THE ANALOGY OF CAMERON-JOHNSON'S THEOREM FOR THE LINEAR $\mathbb{C}$-ANALYTIC EQUATIONS IN THE HILBERTIAN SPACE.

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Abstract. The well-known Cameron-Johnson's theorem affirms that the equation $x' = A(t)x \quad (*)$ with recurrent (almost periodic by Bhor) matrix may be reduced by the Lyapunov-Perron transformation to the equation $y' = B(t)y$ with skew-symmetric matrix $B(t)$, if all solutions of the equation (*) and all its limit equations are bounded on the whole real axis. The generalization of this result on the linear $\mathbb{C}$-analytic equations in the Hilbertian space is given.

The well-known Cameron-Johnson's theorem [1,2] affirms that the equation

$$x' = A(t)x \tag{1}$$

with recurrent (almost periodic by Bhor) matrix may be reduced by the Lyapunov-Perron transformation to the equation $y' = B(t)y$ with skew-symmetric matrix $B(t)$, if all solutions of the equation (1) and all its limit equations are bounded on the whole real axis. The generalization of this result on the linear $\mathbb{C}$-analytic equations in the Hilbertian space is our main scope in the article.

Let $\mathbb{R} = (-\infty, +\infty), \mathbb{C}^m$ $- m-$dimensional complex Euclidean space, G be a domain from $\mathbb{C}^m, \mathbb{H}$ be a real or complex Hilbertian space with scalar product $< \cdot, \cdot >$ and with norm $| \cdot |^2 =< \cdot, \cdot >$. Denote by $\mathbb{H}_w$ a space $\mathbb{H}$ equipped with weak topology, and by $[\mathbb{H}]$ (respectively $[\mathbb{H}_w]$) the family of all linear continuous operators acting into $\mathbb{H}$ (resp. $\mathbb{H}_w$) and equipped with operational norm (with weak topology). Assume that $\mathcal{H}(G, [\mathbb{H}]))$ (resp. $\mathcal{H}(G, [\mathbb{H}_w]), \mathcal{H}(G, \mathbb{C}^m))$ is the family of all holomorphic functions $h : G \to [\mathbb{H}]$ (resp. $h : G \to [\mathbb{H}_w], h : G \to \mathbb{C}^m$) equipped with compact-open topology and consider the system

$$\begin{cases} x' = \mathcal{A}(z)x \\ z' = \Phi(z) \end{cases} \qquad (z \in G), \tag{2}$$

where $\Phi \in \mathcal{H}(G, \mathbb{C}^m)$ and $\mathcal{A} \in \mathcal{H}(G, [\mathbb{H}])$. We suppose that the second equation of system (2) generates the dynamical system $(G, \mathbb{R}, \sigma)$ on G. Denote by $U(t, z)$ the Cauchy's operator of equation

$$x' = \mathcal{A}(zt)x \quad (z \in G), \tag{3}$$

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where $zt = \sigma(t,z)$. It follows from general property of solutions of differential equations (see, for example, [3-5]) that the family of operators $\{U(t,z)|t \in \mathbb{R}, z \in G\}$ satisfies the following conditions:

1. $U(0,z) = I \quad (\forall z \in G)$, where I is a unit operator on $\mathbb{H}$;
2. $U(t+\tau, z) = U(t, z\tau)U(\tau, z) \quad (\forall t, \tau \in \mathbb{R}$ and $z \in G$);
3. the mapping $U : \mathbb{R} \times G \to [\mathbb{H}] \quad (U : (t,z) \mapsto U(t,z))$ is continuous and for every $t \in \mathbb{R}$ the mapping $U(t,\cdot) : G \to [\mathbb{H}]$ is holomorphic.

The following assertion takes place.

Theorem 1. *Suppose that there exists a positive canstant C such that*

$$\|U(t,z)\| \leq C \tag{4}$$

for all $t \in \mathbb{R}$ and $z \in G$, then there exists $P \in \mathcal{H}(G,[\mathbb{H}])$ such that:

a. P- is biholomorphic, i.e. the operator $P(z)$ is invertible for all $z \in G$ and the mapping $P^{-1} : G \to [\mathbb{H}]$ $(P^{-1} : z \mapsto P^{-1}(z))$ is holomorphic;

b. $P_*(z) = P(z)$ for all $z \in G$, where $P_*(z)$ is an operator adjoint for $P(z)$;

c. the operator $P(z)$ is positive definite for all $z \in G$;

d. $C^{-1}|x| \leq |P(z)x| \leq C|x|$ for all $z \in G$ and $x \in \mathbb{H}$;

e. the change of variables $x = P(zt)y$ takes equation (3) into

$$y' = \mathcal{B}(zt)y \tag{5}$$

with skew-Hermitian operator $B \in \mathcal{H}(G,[\mathbb{H}])$, i.e. $\mathcal{B}_*(z) = -\mathcal{B}(z)$ for all $z \in G$.

