Discreteness of the spectrum for some differential operators with unbounded coefficients in $\mathbf{R}^n$

G. Metafune*   D. Pallara*

Abstract

We give sufficient conditions for the discreteness of the spectrum of differential operators of the form $Au = -\Delta u + \langle \nabla F, \nabla u\rangle$ in $L^2_\mu(\mathbf{R}^n)$ where $d\mu(x) = e^{-F(x)}\,dx$ and for Schrödinger operators in $L^2(\mathbf{R}^n)$. Our conditions are also necessary in the case of polynomial coefficients.

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1 Introduction

In this paper we study the discreteness of the spectrum of two strictly related second order elliptic differential operators with unbounded coefficients on $\mathbf{R}^n$. These operators are

$$A = -\Delta + \sum_{i=1}^{n} \frac{\partial F}{\partial x_i}\frac{\partial}{\partial x_i}, \qquad B = -\Delta + V,$$

with $F \in C^2(\mathbf{R}^n)$ and $V \in C(\mathbf{R}^n)$. B is the classical Schrödinger operator, whereas A is a special case of second order operators with (possibly) unbounded coefficients of the first order terms. These operators are of interest when dealing with diffusion processes on all of $\mathbf{R}^n$ in presence of a drift represented by the first order terms. Unlike the case of bounded coefficients, only recently has that of unbounded ones been studied, starting from the prototype of Ornstein-Uhlenbeck operators. Existence and regularity of the associated semigroups that describe the underlying processes have been studied both with stochastic (see [2]) and analytic tools (see [1], [9], [11], [12] and [10]).

To the operator A is canonically associated the measure $d\mu = e^{-F}dx$ on $\mathbf{R}^n$, which is the (unique) invariant measure of the associated Markov process, and therefore it is natural to study A in the Hilbert space $L^2(\mathbf{R}^n, d\mu)$, where it turns out to be self-adjoint and non-negative. Moreover, due to the gradient structure of the coefficients of the first order terms, the operator A is unitarily equivalent to B with $V = (1/4)|\nabla F|^2 - (1/2)\Delta F$. Hence, we can deduce properties of A on $L^2(\mathbf{R}^n, d\mu)$ from those of B on $L^2(\mathbf{R}^n, dx)$.

As regards Schrödinger operators, various conditions on V are known (see [3], [4], [15], [16]) guaranteeing the discreteness of the spectrum, and also a characterisation (see [14], [7])

*Dipartimento di Matematica "Ennio De Giorgi", Università di Lecce, C.P.193, 73100, Lecce, Italy. e-mail: metafune@le.infn.it, pallara@le.infn.it

based on quantitative capacity estimates. We give here a new simple condition for positive potentials based on Sobolev embeddings, which turns out to be also necessary for polynomial potentials V, a case of interest in quantum mechanics (see [8], [17]). In particular, Theorem 3.5 confirms a conjecture of B. Simon's (see [17, §6, Remark 5]).

Notation We use L^2 for $L^2(\mathbf{R}^n)$ with respect to the Lebesgue measure. Similarly, H^k stands for the usual Sobolev space $H^k(\mathbf{R}^n)$. By C_0^k, $(0 \le k \le +\infty)$ we denote the space of all C^k-functions with compact support in $\mathbf{R}^n$. The integration domain is always understood to be $\mathbf{R}^n$, if not otherwise stated. If E is a measurable subset of $\mathbf{R}^n$, we denote by $|E|$ its Lebesgue measure. We denote by $Q(c,d)$ the open cube of $\mathbf{R}^n$ with centre c and side $d > 0$.

