, \quad t > 0,$$

consists of Hilbert-Schmidt operators. Here $Jf = f\restriction_\Sigma$.

Motivation. By Weyl's theorem we get: if $D_\Sigma(t)$ is Hilbert-Schmidt, then

$$\sigma_{\text{ess}}\left(K_0\dot{+}V\right) = \sigma_{\text{ess}}\left(\left(K_0\dot{+}V\right)_\Sigma\right).$$

21. Theorem. *If $\int h_\Sigma^{a+V}(x)^{1/2} dm(x) < \infty$ for some $a > 0$, then the operators $D_\Sigma(t)$ are trace.*

Motivation. If $D_\Sigma(t)$ is trace, then

$$\sigma_{\text{ac}}\left(K_0\dot{+}V\right) = \sigma_{\text{ac}}\left(\left(K_0\dot{+}V\right)_\Sigma\right), \quad \text{and the wave operators}$$

$$\Omega_\pm = \text{s-}\lim_{t\to\infty} \exp\left(\pm it\left(K_0\dot{+}V\right)\right) J^* \exp\left(\mp it\left(K_0\dot{+}V\right)_\Sigma\right)$$

exist and are unitary from $P_{\text{ac}}\left(K_0\dot{+}V\right)_\Sigma L^2(\Sigma, m)$ onto $P_{\text{ac}}\left(K_0\dot{+}V\right) L^2(E, m)$. In the proof of Theorem 21 the following result on trace operators may be used: see Demuth, Stollmann, Stolz, Van Casteren [16].

22. Lemma. *Let K_1 and K_2 be integral operators with kernels $k_1(x,y)$ and $k_2(x,y)$ respectively. Suppose that the integral*

$$\int dz \sqrt{\int \left|k_2(x,z)\right|^2 dx \int \left|k_1(z,x)\right|^2 dx} \quad \textit{is finite.}$$

Then $K_2 \circ K_1$ is a trace operator and its trace norm is dominated by the latter integral.

A formal, but not necessarily rigorous, proof of Lemma 22 reads as follows: Let U be partial isometry for which $|K_2 \circ K_1| = UK_2 \circ K_1$. Then, loosely speaking,

$$\begin{aligned} \left\|K_2\circ K_1\right\|_{\text{trace}} &= \int dx \int dz \left[Uk_2(\cdot, z)\right](x) k_1(z,x) \\ &= \int dz \int dx \left[Uk_2(\cdot, z)\right](x) k_1(z,x). \end{aligned} \tag{4.1}$$

The expression in 4.1 is less than or equal to the integral in Lemma 22. Put

$$D(t) = \exp\left(-t\left(K_0\dot{+}V\right)\right) - \exp\left(-t\left(K_0\dot{+}W\right)\right) \tag{4.2}$$

and let $D(t,x,y)$ be its integral kernel. Put $A(t) = \int_0^t \left(W(X(u)) - V(X(u))\right) du$ and $V_s = (1-s)V + sW$.

Inequality (4.3) is used in the proof of Theorem 24.

23. Lemma. *The following identity and inequality are true:*

$$D(t,x,y) = \int A(t) \int_0^1 ds \exp\left(-\int_0^t V_s(X(u))du\right) d\mu_{0,x}^{t,y};$$

$$|D(t,x,y)| \leq \left(\int A(t)^2 d\mu_{0,x}^{t,y}\right)^{1/2} \left(\int \left(\int_0^1 ds \exp\left(-\int_0^t V_s(X(u))du\right)\right)^2 d\mu_{0,x}^{t,y}\right)^{1/2}. \tag{4.3}$$

24. Theorem. (a) *If $\int dx \mathbf{E}_x\left[A(t)^2\right] < \infty$, then $D(t)$ is Hilbert-Schmidt.*

(b) *If $\int \sqrt{\mathbf{E}_x\left[A(t)^2\right]} dx < \infty$, then $D(t)$ is a trace operator.*

Remark. The latter result is probably also true if $A(t)$ is of the form:

$$A(t) = -i\int_0^t a(b(s).db(s) - \frac{i}{2}\int_0^t div(a(b(s)))ds - \int_0^t V(b(u))du, \tag{4.4}$$

corresponding to a particle under the influence of a magnetic field with vector potential a with Hamiltonian:

$$H(a,V) = \frac{1}{2}\left(-i\nabla - a\right)^2 + V.$$

So $\exp\left(-tH(a,V)\right) - \exp\left(-tH(0,0)\right)$ is a trace operator if $\int dx \sqrt{\mathbf{E}_x\left[A(t)^2\right]}$ is finite.

The proof of the following theorem can be found in [17], and also in [18].

25. Theorem. *Suppose that the unperturbed semigroup $\left\{\exp\left(-tK_0\right) : t \geq 0\right\}$ is L^1-L^∞-smoothing. Then the following assertions hold true.*

(a) *Suppose $W - V$ belongs to $L^1(E,m)$. Then the wave operators*

$$\Omega_\pm = s\text{-}\lim_{s\to\infty} \exp\left(\pm is\left(K_0\dot{+}V\right)\right)\exp\left(\mp is\left(K_0\dot{+}W\right)\right)$$

exist on the range of $P_{\mathrm{ac}}(K_0\dot{+}W)$ and they are complete.

(b) *$W - V \in L^1(E,m)$ does not imply that $D(t)$ is a trace operator.*

(c) *Suppose that the function h_Σ^{a+V} belongs to $L^1(E,m)$. Then the wave operators*

$$\Omega_{\Sigma,\pm} := s\text{-}\lim_{s\to\infty} \exp\left(\pm is\left(K_0\dot{+}V\right)_\Sigma\right) J_\Sigma \exp\left(\mp is\left(K_0\dot{+}V\right)\right)$$

exist and are complete. The operator J_Σ restricts a function to Σ.

The proof is similar to the previous one. It uses a singular version of Duhamel's formula: see 18.

A proof of the results in (1) through (8) can be found in [18]. Throughout it is assumed that the unperturbed semigroup $\left\{\exp\left(-tK_0\right) : t \geq 0\right\}$ is L^1-L^∞-smoothing.

Summary. 1. $\exp\left(-\frac{t}{2}K_0\right)|W-V|^2 \in L^{1/2}(E,m)$ *implies:* $D(t)$, $t>0$, *is a trace operator;*
2. $W-V \in L^2(E,m)$ *implies:* $D(t)$, $t>0$, *is a Hilbert-Schmidt operator;*
3. $W-V \in L^1(E,m)$ *implies that the wave operators exist and are complete;*
4. $h_{\Sigma}^{a+V} \in L^{1/2}(E,m)$ *implies:* $D_{\Sigma}(t)$, $t>0$, *is a trace operator;*
5. $h_{\Sigma}^{a+V} \in L^2(E,m)$ *implies:* $D_{\Sigma}(t)$, $t>0$, *is a Hilbert-Schmidt operator;*
6. $h_{\Sigma}^{a+V} \in L^1(E,m)$ *implies that the wave operators exist and are complete;*
7. $W-V \in C_0(E)$ *implies:* $D(t)$, $t>0$, *is compact;*
8. $h_{\Sigma}^{a+V} \in C_0(E)$ *implies* $D_{\Sigma}(t)$, $t>0$, *is compact;*
9. *In 3, 4 and 6 (extensions) of the classical Pearson estimates are available. We owe this result to Demuth and Eder* [15].

Problem 5. What happens to the results in the above summary if the additive functional $\int_0^t V(X(u))du$ is replaced with more general additive processes, like stochastic integrals? The general result on the existence of Feynman-Kac semigroups can be formulated and proved for a Kato type class of additive processes. This result was obtained by L. Smits (Antwerp). The compactness properties for these more general processes are not yet investigated in its full generality.

