The existence of moments of solutions to transport equations with inelastic scattering and their application in the asymptotic analysis

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Abstract

In this paper we prove the existence of all moments of solutions to the time-depended spatially homogeneous transport equation describing elastic and inelastic scattering of particles. The proof uses Arendt-Batty-Robinson generation theorem for resolvent positive operators. As an application we carry out the asymptotic analysis of the full transport equation with dominant elastic scattering and, using the results of the first part of the paper, we show that its solution can be approximated in the L_1-norm by the solution of the limit equation obtained by formal asymptotic expansion.

Key words: evolution equations, positive semigroups, inelastic scattering, singular perturbations, hydrodynamic limit

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1 Introduction

Recent several years witnessed a renewed interest in transport equations describing both elastic and inelastic scattering of particles. Such models are relevant to the semiconductor processes (see e.g. [17, 9, 18, 19] and many others), and also to various electron and neutron transport processes in gases and solids (see e.g. [16] and references therein, also [8, 3]). The term "inelastic scattering" refers here to the fact that the moving particles can exchange quantized portions of energy with the background – a more detailed description of the physical situation leading to such models is given in Section 2.

Elastic and inelastic scattering in a given model can occur with different intensity or on different time scales. Mathematically this is accounted for by the introduction of dimensionless parameters multiplying appropriate terms of the equation and indicating their relative magnitudes.

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In some cases certain processes dominate the others which justifies analysing the behaviour of the equation when relevant parameters tend to infinity, and studying possible limit cases.

Such an analysis is also important from the computational point of view as the limit equations are usually much simpler than the original ones. Thus, if we know that the difference between the solutions to the limit equation and to the original equation is small, we can use the former, obtained with much less computational effort, as an approximation to the latter.

An exhaustive discussion of various asymptotic limits for semiconductor equations can be found in [9]. The aim of this paper, which is a part of a wider collection of works ([4, 7, 8, 11, 14]), is to investigate possible asymptotic limits for another class of transport equations. These equations contain no external field, which makes them easier than semiconductor equations, but contrary to the latter, contain the unbounded inelastic scattering operator which makes the rigorous mathematical analysis quite involved.

The main aim of this paper is to prove that solutions of time-dependent spatially homogeneous equation with unbounded inelastic scattering operator have all the moments. This is done by careful estimation of the resolvent in weighted L_1 spaces and an application of Arendt-Batty-Robinson theorem for positive semigroups [2].

The relevance of this result is demonstrated in the last section of the paper. We use it to prove that the formal asymptotic expansion yields the correct approximation of the solution of the transport equation, and to provide a sharp estimate of the error of this approximation in physically natural L_1 space.

2 Description of model

We consider a gas of test particles having mass m propagating through a host medium of particles having mass M and undergoing two types of collisions with them: elastic and inelastic ones.

We assume that the host medium is at thermodynamical equilibrium with temperature T and we consider the Lorentz gas limit ($m/M << 1$, in other words, the test particles collide with something like a rigid net – they can be deflected (elastic collisions) or exchange quanta of energy with the background (inelastic collisions), but the classical exchange of kinetic energy is ruled out). For technical reasons we assume that the scattering is isotropic.

Furthermore, the background particles can occur in two energy states: a ground level and an excited level, spaced by an energy gap denoted by ΔE (which throughout the paper will be normalized to one). The number densities of the particles in the ground and in the excited state are assumed to be constant; we denote them by n_1 and n_2, respectively. They are related through the Boltzmann factor $a := n_2/n_1 = e^{-\frac{\Delta E}{kT}}$, where k is the Boltzmann constant.

We are concerned with two scattering operators: inelastic and elastic ones. The elastic scattering operator is given by

$$(C^e f)(\boldsymbol{v}) = -f(\boldsymbol{v})\lambda(v) + \frac{1}{4\pi}\lambda(v)\int\limits_{S^2} f(v\boldsymbol{\omega}')d\boldsymbol{\omega}', \tag{2.1}$$

where f is the density of test particles, S^2 denotes the unit sphere in $\mathbb{R}^3_v$, $\boldsymbol{v} = v\boldsymbol{\omega}$ with $v \in [0,\infty[$ and $\boldsymbol{\omega} \in S^2$. Since we will be working with the Maxwell molecules, we assume that the elastic scattering cross-section λ is a measurable function satisfying:

$$\lambda_{min} \leq \lambda(v) \leq \lambda_{max}$$

for almost all $v \in [0,\infty[$ and some $0 < \lambda_{min} \leq \lambda_{max} < +\infty$.

