M. V. Keldysh Institute of Applied Mathematics
Russian Academy of Sciences

R. Denk, R. Mennicken, and L. R. Volevich

BOUNDARY VALUE PROBLEMS FOR A CLASS OF ELLIPTIC OPERATOR PENCILS

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Abstract

R. Denk, R. Mennicken, and L. R. Volevich[1]. Boundary value problems for a class of elliptic operator pencils.

In this paper operator pencils $A(x, D, \lambda)$ are studied which act on a manifold with boundary and satisfy the condition of N-ellipticity with parameter, a generalization of the notion of ellipticity with parameter as introduced by Agmon and Agranovich–Vishik. Sobolev spaces corresponding to a Newton polygon are defined and investigated; in particular it is possible to describe their trace spaces. With respect to these spaces, an a priori estimate holds for the Dirichlet boundary value problem connected with an N-elliptic pencil, and a right parametrix is constructed.

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1. Introduction

In this paper we consider operator pencils of the form

$$A(x,D,\lambda) = A_{2m}(x,D) + \lambda A_{2m-1}(x,D) + \cdots + \lambda^{2m-2\mu} A_{2\mu}(x,D) \quad (1.1)$$

acting on a smooth manifold M with smooth boundary Γ. Here m and μ are integer numbers with $m > \mu \geq 0$, $A_{2\mu}, \ldots, A_{2m}$ are partial differential operators in M with infinitely smooth coefficients and λ is a complex parameter. We assume that

$$A_j(x,D) = \sum_{|\alpha|\leq j} a_{\alpha j}(x) D^\alpha \quad (j = 2\mu,\, 2\mu+1, \ldots, 2m) \qquad (1.2)$$

is a differential operator of order j with scalar coefficients $a_{\alpha j}(x) \in C^\infty(\overline{M})$. As usual, we use for multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ the notation

$$D^\alpha = D_1^{\alpha_1} \cdots D_n^{\alpha_n}\,, \quad D_j = -i \frac{\partial}{\partial x_j}\,, \quad |\alpha| = \alpha_1 + \ldots + \alpha_n\,. \qquad (1.3)$$

There is a close connection between pencils of the form (1.1) and general parabolic problems which we want to describe briefly. An important tool in the field of elliptic and parabolic problems is the concept of the Newton polygon. For a given polynomial

$$P(\xi,\lambda) = \sum_{\alpha,k} a_{\alpha k} \xi^\alpha \lambda^k\,, \qquad (1.4)$$

where $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$, let $\nu(P)$ be the set of all integer points (i,k) such that an α exists with $|\alpha| = i$ and $p_{\alpha k} \neq 0$. Then the Newton polygon $N(P)$ is defined as the convex hull of all points in $\nu(P)$, their projections on the coordinate axes and the origin. The polynomial $P(\xi,\lambda)$ is called N-parabolic (see [8], Chapter 2) if $N(P)$ has no sides parallel to the coordinate axes and if the inequality

$$|P(\xi,\lambda)| > \delta \sum_{(i,k)\in N(P)} |\xi|^i \, |\lambda|^k \qquad (1.5)$$

holds for all $\lambda \in \mathbb{C}$ with $\operatorname{Im}\lambda < \lambda_0$ where $\delta > 0$ and λ_0 are constants. An important example of such polynomials is the product of polynomials $P_1(\xi,\lambda), \ldots, P_N(\xi,\lambda)$ which are quasi-homogeneous and $2b_j$-parabolic in the sense of Petrovskii with different weights $2b_j$ $(j = 1, \ldots, N)$. Note that in this case $P(\xi,\lambda)$ is no quasi-homogeneous function in (ξ,λ). Similarly (see [5]), the polynomial $P(\xi,\lambda)$ is called N-elliptic with parameter along some ray $\mathcal{L}$ of the complex plane if (1.5) holds for all $\xi \in \mathbb{R}^n$ and all $\lambda \in \mathcal{L}$, $|\lambda| \geq R$, with large enough R. This type of polynomials appears, for instance, if one considers Douglis–Nirenberg systems $A(x,D) - \lambda I$ which are elliptic with parameter. On manifolds without boundary Douglis–Nirenberg systems were investigated by Kozhevnikov [9] and by the authors in [5]. It turned out that an equivalent condition for unique solvability of such a system and sharp a priori estimate is the condition that for every x the determinant

$$P(x,\xi,\lambda) = \det(A(x,D) - \lambda I)$$

satisfies inequality (1.5).

The basic idea of the Newton polygon method for the problems mentioned above is to assign to λ various weights r_j which are defined by the Newton polygon. For each of these weights we obtain a different principal part of $P(\xi,\lambda)$ which we denote by $P_{r_j}(\xi,\lambda)$. On a manifold without boundary there is a finite open covering $\{U_j\}_j$ of the set of all (ξ,λ) and a corresponding partition of unity $\sum_j \psi_j(\xi,\lambda) \equiv 1$ such that $P(D,\lambda)\psi_j(D,\lambda)$ differs from the corresponding principal part $P_{r_j}(D,\lambda)\psi_j(D,\lambda)$ only by a small regular perturbation. This allows estimates and existence results for the operators $P(D,\lambda)$, cf. [5] for N-elliptic systems and [8] for parabolic problems.

Now let us consider the same problems on a manifold with boundary. For instance, let $P(D,\lambda)$ be the product of two operators which are parabolic in the sense of Petrovskii, i.e.

$$P(D,\lambda) = (\lambda + A_{2p}(D))\,(\lambda + A_{2q}(D))\,,$$

where $\lambda + A_{2p}(D)$ and $\lambda + A_{2q}(D)$ are $2p$- and $2q$-parabolic operators, respectively, with $p > q$. If, for instance, we assign to λ the weight $r_1 = 2q$, we obtain the principal part $P_{r_1}(D,\lambda) = A_{2p}(D)A_{2q}(D) + \lambda A_{2p}(D)$ which is of the form (1.1). If we take the weight r_2 with $2q < r_2 < 2p$ the corresponding principal part is $P_{r_2}(D,\lambda) = \lambda A_{2p}(D)$. The operator $P_{r_2}(D,\lambda)$ has to be supplied with p boundary conditions while the operator $P(D,\lambda)$ needs $p+q$ boundary conditions. Thus we can see here that $P(D,\lambda)$ is (after dividing by λ) a singular perturbation of the principal part $P_{r_2}(D,\lambda)$. A similar situation occurs if the weight of λ is larger than $2p+2q$.

So we see from this example that operator pencils of the form (1.1) and singular perturbations naturally arise when we deal with N-parabolic problems on manifolds with boundary. If we consider boundary value problems elliptic in the sense of Douglis–Nirenberg, the situation is the same or even more complicated.

As a first step to handle these problems we consider as a model problem operator pencils of the form (1.1) and the corresponding Dirichlet problem on manifolds with boundary. The aim of this paper is to show that the Newton polygon provides an easy and direct approach to the Sobolev spaces where the pencil acts and to the proof of a priori estimate. In particular, we obtain a description of the trace spaces which is formulated in the general context of Sobolev spaces corresponding to Newton polygons. We hope to study in a subsequent paper boundary value problems for general N-parabolic operators on manifolds with boundary.

For $r,s\in\mathbb{R}$ let the Sobolev space $H^{(r,s)}(\mathbb{R}^n)$ be defined by

$$H^{(r,s)}(\mathbb{R}^n) = \{u\in\mathcal{S}'(\mathbb{R}^n) : (|\xi|^2+1)^{s/2}(|\xi|^2+|\lambda|^2)^{(r-s)/2}Fu(\xi)\in L_2(\mathbb{R}^n)\} \tag{1.6}$$

where Fu denotes the Fourier transform of u. The norm in $H^{(r,s)}(\mathbb{R}^n)$ is given by

$$\|u\|_{r,s} := \left(\int_{\mathbb{R}^n}(|\xi|^2+1)^s(|\xi|^2+|\lambda|^2)^{r-s}|Fu(\xi)|^2\,d\xi\right)^{\frac{1}{2}}. \tag{1.7}$$

Restricting the distributions belonging to $H^{(r,s)}(\mathbb{R}^n)$ to the right half space $\mathbb{R}^n_+ := \{x=(x',x_n)\in\mathbb{R}^n : x_n>0\}$, we obtain the Sobolev space $H^{(r,s)}(\mathbb{R}^n_+)$. See Section 2 for the description of the norm in this Sobolev space. In the standard way we can also define $H^{(r,s)}(M)$ using local coordinates.

For every r and s the operator pencil (1.1) acts continuously from $H^{(r,s)}$ to $H^{(r-2m,s-2\mu)}$. In what follows in connection with the Dirichlet problem for (1.1) we will restrict ourselves to the case $r=m, s=\mu$, i.e. we will realize (1.1) as an operator from $H^{(m,\mu)}$ onto $H^{(-m,-\mu)}$. We will assume this pencil to be elliptic with parameter along the ray $[0,\infty)$ in the following sense: Denote by

$$A_j^{(0)}(x,\xi) := \sum_{|\alpha|=j} a_{\alpha j}(x)\xi^\alpha \quad (j=2\mu,\dots,2m) \tag{1.8}$$

the principal symbol of A_j, where $\xi^\alpha=\xi_1^{\alpha_1}\cdots\xi_n^{\alpha_n}$ for $\xi=(\xi_1,\dots,\xi_n)=(\xi',\xi_n)$, and by

$$A^{(0)}(x,\xi,\lambda) := A^{(0)}_{2m}(x,\xi)+\lambda A^{(0)}_{2m-1}(x,\xi)+\ldots+\lambda^{2m-2\mu}A^{(0)}_{2\mu}(x,\xi) \tag{1.9}$$

the principal symbol of $A(x,D,\lambda)$. Then our main assumption is that the estimate

$$|A^{(0)}(x,\xi,\lambda)| \geq C|\xi|^{2\mu}\,(\lambda+|\xi|)^{2m-2\mu} \quad (\xi\in\mathbb{R}^n, \lambda\in[0,\infty), x\in\overline{M}) \tag{1.10}$$

holds where the constant C does not depend on x,ξ or λ. In the case $\mu = 0$ this is the usual definition of ellipticity with parameter which was introduced by Agmon [1] and Agranovich–Vishik [3]. Therefore we may assume in the following that $\mu > 0$. In this case even for $\lambda \neq 0$ the principal symbol $A^{(0)}(x,\xi,\lambda)$ vanishes for $\xi = 0$ which causes the main difficulties in proving existence results and estimates. Note that the symbol $A^{(0)}(x,\xi,\lambda)$ is homogeneous in ξ and λ of degree $2m$, as it is the case for the problems treated in [3].

We will consider boundary value problems in $\mathbb{R}^n$, $\mathbb{R}^n_+$ and M. For this we will describe the space of traces of functions $u\in H^{(m,\mu)}(\mathbb{R}^n_+)$, i.e. the space

$$\{D_n^{j-1}u(x',0) : u\in H^{(m,\mu)}(\mathbb{R}^n_+)\} \quad \text{for } j=1,\dots,m\,. \tag{1.11}$$

This will be done in a more general context in Section 2 where Sobolev spaces corresponding to Newton polygons are considered. The space $H^{(m,\mu)}(\mathbb{R}^n_+)$ appears to be a special case of the space $H^{\Xi}(\mathbb{R}^n_+)$ where $\Xi(\xi,\lambda)$ is the weight function corresponding to the Newton polygon $N(P)$ of a polynomial $P(\xi,\lambda)$ in $\xi\in\mathbb{R}^n$ and $\lambda\in\mathbb{C}$. It turns out that the trace space $\{D_n^{j-1}u(x',0) : u\in H^{\Xi}(\mathbb{R}^n_+)\}$ is given by $H^{\Xi^{(-j+\frac12)}}(\mathbb{R}^{n-1})$ where $\Xi^{(-j+\frac12)}(\xi',\lambda)$ denotes the weight function corresponding to the Newton polygon which is constructed from $N(P)$ by a shift of length $j-\frac12$ to the left. Cf. Section 2 for details. In particular, in the case of the operator pencil (1.1) the trace spaces have the form $H^{(m_j,\mu_j)}(\mathbb{R}^{n-1})$ where the parameters m_j and μ_j can be seen directly from the corresponding Newton polygon.

In Section 5 we consider the Dirichlet boundary problem

$$A(x,D,\lambda)\,u(x) = f(x) \quad \text{in } M\,, \tag{1.12}$$

$$\Big(\frac{\partial}{\partial\nu}\Big)^{j-1} u(x) = g_j(x) \quad (j=1,\dots,m) \ \text{ on } \Gamma\,. \tag{1.13}$$

where $\frac{\partial}{\partial\nu}$ denotes the derivative in the direction of the inner normal to the boundary. The main theorem states that for every solution $u\in H^{(m,\mu)}(M)$ of the boundary value problem (1.12)–(1.13) the a priori estimate

$$\|u\|_{m,\mu} \leq C\Big(\|f\|_{-m,-\mu} + \sum_{j=1}^{m} \|g_j\|_{m_j,\mu_j} + \lambda^{m-\mu}\|u\|_{L_2(M)}\Big) \tag{1.14}$$

holds for $\lambda \geq \lambda_0$ with a constant C not depending on λ or u. The proof of this theorem is essentially based on estimates of the solution of an ordinary differential equation which arises from (1.12)–(1.13) by fixing $x \in \Gamma$, rewriting the boundary value problem in coordinates corresponding to x and taking the partial Fourier transform with respect to the first $n-1$ variables. Estimates for the fundamental solution of the resulting ordinary differential equation can be found in Section 4 and use the precise knowledge about the zeros of the principal symbol $A^{(0)}(x, \xi, \lambda)$ considered as a polynomial in ξ_n.

These zeros can (for large λ) be arranged in two groups, one group remaining bounded for $\lambda \to \infty$, the other group of zeros being exactly of order $O(\lambda)$ for $\lambda \to \infty$. To obtain this result we have to impose an additional condition on the principal symbol $A^{(0)}(x, \xi, \lambda)$ which is the analogue of the condition of regular degeneration which is known from the theory of singular perturbations (cf. Vishik-Lyusternik [13]). The details can be found in Section 3.