Problem 6. What happens to the compactness results, if the Dirichlet type boundary condition is replaced with a Neumann type boundary condition?

As in §6 for $j=0$, 1, we consider self-adjoint operators $H_j = H_j^* \geq -\omega_j I$, where $\omega_j > -\infty$, (Hamiltonians) in the respective Hilbert space $\mathcal{H}_j$, $j=0$, 1. Let $V_j(t) = \exp(-tH_j)$, $t \geq 0$, be the strongly continuous semigroup generated by H_j, $j=0$, 1. Let $J : \mathcal{H}_1 \to \mathcal{H}_0$ be a continuous linear operator. It is considered as an *identification* operator. Furthermore, let $\Psi : \mathbf{R} \to \mathbf{C}$ and $\Phi : \mathbf{R} \to \mathbf{R}$ be Borel measurable functions with the following properties (the operators $\Phi(K_0)$, $\Psi(K_0)$, and $\Psi(H_1)$ are defined via spectral theory and symbolic calculus):

1. The operators $\Psi(H_j) : \mathcal{H}_j \to \mathcal{H}_j$, $j=0$, 1, are continuous, and $\Psi(H_1)\mathcal{H}_{1,\mathrm{ac}}^{H_1}$ is dense in $\mathcal{H}_{1,\mathrm{ac}}^{H_1}$.
2. There exists an admissible function $\alpha : \mathbf{R} \to \mathbf{R}$ such that $\alpha(\Phi(H_j)) = H_j$, $j=0$, 1.
3. The operator $\Psi(K_0)(\Phi(K_0)J - J\Phi(H_1))\Psi(H_1)$ is of trace class.
4. The operator $(\Psi(K_0)J - J\Psi(H_1))\Psi(H_1)$ is compact.

Here a real-valued function α is said to be *admissible* if there exists a sequence of open, mutually disjoint, intervals $(I_n : n \in \mathbf{N})$ in $\mathbf{R}$ such that

1. the function α is continuously differentiable on $\mathbf{R}$;
2. $\alpha'(x) > 0$, $x \in \mathbf{R}$;
3. on each closed sub-interval of $\bigcup_{n=1}^{\infty} I_n$ the function α' is of bounded variation.

Let $E_1(\cdot)$ be the spectral decomposition of the operator H_1. A vector g belongs to $\mathcal{H}_{a,\mathrm{ac}}^{H_1}$ if the measure $B \mapsto \langle E_1(B)g, g\rangle$ is absolutely continuous with respect to the Lebesgue measure. In 3 we may take $\mathcal{H}_0 = \mathcal{H}_1 = L^2(E,m)$, $J=I$, $\Psi(\lambda) = e^{-\lambda t_0}$, $\Phi(\lambda) = \lambda$ (and hence $\alpha(\lambda) = \lambda$), $K_0 = K_0 \dot{+} V$, $H_1 = K_0 \dot{+} W$. In 4 we may take

$\mathcal{H}_0 = L^2(\Sigma, m)$, $\mathcal{H}_1 = L^2(E, m)$, $Jf = f$ restricted to Σ, $\Psi(\lambda) = \Phi(\lambda) = \alpha(\lambda) = \lambda$, $H_0 = \exp\left(-t_0\left(K_0\dot{+}V\right)_\Sigma\right)$, $H_1 = \exp\left(-t_0\left(K_0\dot{+}V\right)\right)$. In 6 we may take $\mathcal{H}_0 = L^2(\Sigma, m)$, $\mathcal{H}_1 = L^2(E, m)$, $Jf = f$ restricted to Σ, $\Psi(\lambda) = e^{-\lambda t_0}$, $\Phi(\lambda) = \lambda$ (and hence $\alpha(\lambda) = \lambda$), $H_0 = \left(K_0\dot{+}V\right)_\Sigma$, $H_1 = K_0\dot{+}W$.

In [15] the authors prove the following theorem.

26. Theorem. *Let $\beta : \mathbf{R} \to \mathbf{R}$ be any admissible function. Let g belong to $\mathcal{H}_{1,\mathrm{ac}}^{H_1}$. The following Pearson estimate is true:*

$$\begin{aligned}
&\left\|\left(\Omega_\pm\left(\beta\left(H_0\right), J, \beta\left(H_1\right)\right) - J\right)\psi\left(H_1\right)^2 g\right\|_{\mathcal{H}_0}^2 \qquad (4.5)\\
&\leq \Big(16\pi\left\|\Psi\left(H_0\right)J\Psi\left(H_1\right)\right\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_0)}\left\|\Psi\left(H_0\right)\left(\Phi\left(H_0\right)J - J\Phi\left(H_1\right)\right)\Psi\left(H_1\right)\right\|_{\mathrm{trace}(\mathcal{H}_1.\mathcal{H}_0)}\\
&\qquad + \left\|\left(\Psi\left(H_0\right)J - J\Psi\left(H_1\right)\right)\Psi\left(H_1\right)\right\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_0)}^2\Big)\\
&\qquad \times\left(\|g\|_{\mathcal{H}_1}^2 + \left\|\frac{d}{d\lambda}\left\langle E_{\Phi(H_1)}(-\infty,\lambda]g, g\right\rangle\right\|_{L^\infty(\mathbf{R})}\right).
\end{aligned}$$

Here $E_{\Phi(H_1)}(B) = E_1\left(\Phi^{-1}(B)\right)$ denotes the spectral decomposition of $\Phi(H_1)$.

Remark. If in (4.5) we set $\Psi \equiv 1$, and $\Phi(\lambda) = \lambda$, we get

$$\begin{aligned}
&\left\|\left(\Omega_\pm\left(\beta\left(H_0\right), J, \beta\left(H_1\right)\right) - J\right)g\right\|_{\mathcal{H}_0}^2 \qquad (4.5')\\
&\leq 16\pi\|J\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_0)}\left\|H_0J - JH_1\right\|_{\mathrm{trace}(\mathcal{H}_1.\mathcal{H}_0)}\\
&\qquad \times\left(\|g\|_{\mathcal{H}_1}^2 + \left\|\frac{d}{d\lambda}\left\langle E_{\Phi(H_1)}(-\infty,\lambda]g, g\right\rangle\right\|_{L^\infty(\mathbf{R})}\right),
\end{aligned}$$

which is slightly worse than the classical Pearson estimate:

$$\begin{aligned}
&\left\|\left(\Omega_\pm\left(\beta\left(H_0\right), J, \beta\left(H_1\right)\right) - J\right)g\right\|_{\mathcal{H}_0}^2 \qquad (4.5'')\\
&\leq 16\pi\|J\|_{\mathcal{B}(\mathcal{H}_1,\mathcal{H}_0)}\left\|H_0J - JH_1\right\|_{\mathrm{trace}(\mathcal{H}_1.\mathcal{H}_0)}\left\|\frac{d}{d\lambda}\left\langle E_{\Phi(H_1)}(-\infty,\lambda]g, g\right\rangle\right\|_{L^\infty(\mathbf{R})}.
\end{aligned}$$