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2 Reduction to a Schrödinger operator

We consider the differential operator on $\mathbf{R}^n$

$$Au = -\Delta u + \langle \nabla F, \nabla u \rangle = -e^{F(x)} \mathrm{div}\left(e^{-F(x)} \nabla u\right),$$

with $F \in C^2(\mathbf{R}^n)$, and study the compactness of its resolvent operator in the weighted Hilbert space

$$L^2_\mu = \left\{ u : \mathbf{R}^n \to \mathbf{C} : u \text{ measurable and } \int |u|^2 \, d\mu < +\infty \right\}, \tag{2.1}$$

where $d\mu(x) = e^{-F(x)}\, dx$, endowed with the inner product $(u,v)_\mu = \int u \bar{v} \, d\mu$. We introduce the Sobolev space

$$H^1_\mu = \left\{ u \in L^2_\mu : \nabla u \in L^2_\mu \right\} \tag{2.2}$$

endowed with the inner product $(u,v)_1 = (u,v)_\mu + (\nabla u, \nabla v)_\mu$ and we observe that C_0^∞ is dense both in L^2_μ and in H^1_μ.

We define the domain of A as follows

$$D(A) = \left\{ u \in H^1_\mu \cap H^2_{loc} : Au \in L^2_\mu \right\} \subset L^2_\mu; \tag{2.3}$$

clearly $D(A)$ is dense in L^2_μ.

Proposition 2.1 *The operator $(A, D(A))$ is self-adjoint and non-negative in L^2_μ.*

Proof. The bilinear form $a(u,v) = (\nabla u, \nabla v)_\mu$, defined on $H^1_\mu \times H^1_\mu$, is (weakly) coercive on L^2_μ and defines a self-adjoint, non-negative operator $(L, D(L))$ on L^2_μ in the following way

$$D(L) = \left\{ u \in H^1_\mu : \exists f \in L^2_\mu \text{ such that } a(u,v) = (f,v)_\mu, \ \forall v \in H^1_\mu \right\}, \qquad Lu = f.$$

Let us prove that L coincides with A. By local elliptic regularity, $D(L) \subset H^2_{loc}$. If $u \in D(L)$ and $v \in C_0^\infty$, integrating by parts the equality $a(u,v) = (f,v)_\mu$ we obtain

$$-\int \mathrm{div}\left(e^{-F(x)} \nabla u\right) \bar{v} \, dx = \int f \bar{v} \, d\mu$$

and hence $f = Au$. This shows that $D(L) \subset D(A)$ and that $Lu = Au$ if $u \in D(L)$. Conversely, if $u \in D(A)$ and $f = Au$, the equality $a(u,v) = (f,v)_\mu$ clearly holds for every $v \in C_0^\infty$ (by integrating by parts) and, by density, for all $v \in H^1_\mu$. This concludes the proof. □

The analysis of the spectrum of A will be done by transforming it into a suitable Schrödinger operator $B = -\Delta + V$ on L^2, in the vein of [5] and [6]. We briefly recall the definition and the basic properties of these operators. We assume that V is real-valued, continuous on $\mathbf{R}^n$ and bounded from below and define B through the bilinear form

$$b(u,v) = \int \langle \nabla u, \nabla \bar{v} \rangle + V u \bar{v}\, dx,$$

$u, v \in \mathcal{H} = \left\{ u \in H^1 : |V|^{1/2} u \in L^2 \right\}$. More precisely, we define

$$D(B) = \left\{ u \in \mathcal{H} : \exists f \in L^2 \text{ such that } b(u,v) = (f,v),\ \forall v \in \mathcal{H} \right\}, \qquad Bu = f. \tag{2.4}$$

Arguing as in Proposition 2.1 it is easily checked that

$$D(B) = \left\{ u \in \mathcal{H} \cap H^2_{loc} : -\Delta u + Vu \in L^2 \right\};$$

moreover, C_0^∞ is a core of $(B, D(B))$, *i.e.*, B is essentially self-adjoint on C_0^∞ (see [7, Corollary VII.2.7]).

Under additional hypotheses on the function F, the operator A is similar to a suitable Schrödinger operator B. With the notation $\phi = e^{-F/2}$, we obtain that A is unitarily equivalent to B when

$$V = \frac{\Delta \phi}{\phi} = \frac{1}{4} |\nabla F|^2 - \frac{1}{2} \Delta F.$$

This is stated in the following proposition.