As mentioned above, there is a close connection between pencils of the form (1.1) and elliptic boundary value problems with small parameter. Nazarov obtained in [12] a priori estimates under the assumption that the fundamental solutions fulfill some estimates which are similar to those proved in Section 4 below. (The norms used in [12] differ slightly from the norms used in the present paper.) In several papers Frank and other authors investigated singular perturbed problems and corresponding a priori estimates, cf. [6] and the references therein. The use of the Newton polygon method which gives the connection to general parabolic problems as described above, seems to be new even for singular perturbed problems.

2. Newton's polygon and functional spaces corresponding to it

In this section we consider a polynomial $P(\xi, \lambda)$ of the form (1.4) and its Newton polygon $N(P)$ which was defined in the Introduction. For a detailed discussion of the Newton polygon, we refer the reader to Gindikin-Volevich [8], Chapters 1 and 2.

To construct function spaces corresponding to the Newton polygon, we consider the weight function

$$\Xi_P(\xi,\lambda) := \sum_{(i,k)\in N(P)} |\xi|^i\,|\lambda|^k\,, \tag{2.1}$$

where the summation on the right-hand side is extended over all integer points of $N(P)$. The Sobolev space H^Ξ will arise as a special case of the following more general definition which is taken from Volevich-Paneah [15]. It can be seen directly that the function $\sigma(\xi) := \Xi_P(\xi,\lambda)$ satisfies the condition which appears in this definition (cf. also Remark 2.4 below). In the following, the Fourier transform F is defined by

$$Fu(\xi) = \frac{1}{(2\pi)^{\frac{n}{2}}}\int_{\mathbb{R}^n} e^{-ix\cdot\xi}u(x)\,dx \tag{2.2}$$

for $u \in \mathcal{S}(\mathbb{R}^n)$, the definition is extended in the usual way to distributions $u \in \mathcal{S}'(\mathbb{R}^n)$.

Definition 2.1 Let $\sigma(\xi)$ be a continuous function on $\mathbb{R}^n$ with values in $\mathbb{R}_+$ and assume that $\sigma(\xi)\sigma^{-1}(\eta) \le C(1+|\xi-\eta|^N)$ holds for all $\xi,\eta \in \mathbb{R}^n$ with constants C and N not depending on ξ or η. Then H^σ is defined as the space of all distributions u in $\mathcal{S}'(\mathbb{R}^n)$ such that $\sigma(\xi)Fu(\xi) \in L_2(\mathbb{R}^n)$. The space H^σ is endowed with the norm

$$\|u\|_{\sigma,\mathbb{R}^n} := \Big(\int_{\mathbb{R}^n} \sigma^2(\xi)|Fu(\xi)|^2\,d\xi\Big)^{1/2}. \tag{2.3}$$

Proposition 2.2 (See Volevich-Paneah [15].) *Let $\sigma(\xi,\lambda)$ be a continuous function of ξ and assume that*

$$\sigma(\xi,\lambda)\sigma^{-1}(\eta,\lambda) \le C_1(1+|\xi-\eta|^N)$$

holds with a constant C_1 not depeding on ξ, η or λ. Let

$$\sigma_l'(\xi',\lambda) := \Big(\int_{-\infty}^{\infty} \frac{\xi_n^{2l}}{\sigma^2(\xi,\lambda)}\,d\xi_n\Big)^{-1/2} < \infty\,.$$

Then $D_n^l u(x',0)$ is well-defined as an element of $H^{\sigma_l'}(\mathbb{R}^{n-1})$ for every $u \in H^\sigma(\mathbb{R}^n)$, and there exists a constant C, independent of u and λ, such that

$$\|D_n^l u(x',0)\|_{\sigma_l',\mathbb{R}^{n-1}} \le C\|u\|_{\sigma,\mathbb{R}^n}\,. \tag{2.4}$$

We will apply Proposition 2.2 to the case where $\sigma(\xi,\lambda)$ is given by $\Xi_P(\xi,\lambda)$ (see (2.1)).

Let one of the functions $\sigma(\xi,\lambda)$ or $\sigma_1(\xi,\lambda)$ for each λ satisfy the condition of Definition 2.1 and $\sigma(\xi,\lambda) \approx \sigma_1(\xi,\lambda)$. The symbol $\approx$ means that there exist positive constants C_1 and C_2, independent of ξ and λ, such that

$$C_1\sigma(\xi,\lambda) \le \sigma_1(\xi,\lambda) \le C_2\sigma(\xi,\lambda)\,.$$

Then the other function also satisfies the condition of Definition 2.1 and, evidently, the statement of Proposition 2.2 remains valid, if we replace σ by the equivalent function σ_1. In the following we will construct an equivalent function for $\Xi_P(\xi,\lambda)$ (cf. [5], Section 2). For this purpose we introduce some simple geometric notions connected with the Newton polygon (see, e.g., [8], Chapter 1).

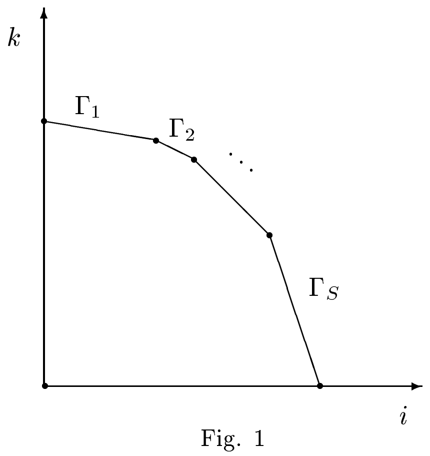

Fig. 1

Let $\Gamma_1,\dots,\Gamma_S$ be the sides of the Newton polygon not lying on the coordinate axes and indexed in the clockwise direction (cf. Fig. 1). Suppose that

$$(0,0),\ (a_1,b_1),\dots,(a_{S+1},b_{S+1})\,,\quad a_1=0,\quad b_{S+1}=0\,,$$

are the vertices of the polygon $N(P)$. Then the side Γ_s is given by

$$\Gamma_s=\{(a,b)\in\mathbb{R}^2 : 1\cdot a + r_s\cdot b = d_s\}\quad (s=1,\dots,S) \qquad (2.5)$$

where $r_s = (a_{s+1} - a_s)/(b_s - b_{s+1})$. The vector $(1, r_s)$ is an exterior normal to the side Γ_s, where we admit $r_1 = \infty$ if Γ_1 is horizontal. Further we have $r_S = 0$ in the case that Γ_S is vertical. In what follows we will suppose that Γ_S is not vertical. Since $N(P)$ is convex, we have

$$\infty \geq r_1 > \ldots > r_S > 0 .$$

The r_s-principal part of P is defined by

$$P_{r_s}(\xi, \lambda) := \sum_{|\alpha| + r_s k = d_s} a_{\alpha k} \xi^\alpha \lambda^k . \tag{2.6}$$

Here d_s is the so-called r_s-degree of P which may be defined by

$$d_s := \max_{(a,b) \in N(P)} (1 \cdot a + r_s \cdot b) . \tag{2.7}$$

Now we set

$$\Xi_{(s)}(\xi, \lambda) = |\xi|^{-a_s} \, |\lambda|^{-b_{s+1}} \sum_{i + r_s k = d_s} |\xi|^i |\lambda|^k .$$

This function will be a polynomial of $|\xi|$ and $|\lambda|$.

Repeating the argument in [8], Theorem 1.1.3, we can prove that

$$\prod_{s=1}^{S} \Xi_{(s)}(\xi, \lambda) = \sum_{s=1}^{S} |\xi|^{a_s} |\lambda|^{b_s} + \ldots , \tag{2.8}$$

where the dots denote the sum of monomials $|\xi|^i |\lambda|^k$ with $(i, k) \in N(P)$. For $|\lambda| \geq 1$ the right-hand side can be estimated from below by

$$1 + \sum_{s=1}^{S} |\xi|^{a_s} |\lambda|^{b_s} .$$

This function can be estimated from below by $\Xi_P(\xi, \lambda)$ (see [5], Subsection 3.2). From this it follows that the left-hand side of (2.8) is equivalent to Ξ_P. Denote by $2m_s$ the largest degree of $|\xi|$ in $\Xi_{(s)}$. It is obvious that $\Xi_{(s)}$ is equivalent to $(|\xi| + |\lambda|^{\frac{1}{r_s}})^{2m_s}$, and consequently

$$\Xi_P(\xi, \lambda) \approx \prod_{s=1}^{S} \left(|\xi|^2 + |\lambda|^{\frac{2}{r_s}} \right)^{m_s} . \tag{2.9}$$

We will suppose further, as in the case of parabolic polynomials (cf. [8], Chapter 2), that $m_1, \ldots, m_S$ are integers.

Remark 2.3 In the case $r_1 = \infty$ (i.e. Γ_1 is horizontal) (2.6) and (2.7) have no sense and (2.6) should be replaced by

$$P_{r_1} := \sum_{|\alpha|=a_2} a_{\alpha b_1} \xi^\alpha \lambda^{b_1} \,.$$

As for the equivalence (2.9), it will be valid for $|\lambda| > \lambda_0$ with arbitrary $\lambda_0 > 0$ and the equivalence constants, of course, depend on λ_0.

Remark 2.4 The fact that $\Xi(\xi, \lambda)$ satisfies the condition of Definition 2.1 is an immediate consequence of (2.9) as this condition is fulfilled for each factor on the right-hand side.

Remark 2.5 From (2.9) it follows that the r_s-degree d_s (cf. (2.7)) is given by

$$d_s = 2\left(\sum_{j=1}^{s} m_j + \sum_{j=s+1}^{S} \frac{r_s}{r_j} m_s\right) . \tag{2.10}$$

To see this, we use the relation

$$\Xi_P(t\xi, t^{r_s}\lambda) = t^{d_s} \Xi_{P_{r_s}}(\xi, \lambda) + o(t^{d_s}), \quad t \to +\infty \,, \tag{2.11}$$

cf. [8], Section 1.1.2. In our case we obtain, denoting the right-hand side of (2.10) by d_s',

$$\begin{aligned}
\Xi_P(t\xi, t^{r_s}\lambda) &= \prod_{j=1}^{S} \left(t^2|\xi|^2 + t^{2\frac{r_s}{r_j}} |\lambda|^{\frac{2}{r_j}} \right)^{m_j} \\
&= t^{d_s'} \prod_{j=1}^{s} \left(|\xi|^2 + t^{2(\frac{r_s}{r_j}-1)} |\lambda|^{\frac{2}{r_j}} \right)^{m_j} \prod_{j=s+1}^{S} \left(t^{2(1-\frac{r_s}{r_j})} |\xi|^2 + |\lambda|^{\frac{2}{r_j}} \right)^{m_j} \\
&= t^{d_s'} \Xi_{P_{r_s}}(\xi, \lambda) + o(t^{d_s'}) \,,
\end{aligned}$$

which shows $d_s = d_s'$.

Now we will describe the trace spaces of the spaces H^Ξ. For this we use the following lemma:

Lemma 2.6 *Let $1 \le a_1 < a_2 < \ldots < a_S < \infty$ and $m_1, \ldots, m_S \in \mathbb{N}$. For $l \in \mathbb{N}$ with $0 \le l < 2(m_1 + \ldots + m_S)$ define the index κ by*

$$2m_1 + \ldots + 2m_{\kappa-1} \le l < 2m_1 + \ldots + 2m_\kappa \,. \tag{2.12}$$

Then there exists a constant $C>0$, independent of $a_1,\dots,a_S$, such that

$$
\begin{aligned}
C^{-1} a_\kappa^{2l+1-4m_1-\ldots-4m_\kappa} \prod_{s=\kappa+1}^{S} a_s^{-4m_s} &\le \int_{-\infty}^{\infty} \frac{t^{2l}}{\prod_{s=1}^{S}(t^2+a_s^2)^{2m_s}}\,dt \\
&\le C a_\kappa^{2l+1-4m_1-\ldots-4m_\kappa} \prod_{s=\kappa+1}^{S} a_s^{-4m_s}\,.
\end{aligned}
\tag{2.13}
$$

In the case $0\le l<2m_1$, we set $m_0=0$ in (2.12).

Proof. Substituting in the integral $t=a_S\tau$, we obtain

$$
\begin{aligned}
I &:= \int_{-\infty}^{\infty} t^{2l} \prod_{s=1}^{S} (t^2+a_s^2)^{-2m_s}\,dt \\
&= 2a_S^{2l+1-4m_1-\ldots-4m_S} \int_0^\infty t^{2l} \prod_{s=1}^{S} \left(t^2+\left(\frac{a_s}{a_S}\right)^2\right)^{-2m_s} dt.
\end{aligned}
$$

For $t\ge 1$ we use

$$
t^{2l}(1+t^2)^{-2m_1-\ldots-2m_S} \le t^{2l} \prod_{s=1}^{S} \left(t^2+\left(\frac{a_s}{a_S}\right)^2\right)^{-2m_s} \le t^{2l-4m_1-\ldots-4m_S}\,.
$$

As $l<2\sum_{s=1}^S m_s$, the left-hand and right-hand side of this inequality are integrable functions over $[1,\infty)$, and we obtain

$$
C_1^{-1} \le \int_1^\infty t^{2l} \prod_{s=1}^{S} \left(t^2+\left(\frac{a_s}{a_S}\right)^2\right)^{-2m_s} dt \le C_1
$$

for some $C_1>0$.

For $0\le t\le 1$ we have $1\le 1+t^2\le 2$, and therefore

$$
\int_0^1 \ldots dt \approx \int_0^1 t^{2l} \prod_{s=1}^{S-1} \left(t^2+\frac{a_s^2}{a_S^2}\right)^{-2m_s} dt\,.
$$

Now we substitute $t=\frac{a_{S-1}}{a_S}\tau$ and get

$$
\int_0^1 \ldots dt \approx \left(\frac{a_{S-1}}{a_S}\right)^{2l+1-4m_1-\ldots-4m_{S-1}} \int_0^{\frac{a_S}{a_{S-1}}} t^{2l} \prod_{s=1}^{S-1} \left(t^2+\frac{a_s^2}{a_{S-1}^2}\right)^{-2m_s} dt\,.
$$

Again we split up $\int_0^{\frac{a_S}{a_{S-1}}} \ldots = \int_0^1 \ldots + \int_1^{\frac{a_S}{a_{S-1}}} \ldots$ and use an estimate of the form $C_2^{-1} \leq \int_1^{\frac{a_S}{a_{S-1}}} \ldots \leq C_2$ for the second integral.