5. Some (abstract) problems related to Neumann semigroups

This section is motivated by the following observation. A general motivation is the following one. Let

$$\left\{\left(\Omega, \mathcal{F}, \mathbf{P}_x\right), \left(X(t), t \geq 0\right), \left(\vartheta_t : t \geq 0\right), \left(\mathbf{R}^\nu, \mathcal{B}\right)\right\}$$

be the Markov process of ν-dimensional Brownian motion. Consider a Neumann initial value problem on an open domain Σ in $\mathbf{R}^\nu$. This means find a solution to the following Cauchy problem:

$$\frac{\partial}{\partial t}u(x,t) = \frac{1}{2}\Delta u(x,t) \quad \text{on} \quad \Sigma, \tag{5.1}$$

where its normal derivative $D_n u(x,t) = 0$, $x \in \partial D$. Assume $u(x,0) = f(x)$, $x \in \Sigma$. Find a sequence of multiplicative processes $M_n(t)$ such that

$$u(x,t) = \lim_{n\to\infty}\mathbf{E}_x\left[M_n(t)f\left(X(t)\right)\right].$$

If we consider Dirichlet semigroups, then such multiplicative functionals can be found. For instance, the semigroup in (3.1) can be written as

$$\exp\left(-t\left(K_0\dot{+}V\right)_\Sigma\right)f(x)=\lim_{n\to\infty}\mathbf{E}_x\left[M_n(t)f(X(t))\right], \tag{5.2}$$

$$\text{where}\quad M_n(t)=\exp\left(-n\int_0^t 1_{E\setminus\Sigma}\left(X(s)\right)ds\right)\exp\left(-\int_0^t V(X(s))ds\right).$$

We extend this kind of problem to a general domain in a locally compact space. Let $v:E\to[0,\infty)$ be a function in $D(K_0)$ with $e^{-v}\in D(K_0)$ as well. We suppose that $K_0 1=0$, i.e. $\int K_0 f(x)dx=0$, for $f\in D\left(K_0\right)$, $K_0 f\in L^1(E,m)$. Put

$$M_v(t)=v(X(t))-v(X(0))+\int_0^t K_0 v(X(s))ds; \tag{5.3}$$

$$\left\langle M_v\right\rangle(t)=\left\langle M_v,M_v\right\rangle(t)=\int_0^t \Gamma_1\left(v,v\right)\left(X(s)\right)ds; \tag{5.4}$$

$$\begin{aligned}Z_v(t)&=M_{-v}(t)-\frac{1}{2}\left\langle M_v\right\rangle(t)\\&=v(X(0))-v(X(t))+\int_0^t e^{v(X(s))}\left[K_0\left(e^{-v}\right)\right](X(s))ds;\end{aligned} \tag{5.5}$$

$$T_v(t)f(x)=\mathbf{E}_x\left[\exp\left(Z_v(t)-\int_0^t V(X(s))ds\right)f(X(t))\right] \tag{5.6}$$

$$\begin{aligned}&=\mathbf{E}_x\Bigg[\exp\left(v(X(0))-v(X(t))\right)\exp\left(\int_0^t e^{v(X(s))}\left[K_0\left(e^{-v}\right)\right](X(s))ds\right)\\&\qquad\times\exp\left(-\int_0^t V(X(s))ds\right)f(X(t))\Bigg]\end{aligned} \tag{5.7}$$

$$=\left[\exp\left(-tK_v\right)f\right](x),\quad\text{where} \tag{5.8}$$

$$K_v f=e^v K_0\left(e^{-v}f\right)+\left(V-e^v K_0\left(e^{-v}\right)\right)f. \tag{5.9}$$

In particular $K_v 1=V$. Here $\Gamma_1(f,g)$ is the carré du champ operator, introduced by Roth [44], but popularized by Bakry (see e.g. [4]):

$$\Gamma_1(f,g)(x)=\lim_{s\downarrow 0}\frac{1}{s}\mathbf{E}_x\left[\left(f(X(s))-f(X(0))\right)\left(g(X(s))-g(X(0))\right)\right], \tag{5.10}$$

and $\langle M_v\rangle=\langle M_v,M_v\rangle$ is the variation process corresponding to the martingale M_v. The family $\{T_v(t):t\geq 0\}$ is a strongly continuous semigroup in $L^2(E,m)$. Put $u(t,x)=T_v(t)f(x)$. Then $u(0,x)=f(x)$ and

$$\frac{\partial u}{\partial t}=-K_0 u-Vu-\Gamma_1(v,u).$$

So the expression $f\mapsto\Gamma_1(v,f)$ is sort of a drift in gradient form. The corresponding quadratic form is given by

$$\begin{aligned}&\mathcal{E}_v^V(f,g)\\&=-\int v\Gamma_1(f,\overline{g})(x)dx+\int\left(2v(x)+1\right)K_0 f(x)\overline{g}(x)dx+\int V(x)f(x)\overline{g}(x)dx.\end{aligned} \tag{5.11}$$

Notice the identity $K_0(fg) + \Gamma_1(f,g) = (K_0 f)g + f(K_0 g)$. Put

$$\mathbf{P}_{v,x}(A) = \mathbf{E}_x \left[\exp\left(v(X(0)) - v(X(t))\right) \exp\left(\int_0^t e^{v(X(s))} \left[K_0 \left(e^{-v} \right) \right] (X(s)) ds \right) \right.$$
$$\left. \times \exp\left(- \int_0^t V(X(s)) ds \right) 1_A \right], \tag{5.12}$$

where A belongs to $\mathcal{F}_t$.

In case K_0 is $-\frac{1}{2}\Delta$, we have $\Gamma_1(f,g) = \nabla f \cdot \nabla g$. Moreover

$$M_v(t) = \int_0^t \nabla v(X(s)) dX(s),$$

(Itô integral) where X is Brownian motion. In addition,

$$\mathbf{E}_x \left[F\left(Y(s) : 0 \le s \le t \right) \exp\left(Z_v(t) \right) \right] = \mathbf{E}_x \left[F\left(X(s) : 0 \le s \le t \right) \right], \tag{5.13}$$

where $Y(s) = X(s) + \int_0^s \nabla v(X(\sigma)) d\sigma$. This is a version of the Girsanov transformation. For an up-to-date account of Girsanov transformations, the reader is referred to Üstünel and Zakai [49]. We want to take (singular) limits in the expressions for $T_v(t)$ and $\mathcal{E}_v^V$, for v tending to $1_{E\setminus\Sigma}$, $0 < a \le \infty$. For $a < \infty$, the quadratic form converges to

$$\mathcal{E}_{\Sigma,a}^V(f,g) := -\, a \int_{E\setminus\Sigma} \Gamma_1(f,\overline{g})(x) dx \tag{5.14}$$
$$+ 2a \int_{E\setminus\Sigma} K_0 f(x)\overline{g}(x) dx + \int K_0 f(x)\overline{g}(x) dx \int V(x) f(x)\overline{g}(x) dx.$$

So that, if $K_0 f\overline{g} - \Gamma_1\left(f,\overline{g}\right)$ vanishes on the complement of Σ,

$$\mathcal{E}_{\Sigma,N}^V(f,g) := \lim_{a\to\infty} \mathcal{E}_{\Sigma,a}^V(f,g) = \int K_0 f(x)\overline{g})(x) dx + \int_\Sigma V(x) f(x)\overline{g}(x) dx. \tag{5.15}$$

Here $\mathcal{E}_{\Sigma,N}^V$ should stand for *Neumann quadratic form*. In the presence of the carré du champ operator we may define a distance on $E \times E$:

$$d(x,y) = \sup\left\{ |\psi(y) - \psi(x)| : \Gamma_1(\psi,\psi) \le 1 \right\}.$$

The *local time* (occupation) the process X (up to time t) spends on the (boundary of the) complement of Σ is then the bounded variation part of the process $d\left(X(t), E\setminus\Sigma\right)$. Suitable logarithmic Sobolev inequalities imply $d(x,y) < \infty$: see e.g. Bakry [4], Théorème 3.2, page 39. A proof of the following result may be based on § 3 of Bakry [4] in combination with the proof of Lemma 3.2.1 in Davies [14], p.83. A detailed proof can be found in [18] Chapter 1, §D.