The inelastic scattering operator is given by

$$\begin{aligned}(C^i f)(\boldsymbol{v}) &= -f(\boldsymbol{v})\left(\frac{n_2 v_+}{v}\nu(v_+) - n_1 H(v_-^2)\nu(v)\right)\\ &+\frac{n_1}{4\pi v}\int\limits_{S^2} v_+\nu(v_+)f(v_+\boldsymbol{\omega}')d\boldsymbol{\omega}' + \frac{n_2}{4\pi}\int\limits_{S^2} H(v_-^2)\nu(v)f(v_-\boldsymbol{\omega}')d\boldsymbol{\omega}'.\end{aligned} \tag{2.2}$$

where H denotes the Heaviside function, and $v_\pm = \sqrt{v^2 \pm 1}$.

As before, the Maxwell molecule hypothesis requires that the inelastic scattering cross-section ν satisfies

$$\nu_{min} \leq \nu(v) \leq \nu_{max} \quad \text{for } v \in [1,\infty[. \tag{2.3}$$

This model is a simplified version of the general inelastic scattering model developed in [16, 10, 3], where a more detailed explanation of the form of the model is provided.

It greatly simplifies the notation if one changes the velocity to the normalized kinetic energy according to the formula $v^2 = \xi$. In what follows, with some abuse of notation, we shall write $g(\xi \pm 1)$ instead of $g(v_\pm)$.

The physically relevant space for our considerations is the space

$$X = L_1(\mathbb{R}^3),$$

as the integral of the density f over $\mathbb{R}^3$ gives the total number of particles. It is useful to note that when we pass to the ξ variable, then the measure $d\boldsymbol{v} = v^2 dv d\boldsymbol{\omega}$ changes into $d\boldsymbol{\xi} = \frac{1}{2}\sqrt{\xi}d\xi d\boldsymbol{\omega}$.

Since we will be working quite a lot with isotropic ($\boldsymbol{\omega}$ independent) functions, it is worthwhile to note that for such functions we have $X = L_1([0,\infty[, v^2dv) = L_1([0,\infty[, \sqrt{\xi}d\xi)$.

We note that the inelastic scattering operator splits in a natural way into two operators: the down-scattering operator C^i_-

$$\begin{aligned}(C^i_- f)(\xi,\boldsymbol{\omega}) &= -\alpha_- f(\xi,\boldsymbol{\omega}) + (K_- f)(\xi) \qquad (2.4)\\ &= -n_1 H(\xi-1)\nu(\xi)f(\xi,\boldsymbol{\omega}) + \frac{n_1}{4\pi}\sqrt{\frac{\xi+1}{\xi}}\nu(\xi+1)\int\limits_{S^2} f(\xi+1,\boldsymbol{\omega})d\boldsymbol{\omega}\end{aligned}$$

which is bounded in X (see [3]) and the up-scattering operator

$$\begin{aligned}(C^i_+ f)(\xi,\boldsymbol{\omega}) &= -\alpha_+ f(\xi,\boldsymbol{\omega}) + (K_+ f)(\xi) \\ &= -n_2\sqrt{\frac{\xi+1}{\xi}}\nu(\xi+1)f(\xi,\boldsymbol{\omega}) + \frac{n_2}{4\pi}H(\xi-1)\nu(\xi)\int\limits_{S^2} f(\xi-1,\boldsymbol{\omega}')d\boldsymbol{\omega}' \end{aligned} \tag{2.5}$$

which is unbounded in X. We consider C^i_+ (and thus C^i and $C^i + C^e$) on the domain $D(C) = L_1(\mathbb{R}^3, (1+v^{-1})d\boldsymbol{v})$. It is clear that $D(C)$ is dense in X. Note that for isotropic functions we have $D(C) = L_1([0,\infty[, (\sqrt{\xi}+1)d\xi)$.