Proceding in this way, we receive

$$\begin{aligned} I \quad &\approx \quad a_S^{2l+1-4m_1-\ldots-4m_S} \left(\frac{a_{S-1}}{a_S}\right)^{2l+1-4m_1-\ldots-4m_{S-1}} \cdot \ldots \\ &\cdot \left(\frac{a_\kappa}{a_{\kappa+1}}\right)^{2l+1-4m_1-\ldots-4m_\kappa} \int_0^{\frac{a_{\kappa+1}}{a_\kappa}} t^{2l} \prod_{s=1}^{\kappa} \left(t^2 + \frac{a_s^2}{a_\kappa^2}\right)^{-2m_s} dt\,. \end{aligned}$$

For the last integral we use

$$\begin{aligned} t^{2l}(t^2+1)^{-2m_1-\ldots-2m_\kappa} \quad &\leq \quad t^{2l} \prod_{s=1}^{\kappa} \left(t^2 + \frac{a_s^2}{a_\kappa^2}\right)^{-2m_s} \\ &\leq \quad t^{2l-4m_1-\ldots-4m_{\kappa-1}}(t^2+1)^{-2m_\kappa}\,. \end{aligned}$$

As $2m_1 + \ldots + 2m_{\kappa-1} \leq l < 2m_1 + \ldots + 2m_\kappa$, the left-hand and the right-hand side of this inequality are integrable functions on $[0,\infty)$. Therefore

$$I \approx a_\kappa^{2l+1-4m_1-\ldots-4m_\kappa} a_{\kappa+1}^{-4m_{\kappa+1}} \cdot \ldots \cdot a_S^{-4m_S}\,.$$

$\square$

Remark 2.7 Using the substitution $t = a_1\tau$, it is easily seen that the condition $a_1 \geq 1$ in Lemma 2.6 may be replaced by $a_1 > 0$.

As in the Introduction, we denote by $\Xi_P^{(-l)}(\xi,\lambda)$ the function corresponding to the Newton polygon which is constructed from $N(P)$ by a shift of length l to the left parallel to the abscissa. We preserve the notation $H^{\Xi_P^{(-l)}}(\mathbb{R}^{n-1})$ for the spaces in $\mathbb{R}^{n-1}$ corresponding to the weight functions $\Xi_P^{(-l)}(\xi',\lambda) := \Xi_P^{(-l)}(\xi',0,\lambda)$.

Lemma 2.8 *Let $\lambda_0 > 0$. Then for $|\lambda| \geq \lambda_0$ we have*

$$\sigma_l'(\xi',\lambda) \approx \Xi^{(-l-\frac{1}{2})}(\xi',\lambda)\,, \tag{2.14}$$

where σ_l' is defined by

$$\sigma_l'(\xi',\lambda) := \left(\int_{-\infty}^{\infty} \frac{\xi_n^{2l}}{\Xi_P^2(\xi,\lambda)}\, d\xi_n\right)^{-\frac{1}{2}}\,. \tag{2.15}$$

Proof. Instead of Ξ_P we use the right-hand side of (2.9). From Lemma 2.6 with $a_s^2 = |\xi'|^2 + |\lambda|^{\frac{2}{r_s}}$ we obtain (see Remark 2.7) that

$$\sigma_l'(\xi',\lambda) \approx \left(|\xi'|^2 + |\lambda|^{\frac{2}{r_\kappa}}\right)^{m_1+\ldots+m_\kappa-\frac{l}{2}-\frac{1}{4}} \prod_{s=\kappa+1}^{S} \left(|\xi'|^2 + |\lambda|^{\frac{2}{r_s}}\right)^{m_s}, \tag{2.16}$$

where κ is chosen according to Lemma 2.6. From Remark 2.5 applied to $\sigma_l'(\xi',\lambda)$ we see that the sides of the Newton polygon corresponding to the weight function (2.16) are given by

$$\Gamma_j = \{(a,b) \in \mathbb{R}^2 : a + r_j b = d_j'\}$$

with $d_j' = d_j - l - \frac{l}{2}$ $(j = \kappa, \ldots, S)$. But this means that the Newton polygon for σ_l' is constructed from $N(P)$ by a shift of $l + \frac{1}{2}$ to the left, i.e. we have $\sigma_l'(\xi',\lambda) \approx \Xi_P^{(-l-\frac{1}{2})}(\xi',\lambda)$. □

The following theorem is an immediate consequence of Proposition 2.2 and Lemma 2.8.

Theorem 2.9 *For every $\lambda_0 > 0$ there exists a constant $C > 0$, independent of u and λ, such that*

$$\|D_n^l u(x',0)\|_{\Xi_P^{(-l-\frac{1}{2})},\mathbb{R}^{n-1}} \le C\|u\|_{\Xi_P,\mathbb{R}^n} \quad (l = 0,\ldots,2m_1+\ldots+2m_S-1) \tag{2.17}$$

holds for $u \in H^{\Xi_P}(\mathbb{R}^n)$ and $\lambda \in \mathbb{C}$ with $|\lambda| \ge \lambda_0$.

In the following, we will also consider the function spaces in the half space $\mathbb{R}^n_+$ which correspond to Newton polygons. Using the binomial formula, it is easily seen that

$$\Xi_P^2(\xi,\lambda) \approx \sum_{l=0}^{M} \xi_n^{2l} \left(\Xi_P^{(-l)}(\xi',\lambda)\right)^2 \tag{2.18}$$

where $M = 2m_1 + \ldots + 2m_S$. From this it follows that we can take

$$\Big(\sum_{l=0}^{M} \int_{-\infty}^{\infty} \|(D_n^l u)(\cdot,x_n)\|^2_{\Xi_P^{(-l)},\mathbb{R}^{n-1}}\, dx_n\Big)^{1/2} \tag{2.19}$$

as an equivalent norm in $H^{\Xi_P}(\mathbb{R}^n)$. Replacing the integral over $\mathbb{R}$ by the integral over $x_n \ge 0$ we define a norm in $H^{\Xi_P}(\mathbb{R}^n_+)$.

To define the space $H^{\frac{1}{\equiv p}}(\mathbb{R}^n_+)$, we use the more general approach which can be found, e.g., in [15]. Let $\sigma(\xi)$ be a weight function fulfilling the condition in Definition 2.1. Denote by $H^\sigma(\mathbb{R}^n)_\pm$ the subspace of $H^\sigma(\mathbb{R}^n)$ consisting of elements with supports in the closure of $\mathbb{R}^n_\pm$. Then we define

$$H^\sigma(\mathbb{R}^n_+) = H^\sigma(\mathbb{R}^n)/H^\sigma(\mathbb{R}^n)_- \tag{2.20}$$

endowed with the natural quotient norm

$$\|f\|_{\sigma,\mathbb{R}^n_+} = \inf_{f_-\in H^\sigma(\mathbb{R}^n)_-} \|f_0 + f_-\|_{\sigma,\mathbb{R}^n}\,, \tag{2.21}$$

where f_0 is an arbitrary representative of the conjugacy class of f.

Suppose that $\sigma(\xi', \xi_n)$ for a fixed $\xi' \in \mathbb{R}^{n-1}$ can be extended as a holomorphic function in ξ_n of polynomial growth in the lower half-plane $\operatorname{Im}\xi_n < 0$. In this case the quotient norm of $f \in H^\sigma(\mathbb{R}^n_+)$ coincides with the norm

$$\|\sigma(D', D_n) f_0\|_{L_2(\mathbb{R}^n_+)} \tag{2.22}$$

which does not depend on the choice of the element f_0 in the conjugacy class. In (2.22) the pseudo-differential operator (ps.d.o.) $\sigma(D', D_n) = \sigma(D)$ is defined by

$$\sigma(D)f := F^{-1}\sigma(\xi)(Ff)(\xi)$$

In the case when

$$\sigma \approx \prod_{j=1}^{S} (|\xi|^2 + |\lambda|^{2/r_j})^{m_j}$$

we replace σ in the definition of $H^\sigma(\mathbb{R}^n_+)$ by

$$\prod_{j=1}^{S} \left(i\xi_n + (|\xi|^2 + |\lambda|^{2/r_j})^{1/2}\right)^{2m_j}.$$

3. The zeros of the symbol

Now we come back to the operator pencil (1.1) and consider the corresponding model problem with constant coefficients and without lower order terms. Let $A(\xi, \lambda)$ be a polynomial in $\xi \in \mathbb{R}^n$ and $\lambda \in \mathbb{C}$ of the form

$$A(\xi, \lambda) = A_{2m}(\xi) + \lambda A_{2m-1}(\xi) + \ldots + \lambda^{2m-2\mu} A_{2\mu}(\xi)\,, \tag{3.1}$$

where $A_j(\xi)$ is a homogeneous polynomial in ξ of degree j.

Definition 3.1 The polynomial $A(\xi,\lambda)$ is called N-elliptic with parameter in $[0,\infty)$ if the estimate

$$|A(\xi,\lambda)| \geq C\,|\xi|^{2\mu}\,(\lambda+|\xi|)^{2m-2\mu} \quad (\xi\in\mathbb{R}^n\,,\lambda\in[0,\infty)) \tag{3.2}$$

holds with a constant C independent of ξ and λ.

Lemma 3.2 *The polynomial $A(\xi,\lambda)$ is N-elliptic with parameter in $[0,\infty)$ if and only if the following conditions are satisfied:*

(i) $A_{2m}(\xi)$ *is elliptic, i.e.* $A_{2m}(\xi)\neq 0$ *for* $\xi\in\mathbb{R}^n\backslash\{0\}$.

(ii) $A_{2\mu}(\xi)$ *is elliptic.*

(iii) $A(\xi,\lambda)\neq 0$ *for* $\xi\in\mathbb{R}^n\backslash\{0\}$ *and* $\lambda\in[0,\infty)$.

Proof. From (3.2) we trivially obtain condition (iii) and, setting $\lambda=0$, condition (i). Taking $\varepsilon=\frac{1}{\lambda}$ and dividing (3.2) by $\varepsilon^{2\mu-2m}$, we receive

$$|A_{2\mu}(\xi)+\varepsilon A_{2\mu+1}(\xi)+\ldots+\varepsilon^{2m-2\mu}A_{2m}(\xi)| \geq C|\xi|^{2\mu}(1+\varepsilon|\xi|)^{2m-2\mu}\,. \tag{3.3}$$

Taking the limit for $\varepsilon\to 0$, we obtain (ii).

Now let conditions (i)–(iii) be fulfilled. For $\xi\in\mathbb{R}^n\backslash\{0\}$ we write $A(\xi,\lambda)$ in the form

$$A(\xi,\lambda)=A_{2\mu}(\xi)B_{2m-2\mu}(\xi,\lambda) \tag{3.4}$$

with

$$B_{2m-2\mu}(\xi,\lambda)=\frac{A_{2m}(\xi)}{A_{2\mu}(\xi)}+\lambda\frac{A_{2m-1}(\xi)}{A_{2\mu}(\xi)}+\ldots+\lambda^{2m-2\mu}\,. \tag{3.5}$$

The coefficients of $B_{2m-2\mu}(\xi,\lambda)$ (considered as a polynomial in λ) are homogeneous functions in $\xi\in\mathbb{R}^n\backslash\{0\}$, and therefore $B(\xi,\lambda)$ is a homogeneous function in (ξ,λ) of degree $2m-2\mu$. From this and from conditions (ii) and (iii) it follows that

$$|A_{2\mu}(\xi)|\geq C|\xi|^{2\mu}\,,\quad |B_{2m-2\mu}(\xi,\lambda)|\geq C(\lambda+|\xi|)^{2m-2\mu}\,. \tag{3.6}$$

Multiplying these estimates, we see that A is N-elliptic with parameter in $[0,\infty)$. □

Denote by $\tau_j(\xi',\lambda)$ $(j=1,\ldots,2m)$ the zeros of the algebraic equation

$$A(\xi',\tau,\lambda)=0 \quad \big(\xi'\in\mathbb{R}^{n-1}\backslash\{0\},\lambda\in[0,\infty)\big)\,. \tag{3.7}$$

Due to Lemma 3.2 (iii), this equation has no real roots. The number m_+ of roots with positive imaginary part is independent of (ξ', λ) and therefore coincides with the corresponding number for $\lambda = 0$. It is easily seen (cf. [4], Section 1.2) that in the case $n > 2$ the set $\{(\xi', \lambda) : \xi' \in \mathbb{R}^{n-1} \setminus \{0\},\ \lambda \in [0, \infty)\}$ is connected, and therefore we have $m_+ = m$. In the case $n \leq 2$ the relation $m_+ = m$ is an additional condition which will be assumed to hold in the following. We denote the roots of $A(\xi', \tau, \lambda)$ with positive imaginary part by $\tau_1(\xi', \lambda), \ldots, \tau_m(\xi', \lambda)$.

To investigate the elliptic pencil corresponding to $A(\xi', \tau, \lambda)$ we will need an additional assumption which is closely related to the condition of regularity of degeneration in the theory of singular perturbations (cf. Vishik-Lyusternik [13], Section 1.1). To formulate this assumption we consider the auxiliary polynomial of degree $2m - 2\mu$ given by

$$Q(\tau) := \tau^{-2\mu} A(0, \tau, 1) . \tag{3.8}$$

From inequality (3.2) with $\xi' = 0$ and $\lambda = 1$ we obtain for $\tau \neq 0$ the estimate

$$|Q(\tau)| \geq C(|\tau| + 1)^{2m-2\mu} \tag{3.9}$$

with a constant independent of τ. By continuity we obtain that $Q(0) \neq 0$, and thus $Q(\tau)$ has no real roots.

Definition 3.3 The polynomial $A(\xi', \tau, \lambda)$ is said to degenerate regularly for $\lambda \to \infty$ if the polynomial $Q(\tau)$ defined in (3.9) has exactly $m - \mu$ roots with positive imaginary part (counted according to their multiplicities).