Proposition 2.2 *If the function $|\nabla F|^2 - 2\Delta F$ is bounded from below in $\mathbf{R}^n$, then the operator $(A, D(A))$ is unitarily equivalent to the Schrödinger operator $(B, D(B))$ with $V = (1/4)|\nabla F|^2 - (1/2)\Delta F$.*

Proof. Let $\phi = e^{-F/2}$ and $T : L^2_\mu \to L^2$ the unitary map defined by $Tf = \phi f$. We define the operator

$$Cu = TAT^{-1}u$$

for $u \in D(C) = T(D(A))$. Clearly C is unitarily equivalent to A and we show that $(B, D(B)) = (C, D(C))$, with the stated choice of V. Since $(B, D(B))$ is essentially self-adjoint on C_0^∞ it is sufficient to prove that $Bu = Cu$ for all $u \in C_0^\infty$. For, a straightforward computation gives for $u \in C_0^\infty$

$$Cu = -\Delta u + \langle \phi \nabla u, \phi^{-1} \nabla F - 2\nabla(\phi^{-1}) \rangle + \left[\langle \nabla F, \phi \nabla(\phi^{-1}) \rangle - \phi \Delta(\phi^{-1}) \right] u.$$

Moreover, since $\phi^{-1} = e^{F/2}$, we obtain

$$\nabla(\phi^{-1}) = (1/2) e^{F/2} \nabla F, \qquad \Delta(\phi^{-1}) = (1/2) e^{F/2} \Delta F + 1/2 \langle \nabla F, \nabla(\phi^{-1}) \rangle,$$

so that $\phi^{-1} \nabla F - 2\nabla(\phi^{-1}) = 0$ and

$$\langle \nabla F, \phi \nabla(\phi^{-1}) \rangle - \phi \Delta(\phi^{-1}) = (\Delta \phi)/\phi = (1/4)|\nabla F|^2 - (1/2)\Delta F.$$

This gives immediately $Cu = Bu$ and concludes the proof. $\square$

Remark 2.3 We observe that under the hypotheses of the above theorem, C_0^2 is a core also for $(A, D(A))$ since it is a core for $(B, D(B))$ and is invariant under the map T.

Remark 2.4 Since the quadratic form $(\nabla u, \nabla u)_\mu$ is non-negative, we obtain that $\int |\nabla u|^2 + V|u|^2\,dx$, with $V = \Delta\phi/\phi$, is non-negative, too. However, the potential V may be negative everywhere. For instance, take $n \geq 3$, $\alpha \in (1 - n/2, 0)$ and $F(x) = -2\alpha \log(1 + |x|^2)$; then $\phi(x) = e^{-F(x)/2} = (1 + |x|^2)^\alpha$ and $V(x) = \alpha(1 + |x|^2)^{-2}[(4\alpha - 4 + 2n)|x|^2 + 2n]$ is negative and bounded from below.

It may also happen that the quadratic form $\int |\nabla u|^2 + V|u|^2\,dx$ is non-negative on C_0^∞ with V unbounded from below. An example is $V(x, y) = y^4 + 4x^2y^2 - 2x$, coming from $F(x, y) = xy^2$ (see also item **a** in Section 4).

3 Discreteness of the spectrum of Schrödinger operators

In this section we study the compactness of the resolvent of the Schrödinger operator $-\Delta + V$, and give sufficient conditions if V is bounded from below, and a characterisation when V is a positive polynomial. We recall that a characterisation of the compactness of $(-\Delta + V)^{-1}$, for positive V, is due to A. M. Molcanov (see [14] and [7, Theorem VIII.4.1]). However, since it involves explicit computations of capacities of arbitrary sets, it is rather difficult to handle. For this reason, it seems to be interesting to prove a sufficient condition based on Sobolev embeddings. Our result embodies the classical case in which the potential goes to $+\infty$ at infinity (see [15, Theorem XIII.67], [3, Section 1.6] or [4, Section 8]): we show that the result is still true if $V \to +\infty$ in a measure-theoretic sense, as $|x| \to +\infty$.

If V is a potential and $M > 0$, we define $E_M = \{x \in \mathbf{R}^n : V(x) < M\}$. If $x \in \mathbf{R}^n$ we set $|x|_\infty = \max_{i=1,\dots,n} |x_i|$, so that $Q(c, d) = \{x : |x - c|_\infty < d\}$. .