We shall consider the time-dependent spatially homogeneous Boltzmann equation for elastic and inelastic scattering

$$\partial_t f = C^e f + C^i f \tag{2.6}$$

with the initial condition

$$f(\boldsymbol{v},0) = \overset{\circ}{f}(\boldsymbol{v}). \tag{2.7}$$

The solvability of (2.6) in X follows from the more general results in [3, 6] where also various effects of the spatial inhomogeneity were taken into account. In this paper, however, we are concerned with the existence of the moments of the solution to (2.6): $(\boldsymbol{v},t) \to v^k f(\boldsymbol{v},t)$, $k = 1, 2\ldots$ or, in other words, with the existence of solutions of (2.6) in the weighted spaces

$$X_k = L_1(\mathbb{R}^3, (1+v^k)d\boldsymbol{v}) = L_1(\mathbb{R}^3, (1+\xi^{k/2})d\boldsymbol{\xi}).$$

Remark 2.1 Before we proceed any further, we note that it seems that this problem cannot be dealt with by the methods developed in [3, 6], as the the latter heavily depended on the conservativity of the collision operators in X. In X_k, however, the conservativity is lost. To show this, let us first note that

$$\xi^{k/2}(C^e f)(\xi,\boldsymbol{\omega}) = C^e(\xi^{k/2}f)(\xi,\boldsymbol{\omega})$$

but, denoting $F = \xi^{k/2}f$, we have

$$\begin{aligned}(\xi^{k/2}C^i f)(\xi,\boldsymbol{\omega}) &= -(\alpha_- + \alpha_+)\xi^{k/2}f(\xi,\boldsymbol{\omega}) \\ &\quad + \frac{n_2 H(\xi-1)\nu(\xi)}{4\pi}\frac{\xi^{k/2}}{(\xi-1)^{k/2}}\int\limits_{S^2}(\xi-1)^{k/2}f(\xi-1,\boldsymbol{\omega}')d\boldsymbol{\omega}' \\ &\quad + \frac{n_1}{4\pi}\sqrt{\frac{\xi+1}{\xi}}\nu(\xi+1)\frac{\xi^{k/2}}{(\xi+1)^{k/2}}\int\limits_{S^2}(\xi+1)^{k/2}f(\xi+1,\boldsymbol{\omega}')d\boldsymbol{\omega}'.\end{aligned}$$

Thus

$$\int\limits_{\mathbb{R}^3}(\xi^{k/2}C^e f)(\xi,\boldsymbol{\omega})d\boldsymbol{\xi} = 0,$$

whereas for the inelastic scattering term upon integration we obtain

$$\int_{\mathbb{R}^3} (\xi^{k/2} C^i_- f)(\xi, \boldsymbol{\omega}) d\boldsymbol{\xi} = n_1 \int_{\mathbb{R}^3} \nu(\xi) H(\xi - 1) \left(\frac{(\xi-1)^{k/2}}{\xi^{k/2}} - 1 \right) \xi^{k/2} f(\xi, \boldsymbol{\omega}') d\boldsymbol{\xi}. \tag{2.8}$$

and similarly,

$$\int_{\mathbb{R}^3} (\xi^{k/2} C^i_+ f)(\xi, \boldsymbol{\omega}) d\boldsymbol{\xi} = n_2 \int_{\mathbb{R}^3} \nu(\xi+1) \sqrt{\frac{\xi+1}{\xi}} \left(\frac{(\xi+1)^{k/2}}{\xi^{k/2}} - 1 \right) \xi^{k/2} f(\xi, \boldsymbol{\omega}') d\boldsymbol{\xi}.$$

To see better the meaning of the above formulas, let us take $k = 2$, $\nu = 1$ and f independent of $\boldsymbol{\omega}$. Then we obtain that

$$\int_{\mathbb{R}^3} \xi^{k/2} (C^i f)(\xi, \boldsymbol{\omega}) d\boldsymbol{\xi} = n_2 \int_0^\infty \sqrt{\xi+1} f(\xi) d\xi - n_1 \int_1^\infty \sqrt{\xi} f(\xi) d\xi.$$