Remark 3.4 a) Suppose that the polynomial $A(\xi, \lambda)$ contains only terms of even order, i.e.

$$\begin{aligned} A(\xi, \lambda) \quad = \quad & A_{2m}(\xi) + \lambda^2 A_{2m-2}(\xi) + \ldots \\ & + \lambda^{2m-2\mu-2} A_{2\mu+2}(\xi) + \lambda^{2m-2\mu} A_{2\mu}(\xi) . \end{aligned} \tag{3.10}$$

Then the polynomial $Q(\tau)$ is a polynomial of degree $m - \mu$ in the variable τ^2 and $A(\xi, \lambda)$ degenerates regularly for $\lambda \to \infty$.

b) (Cf. [13], Lemma 3.4.) Assume that $A(\xi, \lambda)$ is the symbol of a differential operator $\tilde{A}(\frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n}, \lambda)$ with real coefficients. Then the polynomials of even order $A_{2m-2j}(\xi)$ $(j = 0, \ldots, m - \mu)$ are real and the polynomials of odd order $A_{2m-2j-1}(\xi)$ $(j = 0, \ldots, m - \mu - 1)$ are purely imaginary. Assume that $\tilde{A}$ is strongly elliptic, i.e. we have

$$\operatorname{Re} A(\xi, \lambda) \geq C|\xi|^{2\mu}(\lambda + |\xi|)^{2m-2\mu} . \tag{3.11}$$

Then we obtain that $\operatorname{Re} A = A_{2m} + \lambda^2 A_{2m-2} + \ldots + \lambda^{2m-2\mu} A_{2\mu}$ satisfies (3.2), and due to part a) the polynomial $\operatorname{Re} Q(\tau)$ has $m-\mu$ roots with positive imaginary part and $m-\mu$ roots with negative imaginary part. Since the polynomial

$$Q_\delta(\tau) := \operatorname{Re} Q(\tau) + \delta i \operatorname{Im} Q(\tau) \quad (0 \leq \delta \leq 1) \tag{3.12}$$

satisfies

$$\operatorname{Re} Q_\delta(\tau) \geq C(|\tau|+1)^{2m-2\mu} \quad (0 \leq \delta \leq 1)\,, \tag{3.13}$$

the number of roots of Q_δ in the upper half complex plane does not depend on $\delta \in [0,1]$, and $A(\xi,\lambda)$ degenerates regularly for $\lambda \to \infty$.

Lemma 3.5 *Let the polynomial $A(\xi,\lambda)$ in (3.1) be N-elliptic with parameter in $[0,\infty)$ and assume that A degenerates regularly for $\lambda \to \infty$. Then, with a suitable numbering of the roots $\tau_j(\xi',\lambda)$ of $A(\xi',\tau,\lambda)$ with positive imaginary part, we have:*
(i) *Let $S(\xi') = \{\tau_1^0(\xi'),\ldots,\tau_\mu^0(\xi')\}$ be the set of all zeros of $A_{2\mu}(\xi',\tau)$ with positive imaginary part. Then for all $r>0$ there exists a $\lambda_0>0$ such that the distance between the sets $\{\tau_1(\xi',\lambda),\ldots,\tau_\mu(\xi',\lambda)\}$ and $S(\xi')$ is less than r for all ξ' with $|\xi'|=1$ and all $\lambda \geq \lambda_0$.*
(ii) *Let $\tau_{\mu+1}^1,\ldots,\tau_m^1$ be the roots of the polynomial $Q(\tau)$ (cf. (3.8)) with positive imaginary part. Then*

$$\tau_j(\xi',\lambda) = \lambda\tau_j^1 + \tilde{\tau}_j^1(\xi',\lambda) \quad (j=\mu+1,\ldots,m)\,, \tag{3.14}$$

and there exist constants K_j and λ_1, independent of ξ' and λ, such that for $\lambda \geq \lambda_1$ the inequality

$$|\tilde{\tau}_j^1(\xi',\lambda)| \leq K_j |\xi'|^{\frac{1}{k_1}}\, \lambda^{1-\frac{1}{k_1}} \quad (|\xi'| \leq \lambda) \tag{3.15}$$

holds, where k_1 is the maximal multiplicity of the roots of $Q(\tau)$.

Proof. (i) We write $\xi' = \rho\omega$ with $|\omega|=1$ and set $\tilde{\tau} = \frac{\tau}{\rho}$, $\varepsilon = \frac{\rho}{\lambda}$. After division of $A(\xi',\tau,\lambda)$ by $\lambda^{2m-2\mu}\rho^{2\mu}$ we obtain the equation

$$B(\omega,\tilde{\tau},\varepsilon) := A_{2\mu}(\omega,\tilde{\tau}) + \varepsilon A_{2\mu+1}(\omega,\tilde{\tau}) + \ldots + \varepsilon^{2m-2\mu} A_{2m}(\omega,\tilde{\tau}) = 0\,. \tag{3.16}$$

First we fix ω with $|\omega|=1$. Let $\tilde{\tau}_j = \ldots = \tilde{\tau}_{j+p-1}$ be a zero of $B(\omega,\tilde{\tau},0) = A_{2\mu}(\omega,\tilde{\tau})$ of multiplicity p. Then there exists an $\alpha = \alpha(\omega) > 0$ such that

$$\frac{1}{2\pi i}\int_{|z-\tilde{\tau}_j|=\alpha} \frac{\frac{d}{dz}B(\omega,z,\varepsilon)}{B(\omega,z,\varepsilon)}\,dz = \frac{1}{2\pi i}\int_{|z-\tilde{\tau}_j|=\alpha} \frac{\frac{d}{dz}B(\omega,z,0)}{B(\omega,z,0)}\,dz = p \tag{3.17}$$

holds for all $\varepsilon < \varepsilon_0 = \varepsilon_0(\omega)$. Therefore, for every $\varepsilon < \varepsilon_0$ the equation (3.16) has exactly p roots in $\{z \in \mathbb{C} : |z - \tilde{\tau}_j| < \alpha\}$ which we denote by $\tilde{\tau}_j(\omega, \varepsilon), \ldots, \tilde{\tau}_{j+p-1}(\omega, \varepsilon)$. Proceding in this way for all zeros of $A_{2\mu}(\omega, \tilde{\tau})$, we obtain the set $S(\omega, \varepsilon) := \{\tilde{\tau}_1(\omega, \varepsilon), \ldots, \tilde{\tau}_\mu(\omega, \varepsilon)\}$ of zeros of $B(\omega, \tilde{\tau}, \varepsilon)$.

Now we assume that the statement in (i) is false. Then there exists a sequence $(\omega_n)_{n\geq 1}$ with $|\omega_n| = 1$ and a constant $C > 0$ such that $\operatorname{dist}(S(\omega_n), S(\omega_n, \varepsilon_n)) \geq C$ for all $n \geq 1$ where we have set $\varepsilon_n = \frac{1}{n}$. Due to compactness, we may assume that ω_n converges to ω_0. As the zeros of $A_{2\mu}(\omega, \tilde{\tau})$ depend continuously on ω, we obtain for large n that

$$\operatorname{dist}(S(\omega_0), S(\omega_n, \varepsilon_n)) \geq \frac{C}{2}\,. \tag{3.18}$$

But from the same considerations as above we see that for every sufficiently small $\alpha > 0$ there exists an $\varepsilon_0 = \varepsilon_0(\omega_0)$ and an $s > 0$ such that $B(\omega, \tilde{\tau}, \varepsilon)$ has exactly μ roots in $\bigcup_j \{z \in \mathbb{C} : |z - \tilde{\tau}_j(\omega_0)| < \alpha\}$ for all $|\omega - \omega_0| < s$ and $0 < \varepsilon < \varepsilon_0$. Taking $\alpha < \frac{C}{2}$, we obtain a contradiction to (3.18).

(ii) We set $\tilde{\tau} = \frac{\tau}{\lambda}$ and $\varepsilon = \frac{|\xi'|}{\lambda}$ and obtain the equation $B(\omega, \tilde{\tau}, \varepsilon) := A(\varepsilon\omega, \tilde{\tau}, 1) = 0$ with $\omega := \frac{\xi'}{\varepsilon}$. First we fix ω with $|\omega| = 1$. We write

$$0 = B(\omega, \tilde{\tau}, \varepsilon) = A(0, \tilde{\tau}, 1) + \sum_{k=1}^{2m} \Big(\frac{\partial}{\partial\varepsilon}\Big)^k B(\omega, \tilde{\tau}, 0)\, \frac{\varepsilon^k}{k!}\,. \tag{3.19}$$

Let $\tau_j^1 = \ldots = \tau_{j+p-1}^1$ be a zero of $Q(\tau)$ of multiplicity p. Then we know from the theory of algebraic functions that there exist p roots $\tilde{\tau}_j(\omega, \varepsilon), \ldots, \tilde{\tau}_{j+p-1}(\omega, \varepsilon)$ of $B(\omega, \tilde{\tau}, \varepsilon)$ for which we have an expansion (Puiseux series) of the form

$$\tilde{\tau}_s(\omega, \varepsilon) = \tau_j^1 + \sum_{k=1}^{\infty} c_{jk}(\omega)\varepsilon^{k/p} \quad (s = j, \ldots, j+p-1) \tag{3.20}$$

(cf., e.g., [7], Section 7). In formula (3.20) we have to take the p different branches of the function $\varepsilon^{\frac{1}{p}}$ to obtain the zeros $\tilde{\tau}_j(\varepsilon), \ldots, \tilde{\tau}_{j+p-1}(\varepsilon)$. The series on the right-hand side is a holomorphic function in $\varepsilon^{\frac{1}{p}}$ for $|\varepsilon| \leq \varepsilon_1(\omega)$ for some $\varepsilon_1(\omega) > 0$.

From the construction of the Puiseux series (cf. [7], Section 8) we know that the coefficients $c_{jk}(\omega)$ in the series (3.20) depend continuously on the coefficients of the polynomial $B(\omega, \tilde{\tau}, \varepsilon)$ and therefore on ω. Thus there exists an $\varepsilon_1 > 0$, independent of ω, such that the right-hand side of

(3.20) is a holomorphic function in $\varepsilon^{\frac{1}{p}}$ for $|\varepsilon| \le \varepsilon_1$. As the function

$$(\tilde{\tau}_j(\omega,\varepsilon) - \tau_j^1)\varepsilon^{-\frac{1}{p}} = \sum_{k=1}^{\infty} c_{jk}(\omega)\varepsilon^{\frac{k-1}{p}}$$

is continuous in ω and ε for $|\omega| = 1$ and $0 \le \varepsilon \le \varepsilon_0$, it is bounded by some constant K_1, independent of ω and ε, which finishes the proof of part (ii). □

4. Estimates for ordinary differential equations

In this section we consider the polynomial $A(\xi,\lambda)$ given by (3.1) and assume that this polynomial is N-elliptic with parameter in $[0,\infty)$ and degenerates regularly for $\lambda \to \infty$. The Newton polygon corresponding to A has the shape indicated in Figure 2 with $r = 2m$ and $s = 2\mu$.

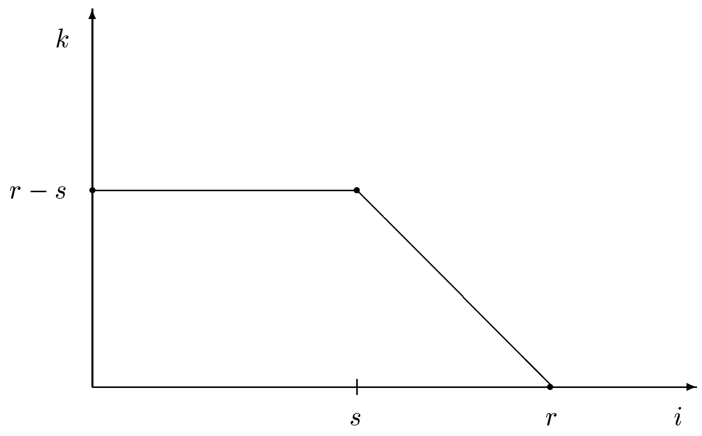

Fig. 2. The Newton polygon $N_{r,s}$.

For fixed $\xi' \in \mathbb{R}^{n-1}$, $\lambda \in [0,\infty)$ and $j = 1,\dots,m$ we consider the ordinary differential equation on the half-line

$$\begin{aligned} A(\xi', D_t, \lambda)\, w_j(t) &= 0 \qquad (t > 0)\,, && (4.1)\\ D_t^{k-1}\, w_j(t)|_{t=0} &= \delta_{jk} \quad (k = 1,\dots,m)\,, && (4.2)\\ w_j(t) &\to 0 \qquad (t \to +\infty)\,. \end{aligned}$$

Here D_t stands for $-i\frac{\partial}{\partial t}$.

Theorem 4.1 *For every $\xi' \in \mathbb{R}^{n-1}\backslash\{0\}$ and $\lambda \in [0,\infty)$ the ordinary differential equation* (4.1)–(4.2) *has a unique solution $w_j(\xi',t,\lambda)$, and the estimate*

$$\|D_t^l w_j(\xi',\cdot,\lambda)\|_{L_2(\mathbb{R}_+)} \le C \begin{cases} |\xi'|^{l-j+\frac{1}{2}}, & j\le\mu, \quad l\le\mu, \\ |\xi'|^{1+\mu-j}(\lambda+|\xi'|)^{l-\mu-\frac{1}{2}}, & j\le\mu, \quad l>\mu \\ |\xi'|^{l-\mu}(\lambda+|\xi'|)^{\mu-j+\frac{1}{2}}, & j>\mu, \quad l\le\mu, \\ (\lambda+|\xi'|)^{l-j+\frac{1}{2}}, & j>\mu, \quad l>\mu, \end{cases} \tag{4.3}$$

holds with a constant C not depending on ξ' and λ.