Theorem 3.1 *Let $B = -\Delta + V$ be defined as in (2.4) and suppose that for every $M > 0$*

$$\lim_{|c|\to+\infty} |E_M \cap Q(c, 1)| = 0. \tag{3.1}$$

Then B has compact resolvent in L^2.

Proof. We may suppose that $V \geq 0$. Since $D(B)$ is contained in H^1 and the embedding of $H^1(Q)$ into $L^2(Q)$ is compact for every cube Q, we have only to show that for every $\varepsilon > 0$ there is a cube Q such that $\displaystyle\int_{\mathbf{R}^n\setminus Q} |u|^2 < \varepsilon$ for every $u \in D(B)$ with $\displaystyle\int |u|^2 + |Bu|^2 \leq 1$.

Let $u \in D(B)$ as above and observe that $\int(|\nabla u|^2 + V|u|^2) \leq 1$. Fix $\varepsilon > 0$, set $M = \varepsilon^{-1}$ and take $R > 0$ such that if $|c|_\infty > R$, then $|E_M \cap Q(c, 1)| < \varepsilon$. Clearly we have

$$\int_{Q(c,1)\setminus E_M} |u|^2 \leq \varepsilon \int_{Q(c,1)\setminus E_M} V|u|^2 \leq \varepsilon \int_{Q(c,1)} V|u|^2. \tag{3.2}$$

To estimate the integral over $Q(c, 1) \cap E_M$ for $|c|_\infty > R$, we take $p = (2n)/(n - 2)$ if $n \geq 3$ and any $p > 2$ if $n = 1, 2$. By the Sobolev embedding we have

$$\Big(\int_{Q(c,1)} |u|^p\Big)^{1/p} \leq C\Big(\int_{Q(c,1)} |u|^2 + |\nabla u|^2\Big)^{1/2}$$

with C independent of c. Then we have

$$\int_{Q(c,1)\cap E_M} |u|^2 \leq |E_M \cap Q(c,1)|^{1-2/p}\Big(\int_{Q(c,1)\cap E_M)} |u|^p\Big)^{2/p} \leq C\varepsilon^\gamma \int_{Q(c,1)} (|u|^2 + |\nabla u|^2),$$

with $\gamma = 1 - 2/p$. From the above inequality and (3.2) we deduce that

$$\int_{Q(c,1)} |u|^2 \leq \varepsilon \int_{Q(c,1)} V|u|^2 + C\varepsilon^\gamma \int_{Q(c,1)} (|u|^2 + |\nabla u|^2)$$

hence, summing over a partition of cubes with centres c satisfying $|c|_\infty \geq R$,

$$\int_{|x|_\infty \geq R} |u|^2 \leq \varepsilon \int_{|x|_\infty \geq R} V|u|^2 + C\varepsilon^\gamma \int_{|x|_\infty \geq R} (|u|^2 + |\nabla u|^2)$$

and finally

$$\int_{|x|_\infty \geq R} |u|^2 \leq \frac{\varepsilon + C\varepsilon^\gamma}{1 - C\varepsilon^\gamma}.$$

□

Let us point out a particular case of the previous result.

Corollary 3.2 *If the potential $V > 0$ satisfies the condition*

$$\lim_{|c|\to+\infty} \int_{Q(c,1)} V^{-\alpha} = 0,$$

for some $\alpha > 0$, then B has compact resolvent.

PROOF. In fact we have $E_M = \{x \in \mathbf{R}^n : V^{-\alpha}(x) > M^{-\alpha}\}$ and

$$|E_M \cap Q(c,1)| \leq M^\alpha \int_{Q(c,1)} V^{-\alpha}.$$

□

Notice that there is no need for choosing cubes of side 1: in fact it is easy to see that if (3.1) is satisfied for some $d > 0$, then it is satisfied for all d. We observe also that condition (3.1) is not necessary even for $n = 1$, as can be checked comparing it with Molcanov's theorem (see the next section).