Clearly, if f has the support in the unit ball, then the right-hand side is positive. On the other hand, if the function f has the support outside the ball of radius $r > \frac{n_2^2}{n_1^2 - n_2^2}$ (recall that $n_2 < n_1$), then the right-hand side is negative. Therefore C^i is not dissipative, let alone conservative, in X_k. ■

Due to the translational character of the operator C^i it is convenient to introduce the reduced energy $\xi \in [0, 1[$ and to re-define all the functions as functional sequences in the following way: for $n = 0, 1, 2 \ldots$ and $\xi \in [0, 1[$

$$\begin{aligned}
f_n(\xi, \boldsymbol{\omega}) &= f(\xi + n, \boldsymbol{\omega}), \\
\lambda_n(\xi) &= \lambda(\xi + n), \\
\nu_n(\xi) &= \nu(\xi + n), \\
\xi_n &= \sqrt{\xi + n} \\
p_n(\xi) &= \frac{\xi_{n+1}}{\xi_n} \nu_{n+1}(\xi) \\
F_n(\xi) &= \frac{1}{4\pi} \int_{S^2} f_n(\xi, \boldsymbol{\omega}) d\boldsymbol{\omega}.
\end{aligned}$$

With this notation the equation (2.6) can be written in the recursive form

$$\begin{aligned}
\partial_t f_0 &= -(\lambda_0 + n_2 p_0) f_0 + \lambda_0 F_0 + n_1 p_0 F_1, \\
&\vdots \\
\partial_t f_n &= -(\lambda_n + n_2 p_n + n_1 \nu_n) f_n + \lambda_n F_n + n_1 p_n F_{n+1} + n_2 \nu_n F_{n-1} \\
&\vdots,
\end{aligned} \tag{2.9}$$

where $n = 1, 2, \ldots$. In this notation the X norm of f is finite if and only if

$$\sum_{j=0}^{\infty} \int_{S^2} \left(\int_0^1 \xi_j |f_j(\xi, \boldsymbol{\omega})| d\xi \right) d\boldsymbol{\omega} \tag{2.10}$$

and $f \in D(C)$ if additionally

$$\int_{S^2} \int_0^1 |f_0(\xi, \boldsymbol{\omega})| d\xi d\boldsymbol{\omega} < \infty.$$

Similarly, $f \in X_k$, that is, f has k-th moment finite, if

$$\|f\|_k = \int_{S^2} \left(\int_0^1 \xi_0 |f_0(\xi)| d\xi + \sum_{j=1}^{\infty} \int_0^1 \xi_j^{k+1} |f_j(\xi, \boldsymbol{\omega})| d\xi \right) d\boldsymbol{\omega} < \infty.$$

Therefore, the function $f \in D(C)$ has finite k-th moment if

$$\int_{S^2} \left(\int_0^1 |f_0(\xi)| d\xi + \sum_{j=1}^{\infty} \int_0^1 \xi_j^{k+1} |f_j(\xi, \boldsymbol{\omega})| d\xi \right) d\boldsymbol{\omega} < \infty. \tag{2.11}$$

We see that the norms in X_k consist of two parts: one is the integral over $S^2 \times [0, 1]$, the other is the summation. It will simplify the notation if we introduce for $f = (f_j)_{j \in \mathbb{N}}$

$$\mid f \mid_k = |f_0| \xi_0 + \sum_{j=1}^{\infty} j^{\frac{k}{2}} \xi_j |f_j|.$$

Thus, we have

$$\|f\|_k = \int_{S^2} \int_0^1 \mid f \mid_k d\xi.$$

3 Generation of the semigroup in X_k

In this section we shall prove the main generation result.