Proof. The existence and the uniqueness of the solution follows immediately from the fact that $A(\xi',\tau,\lambda)$ (considered as a polynomial in τ) has exactly m roots with positive imaginary part. Let $\gamma(\xi',\lambda)$ be a closed contour in the upper half of the complex plane enclosing all roots $\tau_1(\xi',\lambda),\dots,\tau_m(\xi',\lambda)$ with positive imaginary part. Then $w_j(\xi',t,\lambda)$ is given by

$$w_j(\xi',t,\lambda) = \frac{1}{2\pi i}\int_{\gamma(\xi',\lambda)} \frac{M_j(\xi',\tau,\lambda)}{A_+(\xi',\tau,\lambda)} e^{it\tau}\, d\tau \tag{4.4}$$

where

$$A_+(\xi',\tau,\lambda) = \prod_{k=1}^{m}\Big(\tau-\tau_k(\xi',\lambda)\Big) =: \sum_{k=0}^{m} a_k(\xi',\lambda)\tau^k \tag{4.5}$$

and

$$M_j(\xi',\tau,\lambda) = \sum_{k=0}^{m-j} a_k(\xi',\lambda)\tau^{m-j-k}\,. \tag{4.6}$$

(Cf., e.g., [2], Section 1.) The coefficients are given by the formula of Vieta,

$$a_k(\xi',\lambda) = \sum_{1\le l_1<\ldots<l_k\le m} (-1)^k \tau_{l_1}(\xi',\lambda)\cdot\ldots\cdot\tau_{l_k}(\xi',\lambda)\,. \tag{4.7}$$

From (4.4) we see, substituting $\tau = r\tilde{\tau}$, that

$$r^{1-j+l}(D_t^l w_j)(r\xi',\frac{t}{r},r\lambda) = D_t^l w_j(\xi',t,\lambda)\,, \tag{4.8}$$

and therefore

$$\|D_t^l w_j(\xi',\cdot,\lambda)\|_{L_2(\mathbb{R}_+)} = r^{\frac{1}{2}-j+l}\Big\|D_t^l w_j\Big(\frac{\xi'}{r},\cdot,\frac{\lambda}{r}\Big)\Big\|_{L_2(\mathbb{R}_+)}\,. \tag{4.9}$$

If we set $r = |\xi'|$ and $\omega' = \frac{\xi'}{|\xi'|}$ we obtain

$$\|D_t^l w_j(\xi', \cdot, \lambda)\|_{L_2(\mathbb{R}_+)} = |\xi'|^{\frac{1}{2}-j+l} \left\| D_t^l w_j \left(\omega', \cdot, \frac{\lambda}{|\xi'|}\right) \right\|_{L_2(\mathbb{R}_+)} . \tag{4.10}$$

The theorem will be proved if we show that for $|\omega'| = 1$ we have

$$\|(D_t^l w_j)(\omega', \cdot, \Lambda)\|_{L_2(\mathbb{R}_+)} \leq \begin{cases} C\,, & j \leq \mu\,,\ l \leq \mu\,, \\ C\,\Lambda^{l-\mu-\frac{1}{2}}, & j \leq \mu\,,\ l > \mu \\ C\,\Lambda^{\mu-j+\frac{1}{2}}, & j > \mu\,,\ l \leq \mu\,, \\ C\,\Lambda^{l-j+\frac{1}{2}}, & j > \mu\,,\ l > \mu\,, \end{cases} \tag{4.11}$$

for $\Lambda \geq 1$ and that the left-hand side is bounded by a constant for $\Lambda \leq 1$.

The boundedness for $\Lambda \leq 1$ easily follows from the ellipticity of $A(\omega', \tau, \Lambda)$ and inequality (1.10).

For large Λ we write

$$\gamma(\omega', \Lambda) = \gamma^{(1)}(\omega', \Lambda) \cup \gamma^{(2)}(\omega', \Lambda)$$

where $\gamma^{(1)}(\omega', \Lambda)$ encloses the zeros $\tau_1(\omega', \Lambda), \ldots, \tau_\mu(\omega', \Lambda)$ and $\gamma^{(2)}(\omega', \Lambda)$ encloses the zeros $\tau_{\mu+1}(\omega', \Lambda), \ldots, \tau_m(\omega', \Lambda)$. Here we assume that the zeros are numbered according to Lemma 3.5. According to this splitting of the contour γ, we write $w_j(\omega', t, \Lambda) = w_j^{(1)}(\omega', t, \Lambda) + w_j^{(2)}(\omega', t, \Lambda)$ with

$$w_j^{(k)}(\omega', t, \Lambda) := \frac{1}{2\pi i} \int\limits_{\gamma^{(k)}(\omega', t, \Lambda)} \frac{M_j(\omega', \tau, \Lambda)}{A_+(\omega', \tau, \Lambda)}\, e^{it\tau}\, d\tau \quad (k = 1, 2)\,.$$

From Lemma 3.5 we know that

$$\begin{aligned} |\tau_j(\omega', \Lambda)| &\leq C && (|\omega'| = 1,\ \Lambda \geq \Lambda_0, \quad j = 1, \ldots, \mu) \\ |\tau_j(\omega', \Lambda)| &\leq C\Lambda && (|\omega'| = 1,\ \Lambda \geq \Lambda_0, \quad j = \mu+1, \ldots, m)\,. \end{aligned}$$

As $A_{2\mu}$ is elliptic we have, with the notation of Lemma 3.5, $|\tau_j(\omega', \Lambda)| \geq C$ for $j = 1, \ldots, \mu$ and $|\omega'| = 1,\ \Lambda \geq \Lambda_0$. With our additional assumption we also have

$$|\tau_j(\omega', \Lambda)| \geq C\Lambda \quad (|\omega'| = 1,\ \Lambda \geq \Lambda_0, \quad j = \mu+1, \ldots, m),$$

as $\frac{\tau_j(\omega', \Lambda)}{\Lambda} \to \tau_j^1$ and $\operatorname{Im} \tau_j^1 > 0$, cf. Lemma 3.5 (ii). Therefore

$$|A_+(\omega', \tau, \Lambda)| = \prod_{k=1}^m |\tau - \tau_k(\omega', \Lambda)| \geq \begin{cases} C\Lambda^{m-\mu} & \text{on } \gamma^{(1)}\,, \\ C\Lambda^m & \text{on } \gamma^{(2)} \end{cases} \tag{4.12}$$

(note that $|\tau| \approx C$ on $\gamma^{(1)}$ and $|\tau| \approx C\Lambda$ on $\gamma^{(2)}$). Now we have to estimate $|M_j(\omega', \tau, \Lambda)|$ in (4.4). For this we use the fact that according to (4.7)

$$|a_k(\omega', \Lambda)| \leq \sum_{l_1 < \ldots < l_k} |\tau_{l_1}| \cdot \ldots \cdot |\tau_{l_k}| \leq \begin{cases} C\Lambda^k, & k \leq m - \mu\,, \\ C\Lambda^{m-\mu}, & k \geq m - \mu\,. \end{cases} \tag{4.13}$$

On $\gamma^{(1)}$ we have

$$M_j(\omega', \tau, \Lambda)| \leq \begin{cases} C\Lambda^{m-\mu}, & j \leq \mu\,, \\ C\Lambda^{m-j}, & j \geq \mu\,. \end{cases}$$

As length$(\gamma^{(1)}) \leq C$ we obtain

$$\left| \int_{\gamma^{(1)}} (i\tau)^l \frac{M_j(\omega', \tau, \Lambda)}{A_+(\omega', \tau, \Lambda)} e^{it\tau}\, d\tau \right| \leq \begin{cases} C\exp(-Ct)\,, & j \leq \mu\,, \\ C\Lambda^{\mu-j}\exp(-Ct), & j \geq \mu\,, \end{cases} \tag{4.14}$$

and therefore

$$\|(D_t^l w_j^{(1)})(\omega', \cdot, \Lambda)\|_{L_2(\mathbb{R}_+)} \leq \begin{cases} C\,, & j \leq \mu\,, \\ C\Lambda^{\mu-j}, & j \geq \mu\,, \end{cases} \quad (|\omega'| = 1,\ \Lambda \geq \Lambda_0)\,. \tag{4.15}$$

For an estimation on $\gamma^{(2)}$ we first remark that for every $l \geq 0$ we have

$$|\tau^l M_j(\omega', \tau, \Lambda)| \leq \sum_{k=0}^{m-j} |a_k|\, |\tau^{m-j+l-k}| \leq C\Lambda^{m-j+l}\,.$$

Therefore the inequalities

$$|D_t^l w_j^{(2)}(\omega', t, \Lambda)| \leq C\Lambda^{l-j+1}\exp(-C\Lambda t)$$

and

$$\|D_t^l w_j^{(2)}(\omega', \cdot, \Lambda)\|_{L_2(\mathbb{R}_+)} \leq C\Lambda^{l-j+\frac{1}{2}} \quad (l \geq 0) \tag{4.16}$$

hold. To find a sharper estimate in the case $j \leq \mu$ we use the relation

$$\begin{aligned} \tau^l M_j(\omega', \tau, \Lambda) &= \tau^{l-j} \sum_{k=0}^{m-j} a_k(\omega', \Lambda)\tau^{m-k} \\ &= \tau^{l-j} \Big(A_+(\omega', \tau, \Lambda) - \sum_{k=m-j+1}^{m} a_k(\omega', \Lambda)\tau^{m-k} \Big) \end{aligned}$$

which yields

$$D_t^l w_j^{(2)}(\omega', t, \Lambda) = -\frac{1}{2\pi i}\int_{\gamma^{(2)}} \frac{\sum_{k=m-j+1}^{m} a_k(\omega',\Lambda)\tau^{m-k+l-j}}{A_+(\omega',\tau,\Lambda)} e^{it\tau}\,dt\,.$$

Here we used the fact that the contour $\gamma^{(2)}$ does not enclose the origin, and therefore $\tau^{l-j}e^{it\tau}$ is holomorphic inside $\gamma^{(2)}$.

We obtain for the case $j \le \mu$ and for every $l \ge 0$ that

$$\Big|\sum_{k=m-j+1}^{m} a_k \tau^{m-k+l-j}\Big| \le C\Lambda^{m-\mu}\Lambda^{m-(m-j+1)+l-j} = C\Lambda^{m-\mu+l-1}$$

and

$$\|D_t^l w_j^{(2)}(\omega',\cdot,\Lambda)\|_{L_2(\mathbb{R}_+)} \le C\Lambda^{l-\mu-\frac12} \quad (j\le\mu,\ l\ge 0) \tag{4.17}$$

in view of Remark 2.3 for, say, $\Lambda \ge 1$. Now we compare the right-hand sides of (4.15)–(4.17) with the right-hand side of (4.11).

a) For $j,\, l \le \mu$ the norm of $D_t^l w^{(1)}$ is $O(1)$ and the norm of $D_t^l w^{(2)}$ is estimated by $\Lambda^{l-\mu-\frac12} \le \Lambda^{-\frac12}$.

b) For $j \le \mu$ and $l > \mu$ according to (4.17) the norm of $D_t^l w^{(2)}$ is estimated by $\Lambda^{l-\mu-\frac12} \ge \Lambda^{\frac12}$ and the norm of $D_t^l w^{(1)}$ is estimated by a constant.

c) For $j > \mu$ and $l \le \mu$ according to (4.15) and (4.17) the norm of $D_t^l w^{(1)}$ is estimated by $\Lambda^{\mu-j}$ and the norm of $D_t^l w^{(2)}$ is estimated by $\Lambda^{l-j+\frac12} \le \Lambda^{\mu-j+\frac12}$.

d) For $j,\, l > \mu$ the norm of $D_t^l w^{(2)}$ is estimated by $\Lambda^{l-j+\frac12}$ and the norm of $D_t^l w^{(1)}$ is estimated by $\Lambda^{\mu-j} < \Lambda^{l-j+\frac12}$.

Thus the inequality (4.11) holds, which finishes the proof of the theorem. □

5. The main results

Now we want to prove an a priori estimate for the Dirichlet boundary value problem corresponding to the elliptic pencil $A(x, D, \lambda)$ defined in (1.1). First we consider model problems in $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

Let A be a polynomial of the form (3.1). As it was already mentioned at the beginning of Section 4, the Newton polygon $N_{2m,2\mu}$ of $A(\xi,\lambda)$ has the form indicated in Figure 2 with $r = 2m$ and $s = 2\mu$. The a priori estimates which we will obtain below, however, do not use the Sobolev spaces corresponding to this Newton polygon but the "energy spaces" which are defined as the Sobolev spaces corresponding to the Newton polygon $N_{m,\mu}$. For this Newton polygon we have

$$\Xi(\xi,\lambda) := \Xi_{N_{m,\mu}}(\xi,\lambda) \approx (1+|\xi|)^\mu(\lambda+|\xi|)^{m-\mu}\,. \tag{5.1}$$

In the notation of the Introduction, we have $H^\Xi(\mathbb{R}^n) = H^{(m,\mu)}(\mathbb{R}^n)$. As in Section 2, we will denote by $\Xi^{(-l)}(\xi,\lambda)$ the weight function corresponding to the shifted Newton polygon (with a shift of length l to the left). The space $H^{(-m,-\mu)}(\mathbb{R}^n)$ which appears in the Introduction is equal to the space $H^{\frac{1}{\Xi}}(\mathbb{R}^n)$.

From the trace results of Theorem 2.9 we immediately obtain the continuity of the corresponding operators:

Lemma 5.1 a) *The operator $A(D,\lambda)$ acts continuously from $H^\Xi(\mathbb{R}^n)$ to $H^{\frac{1}{\Xi}}(\mathbb{R}^n)$.*
b) *The boundary operator D_n^{j-1} $(j \le m)$ acts continuously from $H^\Xi(\mathbb{R}^n)$ to $H^{\Xi^{(-j+\frac{1}{2})}}(\mathbb{R}^{n-1})$.*

Here and in the following, the continuity of the operator means that the norm of this operator can be estimated by a constant independent of λ.

Proposition 5.2 (A priori estimate in $\mathbb{R}^n$.) *Let $A(\xi,\lambda)$ be N-elliptic with parameter in $[0,\infty)$. Then for every $\lambda_0 > 0$ the inequality*

$$\|u\|_{\Xi,\mathbb{R}^n} \le C\Big(\|A(D,\lambda)u\|_{\frac{1}{\Xi},\mathbb{R}^n} + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)}\Big) \tag{5.2}$$

holds for all $\lambda \ge \lambda_0$ with a constant $C = C(\lambda_0)$ independent of u and λ.