27. Theorem. *Suppose that there exists a continuous function* $m : (0,\infty) \to (0,\infty)$ *with the property that for every* $1 < u < \infty$ *the following logarithmic Sobolev inequality is true:*

$$E_2(f) \le u\mathcal{E}_2(f) + m(u) \left\| f \right\|_2^2 \tag{5.16}$$

for all $f > 0$, $f \in \mathcal{A}$. Let $c(u) > 0$ be a function in $L^1([1,\infty))$ with the property that $\int_1^\infty c(u)du = 1$. The quantity $m_c(t)$ is defined by

$$m_c(t) = \int_1^\infty \frac{m\big(4tc(u)(u-1)\big)}{u^2} du. \tag{5.17}$$

(1) *Let $t > 0$ be such that the corresponding quantity $m_c(t)$ is finite. Then $\exp(-tK_0)$ maps $L^1(E,m)$ to $L^\infty(E,m)$ and*

$$\|\exp(-tK_0)\|_{\infty,1} \leq \exp(m_c(t)).$$

(2) *Put*

$$\eta = \eta_c = \frac{1}{2}\int_1^\infty c(u)\left(u - 1 + \frac{1}{u-1} - 2\right)du$$

and suppose that η is finite. Let ψ be a function in $\mathcal{A}$ with the property that $\Gamma_1(\psi,\psi) \leq 1$. Suppose that $m_c(t/2)$ is finite. Then

$$\exp(-tK_0)(x,y) \leq \exp(m_c(t/2))\exp\left(-\frac{|\psi(y)-\psi(x)|^2}{2t(1+\eta)}\right). \tag{5.18}$$

The quantities $E_p(f)$ ("entropy") and $\mathcal{E}_p(f)$ ("energy"), $1 \leq p < \infty$, $f > 0$, are defined via the following formulae:

$$E_p(f) = \int_E f(x)^p \log\left(\frac{f(x)^p}{\|f\|_p^p}\right)dx; \tag{5.19}$$

$$\mathcal{E}_p(f) = \left\langle K_0 f, f^{p-1}\right\rangle, \quad f \in D(K_0). \tag{5.20}$$

For $p = 2$ these expressions also make sense for complex-valued functions $f \in D(K_0)$.

Remark 1. Let $c(u)$ be as in the theorem and suppose that $\int_2^\infty c(u)du = 1/2$. Define the function $c_1(v)$, $v > 1$, by

$$\begin{cases} c_1(v) = c\left(\dfrac{v}{v-1}\right)\dfrac{1}{(v-1)^2}, & \text{if } 1 < v \leq 2; \\ c_1(v) = c(v), & \text{if } v \geq 2. \end{cases}$$

Then the hypotheses and conclusions of Theorem 27 remain valid with $c_1(u)$ instead of $c(u)$.

Remark 2. The operator K_0 generates a *diffusion* in the following sense: for every C^∞-function $\Phi : \mathbf{R}^\nu \to \mathbf{R}$, with $\Phi(0,\dots,0) = 0$, the following identity is valid:

$$\begin{aligned} &K_0\left(\Phi(f_1,\dots,f_n)\right) \\ &= \sum_{j=1}^n \frac{\partial\Phi}{\partial x_j}(f_1,\dots,f_n)\,K_0 f_j - \frac{1}{2}\sum_{j,k=1}^n \frac{\partial^2\Phi}{\partial x_j \partial x_k}(f_1,\dots,f_n)\,\Gamma_1(f_j,f_k) \end{aligned} \tag{5.21}$$

for all functions $f_1,\dots,f_n$ in a rich enough algebra of functions $\mathcal{A}$, contained in the domain of the generator K_0, as described in Remark 3.

Remark 3. The algebra $\mathcal{A}$ in Theorem 27 has to be "large" enough. To be specific, it is supposed to possess the following properties: It is dense in $L^p(E,m)$ for all $1 \leq p < \infty$ and it is a core for K_0 considered as an operator in $L^2(E,m)$. In addition, it is assumed that $\mathcal{A}$ is stable under composition with C^∞-functions of several variables, that vanish at the origin. Moreover, in order to obtain some nice results a rather technical condition is required: whenever $(f_n : n \in \mathbf{N})$ is a sequence in $\mathcal{A}$ that converges to f with respect to the graph norm of K_0 (in $L^2(E,m)$) and whenever $\Phi : \mathbf{R} \to \mathbf{R}$ is a C^∞-function, vanishing at 0, with bounded derivatives of all orders (including the order 0), then one may extract a subsequence $(\Phi(f_{n_k}) : k \in \mathbf{N})$ that converges to $\Phi(f)$ in $L^1(E,m)$, whereas the sequence $(K_0\Phi(f_{n_k}) : k \in \mathbf{N})$ converges in $L^1(E,m)$ to $K_0\Phi(f)$. Notice that all functions of the form $e^\psi f$, ψ, $f \in \mathcal{A}$, belong to $\mathcal{A}$. This fact was used in the proof of Theorem 27. Also notice that the required properties of $\mathcal{A}$ depend on the generator K_0. In fact we will assume that the algebra $\mathcal{A}$ is also large enough for all operators of the form $f \mapsto e^{-\psi}K_0\left(e^\psi f\right)$, where ψ belongs to $\mathcal{A}$.

Roughly speaking the problem can be described as follows:

Problem 7. What relations, if any, do exist between the following concepts:

1. singular limit of quadratic form;
2. singular limit of Feynman-Kac semigroup;
3. local time spent by the process X in $E \setminus \Sigma$;
4. Girsanov transformation (SDE);
5. reflected Markov process?

Problem 8. A related, not completely understood problem, is to formulate and prove the precise correspondence between convergence of semigroups and of the associated quadratic forms. One ought to consider Γ-convergence of quadratic forms. For the latter see e.g. Dal Maso [36].

Problem 9. If possible, incorpurate Neumann scattering in our discussion.

Problem 10. Suppose the sequence $\{v_n : n \in \mathbf{N}\}$ converges to $a1_{E\setminus\Sigma}$ in a reasonable way ($a = \frac{1}{2}$, or, more generally, $\infty \geq a > 0$). Does it follow that the sequence $(\mathbf{P}_{v_n,x})$ converges or is tight?

Instead of a genuine drift we might also consider an imaginary drift term ($u(0,x) = f(x)$):

$$\frac{\partial u}{\partial t} = -K_0 u + i(K_0 v)u - Vu - i\Gamma_1(v,u).$$

A solution to this equation is given by the Feynman-Kac formula:

$$u(t,x) = \mathbf{E}_x\left[\exp\left(A(t)\right) f(X(t))\right],$$

where

$$A(t) = -iv(X(t)) + iv(X(0)) + \frac{1}{2}\int_0^t \Gamma_1\left(v,v\right)(X(s))ds.$$

Problem 11. What happens to this if we take singular limits, i.e. if we let v tend to $a1_{E\setminus\Sigma}$, $0 < a \leq \infty$?

Problem 12. Let $h_{\Sigma,\mathcal{N}}^{a+V}$ be an $a+V$-harmonic function with "normal derivatives" equal to 1. Is the following conjecture true?