We now consider positive polynomial potentials. Such potentials are analysed in [8] by a "volume counting" argument, which turns out to be equivalent to our condition (3.1). Once a level M has been fixed, Fefferman considers the number $N(M)$ of cubes of side $M^{-1/2}$ centred at $jM^{-1/2}$ ($j \in \mathbf{Z}^n$) contained in E_M. From [8, Theorem II.3], it follows that the spectrum of B is discrete if and only if $N(M)$ is finite for every $M > 0$. Instead, we consider the measure of the intersection of E_M with arbitrary cubes of fixed side. For potentials V of the form

$$V = f \circ p, \tag{3.3}$$

where p is a polynomial and $f : \mathbf{R} \to \mathbf{R}$ is a continuous function satisfying $f(t) \to +\infty$ as $|t| \to +\infty$, we show that condition (3.1) is also necessary for the discreteness of the spectrum of B. We need the following lemma, which shows that a control on the supremum of the polynomial p can be deduced from an estimate of the measure of its sublevels.

Lemma 3.3 *Let $d \in \mathbf{N}$ and $\delta > 0$; then there is constant $C > 0$ such that for every $M > 0$*

$$|\{x \in Q : |p(x)| \leq M\}| \geq \delta \qquad \Longrightarrow \qquad \sup_{x \in Q} |p(x)| \leq CM$$

for every polynomial p of degree less than or equal to d and every unit cube $Q \subset \mathbf{R}^n$.

PROOF. It is sufficient to prove the existence of a constant C for the cube $Q = Q(0,1)$ since then the same constant works for every unit cube, by an elementary translation argument. The linear dependence of the upper bound on M readily follows replacing p with p/M, hence we take $M = 1$. If the statement is false for Q, there is a sequence (q_k) of polynomials of degree at most d such that $|\{x \in Q : |q_k(x)| \leq 1\}| \geq \delta$ and $\|q_k\|_Q := \sup_{x\in Q} |q_k(x)| \to +\infty$. Then, the normalised polynomials $p_k = q_k/\|q_k\|_Q$ satisfy $\|p_k\|_Q = 1$ and $|\{x \in Q : |p_k(x)| \leq \varepsilon_k\}| \geq \delta$, with $\varepsilon_k = 1/\|q_k\|_Q \to 0$, and (up to a subsequence) are uniformly convergent to a polynomial p with $\|p\|_Q = 1$. Setting $F_k = \{x \in Q : |q_k(x)| \leq 1\}$ and $F = \cap_k \cup_{r\geq k} F_r$, we have $F \subset \{x \in Q : p(x) = 0\}$ and $|F| = \lim_k |\cup_{r\geq k} F_r| \geq \delta$, whence $|\{x \in Q : p(x) = 0\}| > 0$ and therefore $p \equiv 0$, in contrast with $\|p\|_Q = 1$. $\square$

Proposition 3.4 *Let V be defined as in (3.3). If the operator $B = -\Delta + V$ has compact resolvent then (3.1) holds.*

PROOF. If condition (3.1) does not hold, we can find M, $\delta > 0$, a sequence of pairwise disjoint cubes $Q(c_k, 1)$ and a positive number R such that

$$|E_M \cap Q(c_k, 1)| = |\{x \in Q(c_k, 1) : |p(x)| \leq R\}| \geq \delta \qquad \forall\ k \in \mathbf{N}.$$

From the preceding lemma we infer that $\sup_{x\in Q(c_k,1)} |p(x)| \leq L$, hence the potential V is uniformly bounded on the sequence $Q(c_k, 1)$. Taking a nonvanishing $u \in C_0^\infty\big(Q(0,1)\big)$, the sequence $\big(u_k(x)\big) = \big(u(x - c_k)\big)$ is bounded in the graph norm and is not relatively compact in L^2, so that B cannot have compact resolvent. $\square$

Finally, we show that the spectrum of $-\Delta + V$, $V = f \circ p$, is discrete if and only if the polynomial p is not independent of some variable, *i.e.* if and only if the only constant vector $c = (c_1, \ldots, c_n)$ such that $\sum_i c_i\, \partial p/\partial x_i \equiv 0$ is $c = (0, \ldots, 0)$.