Proof. By changing the constant in (3.2) we can rewrite the N-ellipticity condition in the form

$$\begin{aligned}\lambda^{2m-2\mu} &+ C_1^{-1}\,\frac{|A(\xi,\lambda)|^2}{(1+|\xi|^2)^\mu(\lambda^2+|\xi|^2)^{m-\mu}} \\ &\ge \lambda^{2m-2\mu} + |\xi|^{4\mu}\,(1+|\xi|^2)^{-\mu}(\lambda^2+|\xi|^2)^{m-\mu}\,.\end{aligned}$$

For $|\xi| \ge 1$ the right-hand side can be estimated from below by

$$(1+|\xi|^2)^\mu(\lambda^2+|\xi|^2)^{m-\mu}$$

For $|\xi| \le 1$ and $\lambda \ge \lambda_0$ the right-hand side can be estimated from below by

$$\begin{aligned}\lambda^{2m-2\mu} &= (1+\lambda^{-2})^{-m+\mu}\,(1+\lambda^2)^{m-\mu} \\ &\ge (1+\lambda_0^{-2})^{-m+\mu}\,2^{-2\mu}\,(1+|\xi|^2)^\mu\,(\lambda^2+|\xi|^2)^{m-\mu}\,.\end{aligned}$$

Combining these estimates we obtain for $\lambda \geq \lambda_0$

$$(1+|\xi|^2)^\mu(\lambda^2+|\xi|^2)^{m-\mu} \leq C(\lambda_0)\Big(\frac{|A(\xi,\lambda)|^2}{(1+|\xi|^2)^\mu(\lambda^2+|\xi|^2)^{m-\mu}}+\lambda^{2m-2\mu}\Big)\,.$$

Multiplying both sides by $|Fu(\xi)|^2$ and integrating with respect to ξ we obtain the inequality

$$\|u\|^2_{\Xi,\mathbb{R}^n} \leq C(\lambda_0)\Big(\|A(D,\lambda)u\|^2_{\frac{1}{\Xi},\mathbb{R}^n}+\lambda^{2m-2\mu}\;\|u\|^2\Big)$$

equivalent to (5.2). □

Theorem 5.3 (A priori estimate in $\mathbb{R}^n_+$.) *Let $A(\xi,\lambda)$ be N-elliptic with parameter in $[0,\infty)$ and degenerate regularly for $\lambda\to\infty$. Then for every $\lambda_0>0$ there exists a constant $C=C(\lambda_0)$ such that for all $\lambda\geq\lambda_0$ and all $u\in H^\Xi(\mathbb{R}^n_+)$ the estimate*

$$\begin{aligned}\|u\|_{\Xi,\mathbb{R}^n_+} \;&\leq\; C\Big(\|A(D,\lambda)u\|_{\frac{1}{\Xi},\mathbb{R}^n_+}\\ &+\;\sum_{j=1}^m \|D_n^{j-1}u\|_{\Xi^{(-j+\frac12)},\mathbb{R}^{n-1}}+\lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n_+)}\Big)\end{aligned}\tag{5.3}$$

holds.

Proof. We will follow a standard plan in elliptic theory. In the first part of the proof we reduce (5.3) to the case $f\equiv 0$. Then using Theorem 4.1, we treat the case of the homogeneous equation.

1) Denote by E a linear operator of extension of functions defined in $\mathbb{R}^n_+$ to functions in $\mathbb{R}^n$. If we use the well-known Hestenes construction then the operator $E: L_2(\mathbb{R}^n_+)\to L_2(\mathbb{R}^n)$ and its restriction $E: H^\Xi(\mathbb{R}^n_+)\to H^\Xi(\mathbb{R}^n)$ are bounded operators. We will denote by R the operator of restriction of functions on $\mathbb{R}^n$ onto $\mathbb{R}^n_+$.

2) Let $\psi(\xi)\in C^\infty(\mathbb{R}^n)$ be a cut-off function, i.e. $\psi(\xi)=1$ for $|\xi|\leq 1$ and $\psi(\xi)=0$ for $|\xi|\geq 2$. We write

$$u=u_1+u_2+v=R\psi(D)Eu+R(1-\psi(D))A^{-1}(D,\lambda)Ef+v \tag{5.4}$$

where we have set $Ef=A(D,\lambda)Eu$.

First of all we show that u_1 and u_2 belong to $H^\Xi(\mathbb{R}^n_+)$ and their norms in this space can be estimated by a constant times

$$\|f\|_{\frac{1}{\Xi},\mathbb{R}^n_+}+\lambda^{m-\mu}\,\|u\|_{L_2(\mathbb{R}^n_+)}\,.$$

3) Since the operator $\psi(D)$ is infinitely smoothing we get for $\lambda \geq \lambda_0$ that

$$\|u_1\|_{\Xi,\mathbb{R}^n_+} \leq \|\psi(D)Eu\|_{\Xi,\mathbb{R}^n} \leq C\lambda^{m-\mu}\|Eu\|_{L_2(\mathbb{R}^n)} \leq C_1\lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n_+)}\,.$$

4) Using the Fourier transform we obtain

$$\begin{aligned}\|u_2\|_{\Xi,\mathbb{R}^n_+} &\leq \|(1-\psi(D))A^{-1}(D,\lambda)Ef\|_{\Xi,\mathbb{R}^n} \\ &= \|\Xi(\xi,\lambda)(1-\psi(\xi))A^{-1}(\xi,\lambda)(FEf)(\xi)\|_{L_2(\mathbb{R}^n)}\,.\end{aligned}$$

Since $1-\psi(\xi)=0$ for $|\xi|\leq 1$, we obtain from the N-ellipticity condition that

$$\Xi(\xi,\lambda)\;|1-\psi(\xi)|\;|A^{-1}(\xi,\lambda)| \leq C\,\Xi^{-1}(\xi,\lambda)$$

and

$$\|u_2\|_{\Xi,\mathbb{R}^n_+} \leq \text{ const } \|Ef\|_{\frac{1}{\Xi},\mathbb{R}^n}\,.$$

If the norm in $H^{\Xi^{-1}}(\mathbb{R}^n)$ is defined by means of the pseudodifferential operator

$$\Big((1+|D'|^2)^{1/2}+iD_n\Big)^{-\mu}\Big((\lambda^2+|D'|^2)^{1/2}+iD_n\Big)^{-m+\mu},$$

then according to Section 2

$$\|Ef\|_{\frac{1}{\Xi},\mathbb{R}^n} = \|f\|_{\frac{1}{\Xi},\mathbb{R}^n_+}\,.$$

5) Now we begin the estimation of v defined in (5.4). We have $v = u-u_1-u_2 \in H^{\Xi}(\mathbb{R}^n_+)$ and

$$\begin{aligned}A(D,\lambda)\,v &= 0\,, &(5.5)\\ D_n^{j-1}v(x)|_{x_n=0} &= h_j(x')\,, &(5.6)\end{aligned}$$

where we set $h_j(x') := D_n^{j-1}u(x',0)-D_n^{j-1}u_1(x',0)-D_n^{j-1}u_2(x',0)$. We shall prove the inequality

$$\|v\|_{\Xi,\mathbb{R}^n_+} \leq \text{ const } \Big(\sum_{j=1}^{m}\|h_j\|_{\Xi^{(-j+1/2)},\mathbb{R}^{n-1}} + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)}\Big) \qquad (5.7)$$

The a priori estimate (5.3) follows from this inequality because, due to Theorem 2.9,

$$\|D_n^{j-1}u_i\|_{\Xi^{(-j+1/2)},\mathbb{R}^{n-1}} \leq \text{ const } \|u_i\|_{\Xi,\mathbb{R}^n_+} \quad (i=1,2)\,.$$

The right-hand side of this inequality is already estimated by the right-hand side of (5.3).

6) We define

$$\Phi(\xi,\lambda) := \sum_{i,k} |\xi|^i \lambda^k \,, \tag{5.8}$$

where the sum extends over all integer points (i,k) belonging to the side of $N_{m,\mu}$ which is not parallel to the coordinate lines. From this definition it follows that

$$\Phi(\xi,\lambda) \approx |\xi|^\mu (\lambda + |\xi|)^{m-\mu} \,. \tag{5.9}$$

and $\|v\|_{\Xi,\mathbb{R}^n_+}$ is equivalent to

$$\|v\|_{\Phi,\mathbb{R}^n_+} + \lambda^{m-\mu}\|v\|_{L_2(\mathbb{R}^n_+)} \,.$$

The second term can be estimated by $\lambda^{m-\mu}(\|u\|_{L_2(\mathbb{R}^n_+)} + \|u_1\|_{L_2(\mathbb{R}^n_+)} + \|u_2\|_{L_2(\mathbb{R}^n_+)}) \le \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n_+)} + \|u_1\|_{\Xi,\mathbb{R}^n_+} + \|u_2\|_{\Xi,\mathbb{R}^n_+}$. Therefore, it is enough to estimate $\|v\|_{\Phi,\mathbb{R}^n_+}$ by the right-hand side of (5.7). Repeating the argument in Section 2 (see (2.19)) we reduce our problem to the estimation of

$$\int_0^\infty \|(D_n^l v)(\cdot, x_n)\|^2_{\Phi^{(-l)},\mathbb{R}^{n-1}}\, dx_n \quad (l = 0,\dots,m)$$

or after the Fourier transform with respect to x'

$$\int_0^\infty \int_{\mathbb{R}^{n-1}} |\Phi^{(-l)}(\xi,\lambda)(D_n^l F'v)(\xi',x_n)|^2\, d\xi' dx_n \quad (l = 0,\dots,m) \,. \tag{5.10}$$

The function $F'v(\xi',x_n) =: w(\xi',x_n)$ is (for almost every $\xi' \in \mathbb{R}^{n-1}$) a solution of

$$\begin{aligned} A(\xi', D_n, \lambda) w(x_n) &= 0\,, & (5.11)\\ D_n^{j-1} w(x_n)|_{x_n=0} &= (F'h_j)(\xi')\,. & (5.12) \end{aligned}$$

Due to Theorem 4.1, this solution is unique and given by

$$w(\xi', x_n) = \sum_{j=1}^m w_j(\xi', x_n, \lambda)(F'h_j)(\xi') \tag{5.13}$$

with $w_j(\xi', x_n, \lambda)$ being the solution of (4.1)–(4.2).

8) To obtain the estimate for $w = F'v$ we reformulate Theorem 4.1. It follows from the definition of $N_{m,\mu}$ that

$$\Phi^{(-r)}(\xi,\lambda) \le \begin{cases} |\xi|^{\mu-r}(\lambda+|\xi|)^{m-\mu}, & r \le \mu, \\ (\lambda+|\xi|)^{m-r}, & r > \mu. \end{cases}$$

From this it follows that

$$\frac{\Phi^{(-j+1/2)}(\xi',\lambda)}{\Phi^{(-l)}(\xi',\lambda)} \le \begin{cases} C|\xi'|^{l-j+\frac{1}{2}}, & l \le \mu, \quad j \le \mu, \\ C|\xi'|^{\mu-j+\frac{1}{2}}(\lambda+|\xi'|)^{l-\mu}, & l > \mu, \quad j \le \mu, \\ C|\xi'|^{l-\mu}(\lambda+|\xi'|)^{\mu-j+\frac{1}{2}}, & l \le \mu, \quad j > \mu, \\ C(\lambda+|\xi'|)^{l-j+\frac{1}{2}}, & l > \mu, \quad j > \mu. \end{cases}$$

Comparing the right-hand sides of these inequalities with the right-hand side of (4.3) we see that

$$\|D_n^l w_j(\xi',x_n,\lambda)\|_{L_2(\mathbb{R}_+)} \le C\frac{\Phi^{(-j+1/2)}(\xi',\lambda)}{\Phi^{(-l)}(\xi',\lambda)}\,.$$

From (5.13) and the last inequality it follows that

$$\begin{aligned} (\Phi^{(-l)}(\xi,\lambda))^2 & \int_0^\infty |D_n^l w(\xi',x_n,\lambda)|^2\,dx_n \\ & \le \quad C \sum |\Xi^{(-j+\frac{1}{2})}(\xi',\lambda)(F'h_j)(\xi')|^2\,. \end{aligned}$$

Integrating this inequality with respect to ξ' we obtain the desired estimate. □

Now we consider the Dirichlet boundary value problem for differential operators with parameter acting on a smooth compact manifold M with smooth boundary Γ. In this case we can choose a finite number of coordinate systems. In each of these systems the operator is of the form (1.1). The principal part of the operator is invariantly defined at each of these systems and at every fixed point $x^0 \in M$ it is of the form

$$A^{(0)}(x^0,D,\lambda) = A^{(0)}_{2m}(x^0,D) + \ldots + \lambda^{2m-2\mu}A^{(0)}_{2\mu}(x^0,D) \qquad (5.14)$$

(here $A^{(0)}_j$ denotes the principal part of A_j). We suppose that for each $x^0 \in \overline{M}$ our operator is N-elliptic with parameter. From the reason of continuity and compactness the constant C in inequality (3.2) can be chosen independent of x^0.

We can suppose without loss of generality that the coefficients of $A(x, D, \lambda)$ are of the form

$$a_{\alpha j}(x) = a_{\alpha j} + a'_{\alpha j}(x), \quad a_{\alpha j} \in \mathcal{D}(\mathbb{R}^n)\,. \tag{5.15}$$

Now we fix a point $x^0 \in \Gamma$ and a coordinate system in the neighborhood of x^0 such that in this system locally the boundary Γ is given by the equation $x_n = 0$. In this case we can define an analog of the polynomial (3.8):

$$Q(x^0, \tau) = \tau^{-2\mu} A^{(0)}(x^0, 0, \tau, 1) \tag{5.16}$$

Suppose that at a point $x^0 \in \Gamma$ and in a fixed coordinate system this polynomial has $m - \mu$ roots in the upper half-plane of the complex plane. It easily follows from this fact that every polynomial (5.16) corresponding to an arbitrary $x^0 \in \Gamma$ has the same property. In this case we say that the operator $A(x, D, \lambda)$ degenerates regularly at the boundary Γ.