Proof. Let $S^t : \mathcal{H}(G,[\mathbb{H}]) \to \mathcal{H}(G,[\mathbb{H}])$ (resp. $\mathcal{H}(G,[\mathbb{H}_w]) \to \mathcal{H}(G,[\mathbb{H}_w])$) be a mapping defined by equality $(S^t f)(z) = U_*(t,z)f(zt)U(t,z)$ for all $z \in G$ and $f \in \mathcal{H}(G,[\mathbb{H}])$ (resp. $f \in \mathcal{H}(G,[\mathbb{H}_w])$). It is easy to verify that the family of mappings $\{S^t|t \in \mathbb{R}\}$ satisies the following conditions:

4. $S^0 = I$, where I is a unit mapping on $\mathcal{H}(G,[\mathbb{H}])$ (resp. $\mathcal{H}(G,[\mathbb{H}_w])$);
5. $S^t S^\tau = S^{t+\tau}$ for all $t, \tau \in \mathbb{R}$;
6. the mapping S^t ($t \in \mathbb{R}$) is linear and continuous on $\mathcal{H}(G,[\mathbb{H}])$ (resp. $\mathcal{H}(G,[\mathbb{H}_w])$).

It follows from 4.-6. that the family of mappings $\{S^t|t \in \mathbb{R}\}$ is a commutative group of linear continuous operators on $\mathcal{H}(G,[\mathbb{H}])$ (resp. $\mathcal{H}(G,[\mathbb{H}_w])$).

Denote by $K = \{\mathcal{A} \in [\mathbb{H}]| \quad \|\mathcal{A}\| \leq C^2\}$ and we note that the set K is weak closed. According to Tihonov's theorem the set K is weakly compact, because every bounded and closed set in $\mathbb{H}$ is weakly compact. It follows from Cauchy's integral formula (see, for example, [6,p.339]) and theorem of Ascoli [6] that the set $\Gamma = \{\gamma \in \mathcal{H}(G,[\mathbb{H}]| \quad \|\gamma(z)\| \leq C^2\}$ is compact in the space $\mathcal{H}(G,[\mathbb{H}_w])$. We suppose that $V = \overline{conv}\{S^tQ|t \in \mathbb{R}\}$, where $Q(z) = I(\forall z \in G), I$ is a unit operator in $\mathbb{H}$ and $\overline{conv}$ is a weak closure of the convex hulle of $\{S^tQ|t \in \mathbb{R}\}$. We note that the set V is invariant, i.e. $S^tV \subseteq V$ for all $t \in \mathbb{R}$, because the family of operators $\{S^t\}$ is linear and continuous in $\mathcal{H}(G,[\mathbb{H}_w])$. It is easy to see that

$$< S^tQ(z)x, x >=< U_*(t,z)U(t,z)x, x >= |U(t,z)x|^2 \tag{6}$$

for all $z \in G, t \in \mathbb{R}$ and $x \in \mathbb{H}$. It follows from (4) and (6) that