28. Conjecture. (a) *If $h_{\Sigma,\mathcal{N}}^{a+V}$ belongs to $L^2(E,m)$, then $D_{\Sigma,\mathcal{N}}(t)$, $t>0$, are Hilbert-Schmidt operators.*

(b) *If $\left(h_{\Sigma,\mathcal{N}}^{a+V}\right)^{1/2}$ belongs to $L^1(E,m)$, then $D_{\Sigma,\mathcal{N}}(t)$, $t>0$, are trace class operators.*

There exist papers related to the problems which we presented above. There is one by Williams et Zheng [57], where reflected Brownian motion is constructed as a limit in law of processes, with a strong drift close to the boundary. Another related paper is [41] by Pardoux and Williams. An older paper is one on one-dimensional stochastic differential equations involving local times by J.-F. Le Gall [26]: see the remark on page 72/73.

6. Operator valued Cauchy semigroups

Let, for $j=0$, 1, $H_j=H_j^*\geq -\omega_j I$, where $\omega_j>-\infty$, be a self-adjoint operator in the Hilbert spaces $\mathcal{H}_j$. Let $T:\mathcal{H}_1\to\mathcal{H}_0$ be a linear operator with domain in $\mathcal{H}_1$ and range in $\mathcal{H}_0$. Let $V_j(t)=\exp\left(-tH_j\right)$, $t\geq 0$, be the strongly continuous semigroup generated by H_j, $j=0$, 1. If the operators

$$(aI+H_0)\,V_0\left(t_0\right)T\left(aI+H_1\right)^{-1} \text{ and } (aI+H_0)^{-1}\,T\left(aI+H_1\right)V_1\left(t_0\right)$$

are compact, (Hilbert-Schmidt, Trace class), then so is the operator

$$\int_0^{t_0} V_0(u)TV_1\left(t_0-u\right)du.$$

The result is applicable if $T=JH_1-H_0J$, where $J:\mathcal{H}_1\to\mathcal{H}_0$ is a bounded linear (identification) operator. In this case $\int_0^{t_0}V_0(u)TV_1(t_0-u)du=V_0\left(t_0\right)J-JV_1\left(t_0\right)$; i.e. the difference of the semigroups. Some convergence and approximation results are presented as well. For example the operator $\int_0^{t_0}V_0(u)TV_1\left(t_0-u\right)du$ is expressed in terms of the operator $t_0V_0\left(t_0/2\right)TV_1\left(t_0/2\right)$. The results are applicable if $H_0=K_0\dot{+}V$ and $H_1=K_0\dot{+}W$, where V and W are Kato-Feller potentials in $\mathcal{H}_0=\mathcal{H}_1=L^2\left(E,m\right)$. This is also the case if $\mathcal{H}_0=L^2\left(E,m\right)$, $\mathcal{H}_1=L^2\left(\Sigma,m\right)$, $H_0=K_0\dot{+}V$, and $H_1=\left(K_0\dot{+}V\right)_\Sigma$: see section 3. The results are taken from [56]. Some of the results can also be found in [54] and [52]. Let $J:\mathcal{H}_1\to\mathcal{H}_0$ be a continuous linear operator (also called *identification operator*) and put $D(t)=V_0(t)J-JV_1(t)$.

29. Definition. Define the *Cauchy* or *Poisson* semigroup

$$\left\{\exp\left(-t\mathcal{A}\right)=\exp\left(-t\mathcal{A}_{H_0,H_1}\right):t\geq 0\right\}$$

via the equality

$$\exp\left(-t\mathcal{A}\right)(T)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{t}{\tau^2+t^2}e^{-i\tau H_0}Te^{i\tau H_1}d\tau. \tag{6.1}$$

A basic identity is the following one:

$$\exp\left(-\frac{t}{2}\mathcal{A}\right)\mathcal{D}(t)T = \int_0^t \exp\left(-u\mathcal{A}\right)V_0(t/2)TV_1(t/2)du, \tag{6.2}$$

where $T : \mathcal{H}_1 \to \mathcal{H}_0$ is a linear operator, and where

$$\mathcal{D}(t)T = \int_0^t V_0(u)TV_1(t-u)du. \tag{6.3}$$

Put

$$\mathcal{U}(T) = \iint \operatorname{sgn}(\xi-\eta)E_0(d\xi)TE_1(d\eta), \tag{6.4}$$

where $H_0 = \int \xi E_0(d\xi)$ and $H_1 = \int E_1(d\xi)$ are the spectral decompositions of the Hamiltonians H_0 and H_1 respectively. The double integral in 6.4 is called a *double Stieltjes operator integral*. They have to be treated very carefully. They are not integrals in the strict sense of the word. For more information the reader is referred to [6, 7, 8, 9]. Some information on this topic is given by Yafaev in [59], page 225–228 as well, and by Gohberg and Krein in [27] on page 66. In [27] Gohberg and Krein study the operator $\mathcal{U}$ in more detail. They relate it to the integral of *triangular truncation*: see Chapter II §10 page 45 and Chapter III §3 page 94 of [27]. It is believed that the operator $\mathcal{U}$ is sort of an operator valued "Hilbert transform". Its properties are not well understood. Put $\mathcal{A}_0T = H_0TH_1$, $T : \mathcal{H}_1 \to \mathcal{H}_0$ is a, not necessarily bounded, linear operator. Then $\mathcal{A}_0 = \mathcal{U}\mathcal{A}$ (polar decomposition on Hilbert-Schmidt operators).

The random variables U_j, $j = 2, 3, \ldots$, are independent copies of the logistically distributed variable U_1, $U_0 \equiv 0$: see Evans [23], §24. In fact each U_j, $j \geq 1$, is distributed according to the law:

$$\mathbf{P}\left[U_j \in B\right] = \frac{\pi}{2}\int_B \frac{1}{(\cosh \pi\tau)^2}d\tau = \pi^2\int_0^\infty \frac{\sinh \pi\tau}{(\cosh \pi\tau)^3}\int_{-\tau}^{\tau} 1_B(\sigma)d\sigma\, d\tau. \tag{6.5}$$

30. Theorem. *Some relevant identities are*

Formula 1:

$$t_0V_0(t_0/2)TV_1(t_0/2) = \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{1}{(\cosh \pi\tau)^2}V_0(i\tau t_0)\mathcal{D}(t_0)TV_1\left(-i\tau t_0\right)d\tau. \tag{6.6}$$

Formula 2: *(the inverse of Formula 1) Let f be a rapidly decreasing function. Then*

$$\begin{aligned}
&\int_{-\infty}^{\infty} f(\sigma)V_0\left(i\sigma t_0\right)\mathcal{D}(t_0)TV_1\left(-i\sigma t_0\right)d\sigma \\
&-\sum_{k=0}^{n}\binom{n+1}{k+1}(-1)^k\int_{-\infty}^{\infty}\mathbf{E}^{\text{logistic}}\left[f\left(\sigma-\sum_{j=0}^{k}U_j\right)\right] \\
&\qquad V_0\left(i\sigma t_0\right)t_0V_0(t_0/2)TV_1(t_0/2)V_1\left(-i\sigma t_0\right)d\sigma \\
&= \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\left(1-\frac{\xi/2}{\sinh\left(\xi/2\right)}\right)^{n+1}\widehat{f}(\xi)e^{i\sigma\xi}d\xi V_0\left(i\sigma t_0\right)\mathcal{D}(t_0)TV_1\left(-i\sigma t_0\right)d\sigma.
\end{aligned} \tag{6.7}$$

The random variables U_j, $j = 2, 3, \ldots$, are independent copies of the logistically distributed variable U_1, $U_0 \equiv 0$: see Evans [23], *§24. Put*

$$\mathcal{Q}(t_0)T = \mathbf{E}^{\text{logistic}}\left[\exp\left(-iU_1t_0H_0\right)T\exp\left(iU_1t_0H_1\right)\right]. \tag{6.8}$$