Theorem 3.5 *Let $B = -\Delta + V$ with V defined in (3.3). Then the resolvent of B is not compact if and only if there is a direction $\omega \in \mathbf{R}^n$ such that $\dfrac{\partial p}{\partial \omega} \equiv 0$.*

PROOF. If $\dfrac{\partial p}{\partial \omega} \equiv 0$ then V is bounded on a strip parallel to ω and the resolvent of B is not compact, by the argument of Proposition 3.4. Suppose now that B has not compact resolvent. Proposition 3.4 yields the existence of a sequence of unit cubes $Q(c_k, 1) \subset \mathbf{R}^n$, with $|c_k| \to +\infty$, on which $|p|$ is uniformly bounded. We write $c_k = t_k \omega_k$ with $|\omega_k| = 1$ and we may assume (up to a subsequence) that $\omega_k \to \omega$, as $k \to \infty$. We show, by induction on $n = \dim \mathbf{R}^n$, that p is constant along the direction ω. Without loss of generality, we may suppose that $\omega = (1, 0, \ldots, 0)$. Of course the statement is true if $n = 1$.

First of all, we observe that for every $d \in \mathbf{N}$ and every multiindex α there is a constant C such that the inequality

$$\sup_Q |D^\alpha r| \leq C \sup_Q |r|$$

holds for every polynomial r of degree at most d and every unit cube Q.

We write

$$p(x) = \sum_{j=0}^{m} x_1^j q_j(x_2, \dots, x_n)$$

with q_j polynomials in the variables $x_2, \dots, x_n$ and $q_m \not\equiv 0$, and show that $m = 0$. Suppose, by contradiction, that $m > 0$; differentiating $(m-1)$-times with respect to x_1 we obtain that the polynomial $p_1(x) = m! x_1 q_m(x_2, \dots, x_n) + (m-1)! q_{m-1}(x_2, \dots, x_n)$ is uniformly bounded on the sequence $\big(Q(c_k, 1)\big)$. Since $q_m \not\equiv 0$, we can find a multiindex α such that $D^\alpha q_m$ is equal to a non-zero constant c. Then the polynomial

$$p_2(x) = D^\alpha p_1 = c_1 x_1 + r_1(x_2, \dots, x_n),$$

with $c_1 = cm!$, would be uniformly bounded on the sequence $\big(Q(c_k, 1)\big)$. Let us write $\omega_k = (\omega_k^1, \omega_k^2) \in \mathbf{R} \times \mathbf{R}^{n-1}$ with $\omega_k^1 \to 1$ and $\omega_k^2 \to 0$, as $k \to \infty$. Since $c_1 x_1$ is unbounded on $\big(Q(c_k, 1)\big)$, $r_1 \not\equiv 0$ and the sequence $(t_k \omega_k^2)$ is unbounded; moreover, the polynomial ∇r_1 is uniformly bounded on $\big(Q(c_k, 1)\big)$ (hence on the $(n-1)$-dimensional unit cubes centred at $t_k \omega_k^2$), as one can see by differentiating p_2 with respect the variables $x_2, \dots, x_n$. By the induction hypothesis we obtain that ∇r_1 is independent of some direction in $\mathbf{R}^{n-1}$, say that of x_2, and hence that $r_1(x_2, \dots, x_n) = c_2 x_2 + r_3(x_3, \dots, x_n)$. Observe now that $c_1 x_1 + c_2 x_2$ is unbounded on $\big(Q(c_k, 1)\big)$ since $\omega_k^1 \to 1$ and $\omega_k^2 \to 0$, as $k \to \infty$. We may therefore iterate the above procedure to obtain finally $p_2(x) = c_1 x_1 + \cdots c_n x_n$. However, this polynomial cannot be uniformly bounded on the sequence $\big(Q(c_k, 1)\big)$ unless $c_1 = 0$ (by the same argument as above). Therefore $m = 0$ and p is independent of x_1. □

We end this section by considering briefly the case of potentials $V = V_+ - V_-$ not necessarily bounded from below. In order to regard V_- as a small perturbation of $-\Delta + V_+$ we assume that the condition

$$\lim_{|c| \to +\infty} \int_{Q(c,1)} V_-^p = 0 \tag{3.4}$$

holds for $p = n/2$ if $n \geq 3$ and for some $p > 1$ if $n = 1, 2$.