$$C^{-2}|x|^2 \leq < S^tQ(z)x, x > \leq C^2|x|^2 \tag{7}$$

and, consequently,

$$C^{-2}|x|^2 \leq < \mathcal{R}(z)x, x > \leq C^2|x|^2 \tag{8}$$

for all $\mathcal{R} \in V, z \in G$ and $x \in \mathbb{H}$. In addition, the equality $\mathcal{R}_*(z) = \mathcal{R}(z)$ takes place for all $z \in G$ and $\mathcal{R} \in V$. According to equality (8) we have $\|\mathcal{R}(z)\| \leq C^2$ for all $z \in G$ and, consequently, $V \subseteq \Gamma$. According to the theorem of Markov-Kakutany [7] the group $\{S^t | t \in \mathbb{R}\}$ admits in V at last one fixe-point, i.e. there exists $\mathcal{R} \in V$ such that $U_*(t,z)\mathcal{R}(zt)U(t,z) = \mathcal{R}(z)$ for all $z \in G$ and $t \in \mathbb{R}$. We note that the operator $\mathcal{R}(z)$ is self-adjoint and according to equality (8) it is positive definite. It follows from Theorem 12.33 [8] (see also [9,p.65]) that there exists a unique revertible self-adjoint and positive definite operator $M(z)$ such that $M^2(z) = \mathcal{R}(z)$ for all $z \in G$ and according to inequality (8) we have

$$C^{-1}|x| \leq |M(z)x| \leq C|x|$$

for all $z \in G$ and $x \in \mathbb{H}$. Finally, according to equality

$$\mathcal{R}^{\alpha}(z) = -\frac{1}{2\pi i} \oint\limits_L \lambda^{\alpha} (\mathcal{R}(z) - \lambda I)^{-1} d\lambda$$

($\alpha = \pm\frac{1}{2}$), where L is a simple contour enclosing the spectrum of operator $\mathcal{R}(z)$, we have that the mapping $M : G \to [\mathbb{H}]$ is biholomorhic.

We note that $M(zt)U(t,z)M^{-1}(z) = M^{-1}(zt){U_*}^{-1}(t,z)M(z)$ because

$$U_*(t,z)M^2(zt)U(t,z) = M^2(z)$$

for all $t \in \mathbb{R}$ and $z \in G$. We put $\mathcal{V}(t,z) = M(zt)U(t,z)M^{-1}(z)$ and we note that the family of operators $\{\mathcal{V}(t,z) | t \in \mathbb{R}, z \in G\}$ satisfies the condition 1.-3. and in addition:

8.$\mathcal{V}_*(t,z) = \mathcal{V}^{-1}(t,z)$ for all $t \in \mathbb{R}$ and $z \in G$;

9. $U(t,z)P(z) = P(zt)\mathcal{V}(t,z)$ for all $t \in \mathbb{R}$ and $z \in G$, where $P(z) = M^{-1}(z)$.

Denote by

$$B(z) = \frac{d}{dt}\mathcal{V}(t,z)|_{t=0} = [\dot{M}(z) + M(z)\mathcal{A}(z)]P(z), \tag{9}$$

where $\dot{M}(z) = \frac{d}{dt}M(zt)|_{t=0} = \frac{dM(z)}{dz}\Phi(z)$, then $\mathcal{B} \in \mathcal{H}(G, [\mathbb{H}])$ and

$$\frac{d\mathcal{V}(t,z)}{dt} = \mathcal{B}(zt)\mathcal{V}(t,z) \tag{10}$$

for all $t \in \mathbb{R}$ and $z \in G$ and, consequently, the operator $\mathcal{V}(t,z)$ is a Cauchy's operator of equation (5). On the other hand, from condition 8. we have

$$\begin{cases} {\mathcal{V}'_*}^{-1}(t,z) = -\mathcal{B}_*(zt){\mathcal{V}_*}^{-1}(t,z) \\ {\mathcal{V}_*}^{-1}(0,z) = I \end{cases} \tag{11}$$

for all $t \in \mathbb{R}$ and $z \in G$. It follows from the equalities (10),(11) and condition 8. that $\mathcal{B}_*(z) = -\mathcal{B}(z)$ for all $z \in G$. Finally, to finish the proof of theorem it is sufficient to remark that the change of variables $x = P(zt)y$ takes equation (3) into equation (5). The theorem ic proved.

Remark. *a. If the point $z \in G$ is stationary (ω-periodic, quasiperiodic, almost periodic etc), then according to [10] the operator-function $P(zt)$ will be also stationary (ω-periodic, quasiperiodic, almost periodic etc).*

b. The Theorem 1 takes place for the system of difference equations

$$\begin{cases} x(k+1) = \mathcal{A}(zk)x(k) \\ z(k+1) = \Phi(zk) \end{cases} \qquad (k \in \mathbb{Z}),$$

where $\mathcal{A} \in \mathcal{H}(G, [\mathbb{H}])$ and $\Phi \in \mathcal{H}(G, \mathbb{C}^m)$ and also for the system of differential and difference equations with multidimensional time.

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