Then

$$\mathcal{Q}(t_0)\mathcal{D}(t_0)T = t_0V_0\left(t_0/2\right)TV_1\left(t_0/2\right); \tag{6.9}$$

$$\mathcal{D}(t_0)T = \sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathcal{Q}(t_0)^j\left(t_0V_0\left(t_0/2\right)TV_1\left(t_0/2\right)\right) + \left(I-\mathcal{Q}(t_0)\right)^{n+1}\mathcal{D}(t_0)T. \tag{6.10}$$

Formula 3: *In* [52] *the central identity was*

$$\exp\left(-\frac{t}{2}\mathcal{A}\right)\mathcal{D}(t)(T) = \int_0^t du\exp\left(-u\mathcal{A}\right)V_0(t/2)TV_1(t/2). \tag{6.11}$$

Formula 4: *(KMS-formula) Suppose that (τ, s) belongs to $\mathbf{R}\times\left(-\frac{1}{2},\frac{1}{2}\right)$, and let $T : \mathcal{H}_1 \to \mathcal{H}_0$ be a linear operator with the property that the operators $V_0(t_0)T$ and $V_1(t_0)T^*$ are densely defined. Then, in form sense,*

$$\begin{aligned}
&t_0V_0\left(-i\tau t_0\right)V_0\left(\left(\frac{1}{2}+s\right)t_0\right)TV_1\left(\left(\frac{1}{2}-s\right)t_0\right)V_1\left(i\tau t_0\right)\\
&= \frac{t_0}{2}\int_{-\infty}^{\infty}\frac{\cos\pi s}{\cosh\pi(\tau-\sigma)-\sin\pi s}V_0\left(t_0\right)V_0\left(-i\sigma t_0\right)TV_1\left(i\sigma t_0\right)d\sigma \\
&\quad+\frac{t_0}{2}\int_{-\infty}^{\infty}\frac{\cos\pi s}{\cosh\pi(\tau-\sigma)+\sin\pi s}V_0\left(-i\sigma t_0\right)TV_1\left(i\sigma t_0\right)V_1\left(t_0\right)d\sigma.
\end{aligned} \tag{6.12}$$

Formula (6.12) follows by virtue of the following observation. The function at the right hand side of (6.12) (Formula 4) is harmonic on the strip $\mathbf{R}\times\left(-\dfrac{1}{2},\dfrac{1}{2}\right)$, and it possesses boundary values

$$\begin{cases} t_0V_0\left(t_0\right)V_0\left(-i\tau t_0\right)TV_1\left(i\tau t_0\right), & \text{for} \quad s=\frac{1}{2}, \quad \text{and}\\ t_0V_0\left(-i\tau t_0\right)TV_1\left(i\tau t_0\right)V_1\left(t_0\right), & \text{for} \quad s=-\frac{1}{2}.\end{cases}$$

The left hand side is harmonic on the same strip (in fact it is holomorphic there), and has the same boundary values. The uniqueness part on the existence of solutions to the classical Dirichlet problem on a strip, yields Formula 4.

Problem 13. For which operator norms $\|\cdot\|$ do we have

$$\lim_{n\to\infty}\left\|\left(I-\mathcal{Q}(t_0)\right)^n\mathcal{D}(t_0)T\right\| = 0?$$

For a result in this direction see Theorem 33 below.

Remark 1. In § 6 of the present paper the basic role of equality (6.11) in [52] is taken over by the equality in **Formula 1**. This is also the case for [56, 54]. It is quite well possible, that with the semigroup in **Formula 3** there can be associated a quantum diffusion. Instead of considering the evolution $\tau \to V_0(i\tau)TV_1(-i\tau)$ one should look at one in the space $\mathcal{H}_0 \times \mathcal{H}_1$ given by $\tau \mapsto V(i\tau) \begin{pmatrix} I & T \\ T^* & I \end{pmatrix} V(-i\tau)$, where $V(i\tau)$, $\tau \in \mathbf{R}$, is the operator matrix $V(i\tau) = \begin{pmatrix} V_0(i\tau) & 0 \\ 0 & V_1(i\tau) \end{pmatrix}$. We notice that the semigroup $T \mapsto \mathbf{E}^{\text{Cauchy}}\left[V\left(iX(t)H\right)TV\left(-iX(t)H\right)\right]$, $t \geq 0$, is *completely positive* in the sense that, for any finite choice $S_1, \dots, S_n$, $T_1, \dots, T_n$ of bounded linear operators from $\mathcal{H} = \mathcal{H}_0 \times \mathcal{H}_1$ to $\mathcal{H}$, the operators

$$\sum_{j,k=1}^{n} S_j^* \mathbf{E}^{\text{Cauchy}}\left[V\left(iX(t)H\right)T_j^* T_k V\left(-iX(t)H\right)\right] S_k, \quad t > 0, \tag{6.13}$$

are positive: see e.g. [42], pp. 250–252. Here $\{(\Omega, \mathcal{F}, \mathbf{P}), (X(t) : t \geq 0)\}$ is a Cauchy process. The same is true for any other Lévy process. For more on quantum probability the reader is referred to e.g. Meyer [37], [42], and [5]. Define for $(f,g) \in \mathcal{H}_0 \times \mathcal{H}_1$, and $T \in \mathcal{B}\left(\mathcal{H}_0 \times \mathcal{H}_1, \mathcal{H}_0 \times \mathcal{H}_1\right)$ the function $F_{f,g}(\tau + is)$, $\tau \in \mathbf{R}$, $-\frac{1}{2} \leq s \leq \frac{1}{2}$ by

$$F_{f,g}(\tau + is) = t_0 \left\langle \mathcal{A}_\tau \left(V\left(\left(\frac{1}{2} + s\right)t_0\right) TV\left(\left(\frac{1}{2} - s\right)t_0\right)\right) \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle. \tag{6.14}$$

The function $F_{f,g}$ is K(ubo)-M(artin)-S(chwinger)-admissible for the operators $V(t_0)$ and T in the sense that it is continuous on the closed strip

$$\left\{\tau + is : \tau \in \mathbf{R},\ -\frac{1}{2} \leq s \leq \frac{1}{2}\right\},$$

and holomorphic on its interior. Moreover, if $T = T^*$, then

$$\begin{aligned} F_{f,g}\left(\tau + \frac{1}{2}i\right) &= t_0 \left\langle V(t_0)\mathcal{A}_\tau(T) \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle; \\ F_{f,g}\left(\tau - \frac{1}{2}i\right) &= t_0 \left\langle \mathcal{A}_\tau(T)V(t_0) \begin{pmatrix} f \\ g \end{pmatrix}, \begin{pmatrix} f \\ g \end{pmatrix} \right\rangle = \overline{F_{f,g}\left(\tau + \frac{1}{2}i\right)}. \end{aligned} \tag{6.15}$$

We also wrote: $\mathcal{A}_\tau(T) = V\left(-i\tau t_0\right)TV\left(i\tau t_0\right)$.