Lemma 5.4 *For $a(x) = a + a'(x)$ with $a' \in \mathcal{D}(\mathbb{R}^n)$ and $f \in H^{\frac{1}{\equiv}}(R^n)$ we have $af \in H^{\frac{1}{\equiv}}(R^n)$, and the following statements hold:*
a) *There exists a constant $C(a)$ depending on a but not on f or λ such that*

$$\|af\|_{\frac{1}{\equiv}, \mathbb{R}^n} \leq C(a) \|u\|_{\frac{1}{\equiv}, \mathbb{R}^n}\,. \tag{5.17}$$

b) *There exists a constant $C'(a)$ depending only on a such that the inequality*

$$\|af\|_{\frac{1}{\equiv}, \mathbb{R}^n} \leq \sup_{x \in \mathbb{R}^n} |a(x)| \|f\|_{\frac{1}{\equiv}, \mathbb{R}^n} + C'(a) \|f\|_{\Psi, \mathbb{R}^n} \tag{5.18}$$

holds, where we have set

$$\|f\|_{\Psi, \mathbb{R}^n} := \left(\int (1 + |\xi|)^{-2\mu - 2} (\lambda + |\xi|)^{-2m + 2\mu} |\hat{f}(\xi)|^2 \, d\xi \right)^{\frac{1}{2}}. \tag{5.19}$$

Proof. Part a) is a special case of the following more general result which is taken from [15], Section I.2.4. Let σ be a weight function which satisfies

$$\sigma(\xi)\sigma^{-1}(\eta) \leq C(1 + |\xi - \eta|^m)\,.$$

Then we have for $a' \in \mathcal{D}(\mathbb{R}^n)$ the inequality

$$\|a'f\|_{H^\sigma(\mathbb{R}^n)} \leq c(a') \|f\|_{H^\sigma(\mathbb{R}^n)} \tag{5.20}$$

with $c(a') := C \int (1 + |\xi|^m) |(Fa')(\xi)| d\xi$.

Part b) can be shown by standard arguments similar to those used in [11], Section 1.7.1, and [8], Lemma 1.4.5. $\square$

Using the above mentioned covering of M by local coordinate systems and a partion of unity subordinated to this covering we can define the spaces $H^{\Xi}, H^{\frac{1}{\Xi}}$ and $H^{\Xi^{(-j+3/2)}}$. From Lemma 5.4 and the trace results for model problems in $\mathbb{R}^n$ and $\mathbb{R}^n_+$ we immediatly obtain

Lemma 5.5 *Let $D_\Gamma(u) := (u|_\Gamma, \frac{\partial}{\partial\nu}u|_\Gamma, \ldots, (\frac{\partial}{\partial\nu})^{m-1}u|_\Gamma)$ be the Dirichlet boundary operator. Then*

$$(A(x,D,\lambda), D_\Gamma):\ H^{\Xi}(M) \longrightarrow H^{\frac{1}{\Xi}}(M) \times \prod_{j=1}^m H^{\Xi^{(-j+\frac{1}{2})}}(\Gamma)$$

is continuous with norm bounded by a constant independent of λ.

Theorem 5.6 *Let $A(x,D,\lambda)$ be an operator pencil of the form (1.1), acting on the manifold M with boundary Γ. Let A be N-elliptic with parameter in $[0,\infty)$ and assume that A degenerates regularly at the boundary Γ. Then for $\lambda \geq \lambda_0$ there exists a constant $C = C(\lambda_0)$, independent of u and λ, such that*

$$\begin{aligned} \|u\|_{\Xi,M} \quad &\leq \quad C\Big(\|A(x,D,\lambda)u\|_{\frac{1}{\Xi},M} + \sum_{j=1}^m \Big\|\Big(\frac{\partial}{\partial\nu}\Big)^{j-1} u\Big\|_{\Xi^{(-j+\frac{1}{2})},\Gamma} \qquad (5.21)\\ &\qquad + \quad \lambda^{m-\mu}\|u\|_{L_2(M)}\Big)\,. \end{aligned}$$

Proof. For the proof we use the standard technique of localization ("freezing the coefficients"). We only indicate the main steps. By means of a partition of unity it is sufficient to prove (5.21) for $u \in H^{\Xi}(M)$ with small support $\operatorname{supp} u \subset U$. In the case $U \cap \Gamma = \emptyset$, we fix $x_0 \in U$ and use local coordinates. We obtain from the a priori estimate for the model problem in $\mathbb{R}^n$ that

$$\begin{aligned} \|u\|_{\Xi,\mathbb{R}^n} \quad &\leq \quad C_1\left(\|A^{(0)}(x_0,D)u\|_{\frac{1}{\Xi},\mathbb{R}^n} + \lambda^{2m-2\mu}\|u\|_{L_2(\mathbb{R}^n)}\right)\\ &\leq \quad C_1\left(\|A(x,D)u\|_{\frac{1}{\Xi},\mathbb{R}^n} + \lambda^{2m-2\mu}\|u\|_{L_2(\mathbb{R}^n)}\right)\\ &\qquad + C_1\|(A(x,D) - A^{(0)}(x_0,D))u\|_{\frac{1}{\Xi},\mathbb{R}^n} \qquad (5.22) \end{aligned}$$

with a constant C_1 independent of u and λ.

We fix $\varepsilon > 0$. From Lemma 5.4 b) we obtain if the support of u is sufficiently small that

$$\|(A(x,D) - A^{(0)}(x_0,D))u\|_{\frac{1}{\Xi},\mathbb{R}^n} \leq \varepsilon\|u\|_{\Xi,\mathbb{R}^n} + C\|u\|_{\Xi^{(-1)},\mathbb{R}^n}\,. \qquad (5.23)$$

Here we have used that

$$\sum_{\alpha,k} \lambda^k \|D^\alpha u\|_{\frac{1}{\Xi},\mathbb{R}^n} \le C\|u\|_{\Xi,\mathbb{R}^n} \tag{5.24}$$

and

$$\sum_{\alpha,k} \lambda^k \|D^\alpha u\|_{\Psi,\mathbb{R}^n} \le C\|u\|_{\Xi^{(-1)},\mathbb{R}^n} \tag{5.25}$$

Now we use the interpolation inequality

$$\|u\|_{\Xi^{(-1)},\mathbb{R}^n} \le \varepsilon\|u\|_{\Xi,\mathbb{R}^n} + C\lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)} \tag{5.26}$$

which is a consequence of the interpolation inequality for the Sobolev spaces $H^s(\mathbb{R}^n)$ because of

$$\|u\|_{\Xi^{(-1)},\mathbb{R}^n} \approx \|u\|_{H^{m-1}(\mathbb{R}^n)} + \lambda^{m-\mu}\|u\|_{H^{\mu-1}(\mathbb{R}^n)}\,. \tag{5.27}$$

If we choose ε with $C_1\varepsilon < 1$ we obtain

$$\|u\|_{\Xi,\mathbb{R}^n} \le C\Big(\|A(x,D,\lambda)u\|_{\frac{1}{\Xi},\mathbb{R}^n} + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)}\Big)\,. \tag{5.28}$$

In the case $U\cap\Gamma \ne \emptyset$ we choose $x_0 \in U\cap\Gamma$, use local coordinates, and obtain in the same way as above

$$\begin{aligned}\|u\|_{\Xi,\mathbb{R}^n_+} \quad \le \quad & C\Big(\|A(x,D,\lambda)u\|_{\frac{1}{\Xi},\mathbb{R}^n_+} + \sum_{j=1}^m \|D_n^{j-1}u\|_{\Xi^{(-j+\frac{1}{2})},\mathbb{R}^{n-1}} \\ & + \lambda^{m-\mu}\|u\|_{L_2(\mathbb{R}^n)}\Big)\,, \end{aligned} \tag{5.29}$$

where we used the a priori estimate for $(A^{(0)}(x_0,D),(D_n^{j-1})_{j=1}^m)$. □

Now to finish the existence theory we present the construction of the right (rough) parametrix of the Dirichlet problem.

Suppose the assumptions of Theorem 5.6 to hold. We will see below that the solution constructed above for constant coefficients is a right parametrix with respect to the Sobolev spaces defined by the Newton polygon. We will define this parametrix, as usual, with the help of local coordinates. First of all we present the construction in the case of model domains $\mathbb{R}^n$ and $\mathbb{R}^n_+$.

Proposition 5.7 *Suppose $A(x,\xi,\lambda)$ satisfies the N-ellipticity condition and the coefficients are of the form* (5.15)*. Then there exists a bounded operator*

$$B : H^{\frac{1}{\Xi}}(\mathbb{R}^n) \to H^{\Xi}(\mathbb{R}^n) \tag{5.30}$$

such that

$$A(x, D, \lambda)B = I + T \tag{5.31}$$

where I denotes the identity operator in $H^{\frac{1}{\Xi}}(\mathbb{R}^n)$ and

$$T : H^{\frac{1}{\Xi}}(\mathbb{R}^n) \to H^{\Theta}(\mathbb{R}^n) \tag{5.32}$$

is continuous with norm bounded by a constant independent of λ. Here we posed $\Theta(\xi, \lambda) = (1 + |\xi|)/\Xi(\xi, \lambda)$.

Proof. We define B as a classical ps.d.o with symbol

$$B(x, \xi, \lambda) := \psi(\xi) \frac{1}{A_0(x, \xi, \lambda)}, \tag{5.33}$$

where $\psi \in C^{\infty}(\mathbb{R}^n)$ is a cut-off function with $\psi \equiv 0$ for $|\xi| \leq 1$ and $\psi \equiv 1$ for $|\xi| \geq 2$.

The continuity of operator (5.30) is equivalent to the statement that the $L_2 - L_2$ norm of the operator

$$(1 + |D|^2)^{\frac{\mu}{2}} (\lambda^2 + |D|^2)^{\frac{m-\mu}{2}} B(x, D, \lambda)(1 + |D|^2)^{\frac{\mu}{2}} (\lambda^2 + |D|^2)^{\frac{m-\mu}{2}}$$

can be estimated by a constant independent of λ. Using standard results on the L_2-boundedness of ps.d.o. (cf. [10], Section 2.4) we have to show the inequalities

$$\psi(\xi) \left| D_x^{\alpha} A_0^{-1}(x, \xi, \lambda) \right| \leq C_{\alpha} (1 + |\xi|)^{-2\mu} (\lambda + |\xi|)^{-2m+2\mu} .$$

For $|\alpha| = 0$ this inequality directly follows from N-ellipticity with parameter, to prove it for arbitrary α we must use the chain rule.

To prove (5.31)–(5.32) we write the operator T in the form

$$T = \tilde{T} + (A(x, D, \lambda) - A_0(x, D, \lambda))B$$

with $\tilde{T}u = A_0(x, D, \lambda)Bu - u$. Noting that

$$A(x, D, \lambda) - A_0(x, D, \lambda) : H^{\Xi}(\mathbb{R}^n) \to H^{\Theta}(\mathbb{R}^n)$$

is continuous, it is sufficient to prove (5.32) with T replaced by $\tilde{T}$. As above, this is equivalent to the uniform $L_2 - L_2$ boundedness of

$$(1 + |D|^2)^{-\frac{\mu-1}{2}} (\lambda^2 + |D|^2)^{-\frac{m-\mu}{2}} \tilde{T}\, (1 + |D|^2)^{\frac{\mu}{2}} (\lambda^2 + |D|^2)^{\frac{m-\mu}{2}} .$$

For this it is enough to show that the symbol $\tilde{T}(x, \xi, \lambda)$ of $\tilde{T}$ satisfies

$$(1 + |\xi|) \left| D_x^{\beta} \tilde{T}(x, \xi, \lambda) \right| \leq C_{\beta} . \tag{5.34}$$

The last inequality follows easily from the fact that for $|\xi| \geq 2$ we have

$$\tilde{T}(x, \xi, \lambda) = \sum_{0<|\alpha|\leq 2m} \frac{1}{\alpha!} \partial_\xi^\alpha A_0(x, \xi, \lambda)\; D_x^\alpha \,\frac{1}{A_0(x, \xi, \lambda)} \tag{5.35}$$

and from the estimates

$$\left| D_x^\beta \,\frac{1}{A_0(x, \xi, \lambda)} \right| \leq C\; (\Xi_P(\xi, \lambda))^{-1} \quad (|\xi| \geq 2) \tag{5.36}$$

and

$$|D_x^\gamma \partial_\xi^\alpha A_0(x, \xi, \lambda)| \leq C\; \Xi_P^{(-|\alpha|)}(\xi, \lambda) \quad (0 \leq |\alpha| \leq 2m)\,. \tag{5.37}$$

□

Proposition 5.8 *Suppose the conditions of Theorem* 5.6 *are satisfied and the coefficients of* $A(x, \xi, \lambda)$ *are of the form* (5.15). *Then there exists a bounded operator*

$$B : H^{\frac{1}{\Xi}}(\mathbb{R}^n_+) \times \prod_{j=1}^m H^{\Xi^{(-j+1/2)}}(\mathbb{R}^{n-1}) \to H^{\Xi}(\mathbb{R}^n_+) \tag{5.38}$$

such that

$$(A, \gamma_0, \gamma_0 D_n, \ldots, \gamma_0 D_n^{m-1})^t B = I + T\,, \tag{5.39}$$

where I *denotes the identity operator in*

$$H^{\frac{1}{\Xi}}(\mathbb{R}^n_+) \times \prod_{j=1}^m H^{\Xi^{(-j+1/2)}}(\mathbb{R}^{n-1})\,,$$

γ_0 *is the operator of taking the trace of the function at* $x_n = 0$, *and*

$$T : H^{\frac{1}{\Xi}}(\mathbb{R}^n_+) \times \prod_{j=1}^m H^{\Xi^{(-j+1/2)}}(\mathbb{R}^{n-1}) \to H^{\Theta}(\mathbb{R}^n_+) \times \prod_{j=1}^m H^{\Xi^{(-j+3/2)}}(\mathbb{R}^{n-1}) \tag{5.40}$$

is continuous with norm bounded by a constant independent of λ.

Proof. We set

$$B(f, g_1, \ldots, g_m) := B_0 f + \sum_{j=1}^m B_j(g_j - \gamma_0 D_n^{j-1} B_0 f)\,. \tag{5.41}$$

Here (compare Proposition 5.7)

$$B_0 f := R\psi(D)A_0^{-1}(x, D, \lambda)Ef \tag{5.42}$$

for $f \in H^{\frac{1}{\Xi}}(\mathbb{R}^n_+)$. As in (5.4) R and E are restriction and extension operators, respectively, $\psi(\xi)$ is the cut-off function from the proof of Proposition 5.7 and B_j for $j = 1, \ldots, m$ is a ps.d.o. in $\mathbb{R}^{n-1}$ (with x_n as parameter) defined by

$$(B_j g_j)(x', x_n) := \psi'(D')w_j(x', x_n, D', \lambda)g_j\,. \tag{5.43}$$

The symbol w_j in (5.43) is given by (compare (4.4))

$$w_j(x', x_n, \xi', \lambda) := \frac{1}{2\pi i}\int_{\gamma(\xi',\lambda)} \frac{M_j(x', 0, \xi', \tau, \lambda)}{A_+(x', 0, \xi', \tau, \lambda)} e^{ix_n\tau}\, d\tau\,, \tag{5.44}$$

where $A_+(x', 0, \xi', \tau, \lambda)$ and $M_j(x', 0, \xi', \tau, \lambda)$ are given by (4.5)–(4.6) with $A(\xi', \lambda)$ replaced by $A^{(0)}(x', 0, \xi', \lambda)$. The function $\psi'(\xi') \in C^\infty(\mathbb{R}^{n-1})$ is defined by $\psi'(\xi') := \psi(\xi', 0)$.