We refer to [16] for a discussion of various conditions that allow to apply perturbation methods to general potentials and we only point out that (3.4) is a weak form of the classical condition $V_- \in L^{n/2}$.

Let $\varepsilon > 0$ and take $u \in C_0^\infty$; since $2p' = 2n/(n-2)$ $(n \geq 3)$, using Sobolev embedding, as in the proof of Theorem 3.1, we obtain from (3.4)

$$\int_{Q(c,1)} V_- |u|^2 \leq \Big(\int_{Q(c,1)} V_-^p \Big)^{1/p} \Big(\int_{Q(c,1)} |u|^{2p'} \Big)^{1/p'} \leq \varepsilon \Big(\int_{Q(c,1)} |\nabla u|^2 + (V_+ + 1)|u|^2 \Big),$$

for $|c|$ sufficiently large. Using the boundedness of V_- on compact sets of $\mathbf{R}^n$ we infer that there is a constant $C_\varepsilon > 0$ such that the inequality

$$\int V_- |u|^2 \leq \varepsilon \Big(\int |\nabla u|^2 + V_+ |u|^2 \Big) + C_\varepsilon \int |u|^2$$

holds for every $u \in C_0^\infty$. By density, the same inequality holds if $u \in D\big((-\Delta + V_+)^{1/2}\big) = \big\{ u \in H^1 : |V_+|^{1/2} u \in L^2 \big\}$. The quadratic form

$$q(u) = \int |\nabla u|^2 + (V_+ - V_-)|u|^2$$

is therefore closed and bounded from below on the domain $D\big((-\Delta+V_+)^{1/2}\big)$ and defines a semibounded, self-adjoint operator $-\Delta+V$ (see [3, Theorem 1.8.2]. We generalise Theorem 3.1 in the following proposition.

Proposition 3.6 *Assume that conditions (3.1) and (3.4) hold; then the operator $-\Delta+V$ has compact resolvent.*

PROOF. By Theorem 3.1, $-\Delta+V_+$ has compact resolvent and hence the embedding of $D\big((-\Delta+V_+)^{1/2}\big)$ (endowed with the graph norm) in L^2 is compact. Since the domain of the quadratic form of $-\Delta+V$ is $D\big((-\Delta+V_+)^{1/2}\big)$, the statement follows. □

4 Applications and examples

Let us present some concrete examples of application of the results of the previous sections.

a) Theorem 3.5 shows that the operators $-\Delta+x^2y^2$ and $-\Delta+\big[\sum_{i<j}(x_iy_j-x_jy_i)^2\big]^{1/2}$ have discrete spectra in $L^2(\mathbf{R}^2)$ and $L^2(\mathbf{R}^{2n})$, respectively, even though the potentials do not tend to $+\infty$, as the variables go to ∞ (see [17] for different proofs). Another example to which Theorem 3.5 applies and which seems to be worth mentioning is $-\Delta+|y-x^2|$ in $L^2(\mathbf{R}^2)$.

Using Proposition 3.6, we can even construct polynomial potentials, unbounded from below, such that the corresponding Schrödinger operators have compact resolvents. In fact, we consider in $\mathbf{R}^2$ the polynomials $V(x,y)=x^{2k}y^2-x$ with $k\geq 2$, and observe that (3.1) holds and that $\{V<0\}=\{(x,y)\in\mathbf{R}^2 : x>0,\ |y|<x^{-k+1/2}\}$. For $1<p<k-1/2$ the integral

$$\int_{-x^{-k+1/2}}^{x^{-k+1/2}} |V(x,y)|^p\,dy$$

converges to 0 as $x\to+\infty$ and from this condition (3.4) easily follows. Proposition 3.6 yields the compactness of the resolvent of $-\Delta+V$ in $L^2(\mathbf{R}^2)$.