Remark 2. The main formula (6.6) is almost trivial from the point of view of double Stieltjes operator integrals (and if one does not really care about Fubini's theorem). In fact we have

$$\begin{aligned} &\frac{\pi}{2}\int_{-\infty}^{\infty} \frac{1}{\left(\cosh \pi\tau\right)^2} V_0(i\tau t_0) \int_0^{t_0} V_0(u)TV_1(t_0 - u)du V_1(-i\tau t_0)d\tau \\ &= \iint \frac{\pi}{2}\int_{-\infty}^{\infty} \frac{\exp\left(-i\tau t_0(\xi - \eta)\right)}{\left(\cosh \pi\tau\right)^2} d\tau \frac{\exp(-t_0\eta) - \exp(-t_0\xi)}{\xi - \eta} E_0(d\xi)TE_1(d\eta) \\ &= \iint \frac{\frac{1}{2}t_0(\xi - \eta)}{\sinh\left(\frac{1}{2}(\xi - \eta)t_0\right)} \frac{\sinh\left(\frac{1}{2}t_0(\xi - \eta)\right)}{\frac{1}{2}(\xi - \eta)} \exp\left(-\frac{1}{2}t_0(\xi + \eta)\right) E_0(d\xi)TE_1(d\eta) \\ &= t_0 \iint \exp\left(-\frac{1}{2}t_0(\xi + \eta)\right) E_0(d\xi)TE_1(d\eta) = t_0 V_0(t_0/2)TV_1(t_0/2). \end{aligned} \tag{6.16}$$

A proof without double operator integrals may be based on Cauchy's theorem from complex analysis: see the results of section 5 in [54] as well. The basic Formulae 1 and 2 are proved there.

Remark 3. Property (6.13) is not shared by Cauchy semigroups as defined in (6.11). Indeed a Cauchy semigroup should be thought of as an image of the group

$$t \mapsto \exp(-itH_0)\, T \exp(itH_1), \text{ under a Cauchy (or Poisson) transform.}$$

The interesting fact is that this transform associates a generator with the resulting semigroup. As far as we know, the ideas of Poisson and Weierstrass transforms have been studied for the first time by Hille in 1935, but they still enjoy interesting unexplored properties!

Problem 14. As a mathematical problem R. Rebolledo (private communiation) proposed to find a link between the following concepts. As above H_0 and H_1 are self-adjoint Hamiltonians in the respective Hilbert spaces $\mathcal{H}_0$ and $\mathcal{H}_1$, and $J : \mathcal{H}_1 \to \mathcal{H}_0$ is a bounded *identification operator*.

(i) *Quantum mechanical scattering theory*:

$$\text{s-}\lim_{t\to\infty} \exp(\pm itH_0)\, J \exp(\mp itH_1).$$

(ii) *Kullback information*, which now is defined by

$$K((\psi_t), \varphi) = \sup_{0<u<\infty,\ u\in C_b(\mathbf{R})} \left\{ \limsup_{t,s\to\infty} \Re \langle \log u(H_0)\psi_t, \psi_s\rangle - \log \langle u(H_1)\varphi, \varphi\rangle \right\}.$$

Here ψ_t denotes $\psi_t = \exp(-itH_0)\, J \exp(itH_1)\, \varphi$. This Kullback information is inspired by the classical Kullback (Shannon) information. Let μ and ν be probability measures on a Polish space E. The expression

$$K(\mu,\nu) = \sup_{0<u<\infty, u\in C_b(E)} \left\{ \int \log u d\mu - \log \int u d\nu \right\} = \int \left(\log \frac{d\mu}{d\nu}\right) \frac{d\mu}{d\nu} d\nu$$

then defines the mutual information between μ and ν. Then the following assertions are true: $K(\mu,\nu) = 0$ if and only if $\mu = \nu$, and $K(\mu,\nu) < \infty$ if and only if μ is absolutely continuous with respect to ν.

(iii) Let $(X, \mathbf{P})$ be a real valued Markov process: for example Brownain motion. The following expression connects (for $t \to \infty$) "quantum probability" and "scattering theory":

$$\mathbf{E} \begin{pmatrix} \exp(iX(t)H_0) & 0 \\ 0 & \exp(iX(t)H_1) \end{pmatrix} \begin{pmatrix} I & J \\ 0 & I \end{pmatrix} \begin{pmatrix} \exp(-iX(t)H_0) & 0 \\ 0 & \exp(-iX(t)H_1) \end{pmatrix}.$$

(iv) Contiguity of the Hamiltonian H_0 with respect to H_1. A possible definition of this notion could be

$$\text{w-}\lim_{t\to\infty} \exp(-tH_0)\, \pi_t \exp(-tH_1) = 0,$$

whenever $\pi_t^*\pi_t$ and $\pi_t\pi_t^*$ are orthogonal projections which converge weakly to 0. In [43] Rebolledo employs a similar notion for quantum dynamical semigroups in Fock space. Another definition, which is closer to wave operators and scattering theory is

$$\text{w-}\lim_{t\to\pm\infty} \exp\left(-itH_0\right)\pi_t \exp\left(itH_1\right) = 0,$$

whenever $\pi_t^*\pi_t$ and $\pi_t\pi_t^*$ are orthogonal projections which converge weakly to 0.

Remark. Another problem might be to define *contiguity* of one (sequence) spectral measure(s) $E_n^0(\cdot)$ with respect another $E_n^1(\cdot)$. For two Hamiltonians H_0 with corresponding respective spectral measures E_0 and E_1 two notions are suggested in Problem 14. The first one is based on the semigroups $\{\exp\left(-tH_j\right) : t \geq 0\}$, $j = 0$, 1, and the second one is based on the wave oerators $\{\exp\left(itH_j\right) : t \in \mathbf{R}\}$, $j = 0$, 1. In the classical commutative context we have a sequence of probability measures $\left(\mathbf{Q}_n\right)_n$ is contiguous to the sequence $\left(\mathbf{P}_n\right)_n$ if $\lim_{n\to\infty}\mathbf{Q}_n\left(A_n\right) = 0$, whenever $\lim_{n\to\infty}\mathbf{P}_n\left(A_n\right) = 0$. We notice that the sequence $\left(\mathbf{Q}_n\right)_n$ is contiguous to the sequence $\left(\mathbf{P}_n\right)_n$ if and only the following two conditions are satisfied:

(i) $\inf_N \sup_n \int_{\{Z_n > N\}} Z_n d\mathbf{P}_n = 0$;
(ii) $\lim_n \mathbf{Q}_n\left[Z_n = \infty\right] = 0$.

Problem 15. Let A be a C^*-algebra, and let $\pi : A \to \mathcal{L}(\mathcal{H})$ be a faithful representation (obtained e.g. via a Gelfand-Naimark-Segal construction). Define, for every compact operator $T : \mathcal{H} \to \mathcal{H}$ the semi-norm $p_T : A \to [0,\infty)$ by $p_T(x) = \|T\pi(x)\|$, $x \in A$. The topology τ_β inducee by these seminorms may be called the *strict topology*. Is the following theorem true?

31. Theorem. *Suppose that the C^*-algebra A is equipped with the strict topology. Let $\mathcal{L}$ be a linear operator with domain and range in A. The following assertions are equivalent:*

(1) *the operator $\mathcal{L}$ possesses the following properties:*
  (i) *For every $\lambda > 0$ the operator $\left(\lambda I - \mathcal{L}\right)^{-1}$ exists as a bounded, everywhere defined linear operator and is completely positive;*
  (ii) *The domain of $\mathcal{L}$ is dense for the strict topology on A.*
(2) *The operator $\mathcal{L}$ generates a τ_β-contiuous semigroup consisting of completely positive operators.*

Here an operator $\mathcal{T} : A \to A$ is said to be completely positive, if for every finite choice $x_1, \dots, x_n$ and $y_1, \dots, y_n$ in A the element $\sum_{j,k=1}^n x_j^*\mathcal{T}\left[y_j^* y_k\right] x_k$ is positive in A. Let $\mathcal{L}$ be as in Theorem 31 (2). Suppose $\mathcal{L}(I) = I$. Let $\exp\left(\tau\widetilde{\mathcal{L}}\right)$ be its extension (dilation) of $\exp\left(\tau\mathcal{L}\right)$ to (symmetric) Fock space. Make a systematic study of the following limits:

$$\begin{aligned}
&\text{s-}\lim_{\tau\to\pm\infty}\text{s-}\lim_{s\to\infty}\frac{1}{s}\int_0^s \exp\left(\left(\sigma + i\tau\right)\widetilde{\mathcal{L}}\right)d\sigma, \quad \text{and} \\
&\text{s-}\lim_{s\to\infty}\text{s-}\lim_{\tau\to\pm\infty}\frac{1}{s}\int_0^s \exp\left(\left(\sigma + i\tau\right)\widetilde{\mathcal{L}}\right)d\sigma. \qquad (6.17)
\end{aligned}$$

Problem 14 and 15 should be compared to the following result, which can be found in [54].