First of all we check that the operator (5.41) is bounded. The boundness of B_0 follows easily from the proof of Proposition 5.7. Now we proof the continuity of operators

$$B_j : H^{\Xi^{(-j+1/2)}}(\mathbb{R}^{n-1}) \to H^{\Xi}(\mathbb{R}^n_+)\,. \tag{5.45}$$

If the condition (5.15) is satisfied, then the operator B_j can be represented in the form $B_j = B_j^0 + B_j'$, where B_j^0 is a ps.d.o with symbol independent of x and the symbol of B_j' is independent of x, when the modulus of x is large enough.

In fact, using the norm

$$\Big(\sum_{l=0}^{m}\int_0^\infty \|D_n^l u(\cdot, x_n)\|^2_{\Xi^{(-l)},\mathbb{R}^{n-1}}\, dx_n\Big)^{1/2} \tag{5.46}$$

in $H^{\Xi}(\mathbb{R}^n_+)$ and Theorem 4.1 we can easily prove the continuity of operator B_j^0. To prove the continuity of B_j' we show the estimates

$$\psi'(\xi')\Big(\int_0^\infty \Big|D_{x'}^{\beta'} D_n^l w_j(x', x_n, \xi', \lambda)\Big|^2 dx_n\Big)^{1/2} \le C_{\beta'}\ \frac{\Xi^{(-j+1/2)}(\xi', \lambda)}{\Xi^{(-l)}(\xi', \lambda)}\,. \tag{5.47}$$

The case $\beta' = 0$ was treated in the proof of Theorem 4.1, the general case follows by the same method after differentiating in (5.44) under the integral sign.

Now we prove the continuity of the operator (5.40). The main step of the proof is reduced to showing that $AB_j = 0, \quad j = 1, \ldots, m$, up to a lower order operator, i.e. the operator

$$A(x, D, \lambda)B_j : H^{\Xi^{(-j+1/2)}}(\mathbb{R}^{n-1}) \to H^{\Theta}(\mathbb{R}^n_+) \tag{5.48}$$

is continuous.

Before proving this statement we finish the proof of (5.39). Denote by $T_0, T_1, \ldots, T_m$ the components of operator T. Operator T_0 maps the space on the left-hand side of (5.40) into $H^{\Theta}(\mathbb{R}^n_+)$ and $T_k,\ k > 0$, maps the space on the left-hand side of (5.40) into $H^{\Xi^{(-k+3/2)}}(\mathbb{R}^{n-1})$. We have

$$T_0\Big\{f, g_1, \ldots, g_m\Big\} = (AB_0 f - f) + \sum_{j=1}^{m} AB_j(g_j - \gamma_0 D_n^{j-1} B_0 f)\,.$$

The first term on the right-hand side belongs to $H^{\Theta}(\mathbb{R}^n_+)$ according to Proposition 5.7, the sum belongs to this space according to (5.48).

Turning to estimates of other components of T we note that according to (4.2)

$$\gamma_0 D_n^{k-1} B_j(x', x_n, D_n, \lambda)h = \delta_{jk}\psi'(D')h\,.$$

From this follows that for $k \geq 1$ we have

$$\begin{aligned} & T_k\Big\{f, g_1, \ldots, g_m\Big\} \\ = \quad & \gamma_0 D_n^{k-1} B_0 f + \sum_{j=1}^{m} \gamma_0 D_n^{k-1} B_j(g_j - \gamma_0 D_n^{k-1} B_0 f) - g_k \\ = \quad & (1 - \psi'(D'))(\gamma_0 D_n^{k-1} B_0 f - g_k)\,. \end{aligned} \tag{5.49}$$

The function $1 - \psi'(\xi')$ belongs to $\mathcal{D}(\mathbb{R}^{n-1})$ and, consequently, function (5.49) belongs to $H^{\infty}(\mathbb{R}^{n-1})$ for arbitrary $\{f, g_1, \ldots, g_m\}$ from the space on the left-hand side of (5.40).

To prove (5.48) we can suppose, without loss of generality, that the operator A is replaced by its principal part

$$A^{(0)}(x, D, \lambda) = A^{(0)}(x', 0, D, \lambda) + \Big(A^{(0)}(x, D, \lambda) - A^{(0)}(x', 0, D, \lambda)\Big)\,.$$

The composition of $A^{(0)}(x', 0, D, \lambda)$ and B_j is a ps.d.o. in $\mathbb{R}^{n-1}$ (depending on x_n) with the symbol

$$\begin{aligned} & C_j(x', x_n, \xi', \lambda) \\ = \quad & \psi'(\xi') \sum_{|\alpha'|=1}^{2m} \frac{1}{(\alpha')!} \partial_{\xi'}^{\alpha'} A^{(0)}(x', 0, \xi', D_n, \lambda) D_{x'}^{\alpha'} w_j(x', x_n, \xi', \lambda)\,. \end{aligned}$$

The summand corresponding to $\alpha' = 0$ is identically zero due to the definition of w_j. We can rewrite the symbol above in the form

$$\sum_{l=0}^{2m-1} c_l(x', \xi', \lambda) D_n^l w_j(x, \xi', \lambda),$$

where the coefficients $c_l(x', \xi', \lambda)$ and their derivatives with respect to x' admit the estimates

$$\left| D_{x'}^{\beta'} c_l(x', \xi', \lambda) \right| \leq C\ \Xi_P^{(-l-1)}(\xi', \lambda)\,.$$

Differentiating under the integral sign and repeating the argument of Theorem 4.1 we obtain

$$\Big(\int_0^\infty \left| D_{x'}^{\beta'} D_n^l w_j(x', x_n, \xi', \lambda) dx_n \right| \Big)^{1/2} \leq \frac{\Xi^{(-j+\frac{1}{2})}(\xi', \lambda)}{\Xi^{(-l)}(\xi', \lambda)}\,.$$

Remembering that $\Xi_P(\xi, \lambda) \equiv \Xi^2(\xi, \lambda)$, we easily deduce

$$\Big(\int_0^\infty \left| D_{x'}^{\beta'} C(x', x_n, \xi', \lambda) \right|^2 dx_n \Big)^{\frac{1}{2}} \leq C\ (1+|\xi'|)^{-1} \Xi^{(-j-\frac{1}{2})}(\xi', \lambda)\ \Xi(\xi', \lambda)\,.$$

From this follows the continuity of operator (5.48) with A replaced by $A^{(0)}(x', 0, D, \lambda)$.

Applying the Lagrange formula to the coefficients of the operator $A^{(0)}(x, D, \lambda) - A^{(0)}(x', 0, D, \lambda)$, we can rewrite this operator in the form

$$\sum_{l=0}^{2m} c'_l(x, \xi', \lambda) x_n D_n^l\,,$$

where

$$|D_{x'}^{\beta'} c'_l(x, \xi', \lambda)| \leq C\ \Xi(\xi', \lambda)\ \Xi^{(-l)}(\xi', \lambda)\,.$$

Using the relation

$$x_n \exp(ix_n\tau) = -i \frac{\partial}{\partial \tau} \exp(ix_n\tau)$$

and integrating under the sign of the contour integral in (4.4) we obtain that

$$x_n D_n^l w_j(x', x_n, \xi', \lambda) = il\ D_n^{l-1} w_j(x', x_n, \xi', \lambda).$$

Now we easily come to the inequalities

$$\Big(\int_0^\infty \left| x_n D_{x'}^{\beta'} D_n^l w_j(x', x_n, \xi', \lambda) \right|^2 dx_n \Big)^{\frac{1}{2}} \leq C\ \frac{\Xi^{(-j+\frac{1}{2})}(\xi', \lambda)}{(1+|\xi'|)\Xi^{(-l)}(\xi', \lambda)}\,.$$

The above estimates permit us to finish the proof of the Proposition. □

Now we return to the case of a manifold as considered in Theorem 5.6. To construct a right parametrix for (A, D_Γ), we will use local coordinates.

Let $\{\mathcal{O}_j\}_{j=1,\dots,N}$ be a covering of M with coordinate neighborhoods where $\mathcal{O}_j$ is homeomorphic to an open ball in $\mathbb{R}^n$ for $j = 1, \dots, N'$ and homeomorphic to a semi-ball $\{x \in \mathbb{R}^n : |x| < r_j,\ x_n > 0\}$ for $j = N'+1, \dots, N$. In local coordinates corresponding to the neighborhood $\mathcal{O}_k$ we obtain an operator A_k in $\mathbb{R}^n$ for $k = 1, \dots, N'$ and in $\mathbb{R}^n_+$ for $k = N'+1, \dots, N$. We may assume that the coefficients of A in these local coordinates have the form (5.15).

For $k = 1, \dots, N'$ we define a local right parametrix B_k to the operator A_k according to Proposition 5.7; for $k = N'+1, \dots, N$ this can be done due to Proposition 5.8. We then set

$$B := \sum_{j=1}^{N} \psi_k B_k(\phi_k\ \cdot)$$

where $\{\phi_k\}_k$ is a partition of unity subordinated to the covering of M and ψ_k has support in $\mathcal{O}_k$ and is equal to 1 in a neighborhood of $\operatorname{supp}\phi_k$. From the previous two propositions we obtain the following theorem.

Theorem 5.9 *The operator B defined above is a right rough parametrix to (A, D_Γ) in the sense that*

$$(A, D_\Gamma)B = I + T\,, \tag{5.50}$$

where I denotes the identity operator in $H^{\frac{1}{\Xi}}(M) \times \prod_{j=1}^m H^{\Xi^{(-j+1/2)}}(\Gamma)$ and

$$T : H^{\frac{1}{\Xi}}(M) \times \prod_{j=1}^m H^{\Xi^{(-j+1/2)}}(\Gamma) \to H^{\Theta}(M) \times \prod_{j=1}^m H^{\Xi^{(-j+3/2)}}(\Gamma) \tag{5.51}$$

is continuous with norm bounded by a constant independent of λ.

As the norms considered in (5.51) are for every fixed λ equivalent to the corresponding norms in the standard Sobolev spaces, we obtain the compactness of T and the Fredholm property of (A, D_Γ). Note also that for fixed λ the operator (A, D_Γ) is elliptic in the usual sense. From the last remark we also see that the index of this operator is equal to zero.

References

[1] Agmon, S.: On the eigenfunctions and on the eigenvalues of general elliptic boundary value problems. *Comm. Pure Appl. Math.* **15** (1962), 119-147.

[2] Agmon, S., Douglis, A., Nirenberg, L.: Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions. I. *Comm. Pure Appl. Math.* **22** (1959), 623-727.

[3] Agranovich, M. S., Vishik, M. I. (1964): Elliptic problems with parameter and parabolic problems of general form (Russian). *Uspekhi Mat. Nauk* **19** (1964), No. 3, 53-161. English transl. in *Russian Math. Surv.* **19** (1964), No. 3, 53-157.

[4] Agranovich, M. S.: Elliptic boundary problems. *Encyclopaedia Math. Sci.* **79** (1997), 1–144.

[5] Denk, R., Mennicken, R., Volevich, L.: The Newton polygon and elliptic problems with parameter. *Math. Nachr.* **192** (1998), 125-157.

[6] Frank, L.: Coercive singular perturbations. I. A priori estimates. *Ann. Mat. Pura Appl.* (4) **119** (1979), 41–113.

[7] Fuks, B. A., Levin, V. I.: *Functions of a Complex Variable and some of their Applications.* Vol. II. Pergamon Press, London etc., 1961.

[8] Gindikin, S. G., Volevich, L. R.: *The Method of Newton's Polyhedron in the Theory of Partial Differential Equations*, Math. Appl. (Soviet Ser.) **86**, Kluwer Academic, Dordrecht, 1992.

[9] Kozhevnikov, A.: Asymptotics of the spectrum of Douglis–Nirenberg elliptic operators on a closed manifold. Math. Nachr. **182** (1996), 261–293.

[10] Kumano-go, H.: *Pseudo-Differential Operators.* MIT Press, Cambridge etc., 1981.

[11] Lions, J. L., Magenes, E.: *Non-Homogeneous Boundary Value Problems and Applications*, Vol. I, Springer-Verlag, Berlin etc., 1972.

[12] Nazarov, S. A.: The Vishik–Lyusternik method for elliptic boundary value problems in regions with conic points. I. The problem in a cone (Russian). *Sibirsk. Mat. Zh.* **22** (1981), No. 4, 142-163. English transl. in *Siberian Math. J.* **22** (1982), 594-611.

[13] Vishik, M. I., Lyusternik, L. A.: Regular degeneration and boundary layer for linear differential equations with small parameter (Russian). *Uspehi Mat. Nauk (N.S.)* **12** (1957), No. 5 (77), 3-122. English transl. in *Amer. Math. Soc. Transl.* (2) **20** (1962), 239-364.

[14] Vainberg, B. R., Grushin, V. V.: Uniformly nonelliptic problems II. Uniformly non-coercive boundary problems for elliptic equations and systems (Russian). *Mat. Sb.* **73** (1967), No. 1, 126-154. English transl. in *Math. USSR-Sb.* **2** (1967), 111-133.

[15] Volevich, L. R., Paneah, B. P.: Some spaces of generalized functions and embedding theorems (Russian). *Uspehi Mat. Nauk* **20** (1965), No. 1 (121), 3-74. English transl. in *Russian Math. Surv.* **20** (1964), No. 1, 1-73.

[16] Volevich L. R., Shirikyan, A. R.: Remarks on Calderón projections for elliptic equations with parameter. *Keldysh Inst. Appl. Math. Preprint* **48**, 1997.