b) We come back now to the operator

$$Au=-\Delta u+\langle\nabla F,\nabla u\rangle=-e^{F(x)}\mathrm{div}\big(e^{-F(x)}\nabla u\big)$$

of Section 2 and assume that the function $|\nabla F|^2-2\Delta F$ is bounded from below in $\mathbf{R}^n$. By Proposition 2.2 the operator $(A,D(A))$ is unitarily equivalent to the Schrödinger operator $(B,D(B))$ with $V=(1/4)|\nabla F|^2-(1/2)\Delta F$ and hence we obtain the compactness of the resolvent of $(A,D(A))$ in L^2_μ if condition (3.1) is satisfied by V. We specialise our results in the polynomial case.

Proposition 4.1 *Let F be a polynomial such that $|\nabla F|^2-2\Delta F$ is bounded from below in $\mathbf{R}^n$. Then the resolvent of $(A,D(A))$ is not compact in L^2_μ if and only if there is $\omega\in\mathbf{R}^n$, $|\omega|=1$, such that F can be written in the form $F(t\omega+z)=ct\omega+G(z)$, for all $t\in\mathbf{R}$ and $z\perp\omega$, where G is a polynomial in $(n-1)$ variables and $c\in\mathbf{R}$.*

PROOF. We show that the stated representation of F holds if and only if $\dfrac{\partial V}{\partial\omega}\equiv 0$ and conclude, using Theorem 3.5.

If $F(t\omega + z) = c\omega + G(z)$, it is immediate that $\dfrac{\partial V}{\partial \omega} \equiv 0$. Suppose, conversely, that this last equality holds and assume that $\omega = (1, 0, \ldots, 0)$. We write

$$F(x) = \sum_{j=0}^{m} x_1^j q_j(x_2, \ldots, x_n)$$

with q_j polynomials in the variables $x_2, \ldots, x_n$ and $q_m \not\equiv 0$. By assumption, the polynomial $(1/4)|\nabla F|^2 - (1/2)\Delta F$ does not depend on x_1. Comparing the coefficients of maximum degree of the variable x_1 in $(1/4)|\nabla F|^2$ and $(1/2)\Delta F$, one easily obtains that $m \leq 1$ and that q_1 is constant. □

c) Let us point out some one-dimensional examples. In this case it is easy to state Molcanov's characterisation of compactness, which reads as follows: the operator $B = -D^2 + V$, $V \geq 0$, has compact resolvent in L^2 if and only if

$$\lim_{|c| \to +\infty} \int_c^{c+d} V(x)\, dx = +\infty, \qquad \forall d > 0. \tag{4.1}$$

Proposition 4.1 implies the discreteness of the spectrum of the operators $-D^2 + p(x)D$ in L^2_μ for every non-costant polynomial p. In particular, if $p(x) = x^{2k-1}$, $k \in \mathbf{N}$, the measure $d\,\mu(x) = \exp(-x^{2k}/2k)\, dx$ is finite and, if $k = 1$, we obtain the one-dimensional Ornstein-Uhlenbeck operator for which, however, the result is well-known.

Necessary and sufficient conditions for the compactness of the resolvent of the one-dimensional operators $-\alpha D^2 + \beta D$ have been proved also in [13] both in weighted L^2-spaces and in spaces of continuous functions, for general α, β. The methods in [13] are different and do not extend to the multidimensional case. For example, let $\alpha \equiv 1$ and F a primitive of β; if $e^{-F} \in L^1$, so that the measure $d\,\mu(x) = e^{-F(x)}\, dx$ is finite, the operator $-D^2 + \beta D$ has compact resolvent in L^2_μ if and only if

$$\lim_{x \to -\infty} \left(\int_{-\infty}^{x} e^{-F(t)}\, dt \right) \left(\int_0^x e^{F(t)}\, dt \right) = \lim_{x \to +\infty} \left(\int_x^{+\infty} e^{-F(t)}\, dt \right) \left(\int_0^x e^{F(t)}\, dt \right) = 0,$$

a condition that turns out to be equivalent to (4.1) with $V = (1/4)|F'|^2 - (1/2)F''$, even though this does not seem evident at first sight.

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