32. Theorem. *Suppose that for all bounded intervals A_0 and A_1 the following conditions are satisfied:*

(i) *the wave operators $\Omega_\pm\left(H_0, E_0(A_0)JE_1(A_1), H_1\right)$ exist;*
(ii) *the operator $E_0(A_0)\left(H_0J - JH_1\right)E_1(A_1)$ is compact.*
(iii) *The operators $E_0(A_0)\left(JJ^* - I\right)E_0(A_0)$ and $E_1(A_1)\left(J^*J - I\right)E_1(A_1)$ are compact.*

Then the wave operators $\Omega_\pm\left(H_0, J_\varphi, H_1\right)$, $\varphi \in L^1(\mathbf{R})$, exist. Here

$$J_\varphi = \int_{-\infty}^{\infty} \varphi(\sigma)\exp\left(-i\sigma H_0\right) J \exp\left(i\sigma H_1\right) d\sigma.$$

Moreover, for φ_1 and $\varphi_2 \in L^1(\mathbf{R})$,

$$\int \varphi_1(\sigma)d\sigma\, \Omega_\pm\left(H_0, J_{\varphi_2}, H_1\right) = \int \varphi_2(\sigma)d\sigma \;= \int \varphi_2(\sigma)d\sigma\, \Omega_\pm\left(H_0, J_{\varphi_1}, H_1\right).$$

If, in addition to (i) and (ii), $\lim_{n\to\infty}\left\|E_0[n,\infty)JE_1(A_1)\right\| = 0$ for all bounded intervals A_1, then the wave operators $\Omega_\pm\left(H_0, J, H_1\right)$ exist.

If moreover, $\lim_{n\to\infty}\left\|E_0(A_0)JE_1[n,\infty)\right\| = 0$, then $D(t_0) = V_0(t_0)J - JV_1(t_0)$ is compact and the scattering system (H_0, J, H_1) is complete in the sense that the wave operators $\Omega_\pm\left(H_0, J, H_1\right)$ are unitary operators from $P_{\mathrm{ac}}(H_1)\mathcal{H}_1$ onto $P_{\mathrm{ac}}(H_0)\mathcal{H}_0$.

The theorems 33 and 34 are proved in [54] and [56].
Next we put

$$F(\tau) = \exp\left(it_0\tau H_0\right)\left(TV_1\left(t_0\right) - V_0\left(t_0\right)T\right)\exp\left(-it_0\tau H_1\right).$$

Then

$$\begin{aligned}&\int_0^{t_0} V_0(s)TV_1\left(t_0 - s\right)ds - \sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j \mathcal{Q}\left(t_0\right)^j\left[t_0V_0\left(t_0/2\right)TV_1\left(t_0/2\right)\right] \\ &= t_0\int_0^\infty E_n(\tau)\left(F(\tau) - F(-\tau)\right)d\tau,\end{aligned} \tag{6.18}$$

where the numerical function $E_n(\tau)$ satisfies

$$\left|E_n(\tau)\right| \le C\int_0^\infty \left(1 - \widehat{\psi}\left(\frac{\xi}{\tau}\right)\right)^{\frac{1}{2}n}\frac{1}{1+\xi^2}d\xi\frac{1}{1+\tau^2}.$$

Here $\widehat{\psi}\left(\xi\right) = \dfrac{\frac{1}{2}\xi}{\sinh\left(\frac{1}{2}\xi\right)}$ denotes the characteristic function of the logistic distribution. In fact the function $E_n(\tau)$ is given by:

$$E_n(\tau) = \frac{-1}{\pi}\int_0^\infty C_n'(\xi)\left(\int_{\tau\xi}^\infty \frac{\sin\eta}{\eta}d\eta\right)d\xi\frac{1}{1+\tau^2},$$

and where

$$C_n(\xi) = \xi\left(1-\left(\frac{\partial}{\partial\xi}\right)^2\right)\left(\left(1-\widehat{\psi}(\xi)\right)^{n+1}\frac{1}{\xi}\right).$$

33. Theorem. *Suppose that the operator $TV_1(t_0) - V_0(t_0)T$ is bounded (Hilbert-Schmidt, trace class). Then, in operator norm (Hilbert-Schmidt norm, trace class norm) sense, the difference*

$$\int_0^{t_0} V_0(s)TV_1(t_0-s)\,ds - \sum_{j=0}^{n}\binom{n+1}{j+1}(-1)^j\mathfrak{Q}(t_0)^j\left(t_0V_0\left(\frac{t_0}{2}\right)TV_1\left(\frac{t_0}{2}\right)\right)$$

converges to zero if n tends to ∞.

Here $\mathfrak{Q}(t_0)(T) = \mathbf{E}^{\text{logistic}}\left[\exp\left(-iU_1t_0H_0\right)T\exp\left(iU_1t_0H_1\right)\right]$.

34. Theorem. *Suppose that the operators*

$$(aI+H_0)^{-1}TV_1(t_0) + V_0(t_0)T(aI+H_1)^{-1} \quad \text{and} \tag{6.19}$$

$$(aI+H_0)^{-1}T(aI+H_1)V_1(t_0) - (aI+H_0)V_0(t_0)T(aI+H_1)^{-1} \tag{6.20}$$

are compact (Hilbert-Schmidt, trace class). Then the operator

$$\int_0^{t_0} V_0(u)TV_1(t_0-u)\,du$$

is compact (Hilbert-Schmidt, trace class) if and only if $V_0\left(\frac{1}{2}t_0\right)TV_1\left(\frac{1}{2}t_0\right)$ is so. Moreover, if (6.19) and (6.20) are satisfied, then the Riemann approximation

$$\frac{t_0}{n}\sum_{j=0}^{n-1}V_0\left(\frac{2j+1}{2n}t_0\right)TV_1\left(\frac{2n-2j-1}{2n}t_0\right) \quad \textit{converges to} \quad \int_0^{t_0}V_0(u)TV_1(t_0-u)\,du$$

with respect to the operator norm (in the compact case), in the Hibert-Schmidt norm (in the Hilbert-Schmidt case), and in the trace norm (in the trace situation).

As a corollary we obtain the following, where $T = H_1J - JH_0$.

35. Corollary. *The semigroup differences $V_0(t)J - JV_1(t)$ are compact if and only if the resolvent differences $(aI+H_0)^{-1}J - J(aI+H_1)^{-1}$ are compact. Of course, $-a$ is supposed to be in the intersection of the resolvent set of H_0 and of H_1.*

Remark (Concluding remarks). In this paper we proved some theorems about the close connection that exists between probability theory and analysis. We mentioned some (open) problems in connection with Neumann type semigroups. We hope that the readers are inspired by some of the problems and results which are presented. It presents some rather general techniques (Markov processes, Feynman-Kac formula, martingale theory, squared gradient operator, classical harmonic analysis) to prove results in operator theory: unique Markov extension, some results in connection with scattering and spectral theory, a result on heat kernel estimate, compactness properties of differences of self-adjoint semigroups.

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