Theorem 3.1 *The operator $C = C^i + C^e$ with the domain defined by the condition (2.11) generates a semigroup in X_k, for any $k \in \mathbb{N}$.*

Proof. In the first step we note that the operators C^e and C^i_- are bounded in any X_k. The statement for C^e follows from the fact that it commutes with the multiplication by $\xi^{k/2}$. For

C^i_- we have from (2.8) and (2.3) that for nonnegative f

$$\begin{aligned}\int\limits_{\mathbb{R}^3}(1+\xi^{k/2})(K_-f)(\xi,\boldsymbol{\omega})d\boldsymbol{\xi} &= n_1\int\limits_{\mathbb{R}^3}\nu(z)H(z-1)\frac{(z-1)^{k/2}}{z^{k/2}}f(z\boldsymbol{\omega}')(1+z^{k/2})\sqrt{z}dzd\boldsymbol{\omega}'\\ &\leq M\int\limits_{\mathbb{R}^3}(1+\xi^{k/2})f(\xi,\boldsymbol{\omega})d\boldsymbol{\xi} = M\|f\|_{X_k}.\end{aligned}$$

Since K_- is a positive operator, we have $|K_-f|\leq K_-|f|$ and from the above we get

$$\|K_-f\|_{X_k}\leq\|K_-|f|\|_{X_k}\leq M\|f\|_{X_k}.$$

Since the operator of multiplication by α_- commutes with the multiplication by $\xi^{k/2}$, we obtain that C^i_- is bounded in X_k. Thus, if we prove that the operator C^i_+ generates a semigroup, then C^i+C^e will also generate a semigroup by the Bounded Perturbation Theorem.

To do this, first we shall find the resolvent $R(s,C^i_+)$ of C^i_+. Using the representation (2.9) we obtain the system of equations

$$\begin{aligned} g_0 &= (s+n_2p_0)f_0\\ \vdots\ &\quad \vdots\\ g_n &= (s+n_2p_n)f_n+n_2\nu_nF_{n-1}\\ \vdots\ &\quad \vdots, \end{aligned}\tag{3.1}$$

where $n=1,2,\ldots$. Let us split all the functions as $g_k=g_{k0}+g_{k1}$, where $\int\limits_{S^2}g_{k1}d\boldsymbol{\omega}=0$. Then the system (3.1) decouples into

$$\begin{aligned} g_{00} &= (s+n_2p_0)f_{00}\\ \vdots\ &\quad \vdots\\ g_{n0} &= (s+n_2p_n)f_{n0}+n_2\nu_nf_{n-1,0}\\ \vdots\ &\quad \vdots, \end{aligned}\tag{3.2}$$

and

$$\begin{aligned} g_{01} &= (s+n_2p_0)f_{01}\\ \vdots\ &\quad \vdots\\ g_{n1} &= (s+n_2p_n)f_{n1}\\ \vdots\ &\quad \vdots, \end{aligned}\tag{3.3}$$

where $n=1,2,\ldots$. The second system is diagonal, so its solvability renders no difficulties. Let us concentrate then on (3.2). We claim that for $n=0,1\ldots$ and $s>0$

$$f_{n0}=\frac{1}{\xi_n(n_2p_n+s)}\sum_{i=0}^{n}\xi_ig_{i0}\left(\prod_{l=0}^{n-i-1}\frac{n_2p_{n-l-1}}{n_2p_{n-l-1}+s}\right)\tag{3.4}$$

with the convention that the product is equal to 1 if $n-i-1<0$. In fact, $n=0$ the formula (3.4) gives

$$f_{00} = \frac{1}{\xi_0(n_2p_0+s)}\sum_{i=0}^{0}\xi_i g_{i0}\left(\prod_{l=0}^{-i-1}\frac{n_2p_{n-l-1}}{n_2p_{n-l-1}+s}\right) = \frac{g_{00}}{n_2p_0+s}$$

which agrees with the direct solution to the first equation of (3.2). The basic recurence formula which will enable us to prove (3.4) is

$$\frac{\xi_n}{\xi_{n+1}}p_n = \nu_{n+1}, \qquad n = 0, 1 \dots . \tag{3.5}$$

Assuming that (3.4) is correct for $n-1$, we find from (3.2) that

$$\begin{aligned}
f_{n0} &= \frac{1}{n_2p_n+s}(g_{n0}+n_2\nu_n f_{n-1,0}) \\
&= \frac{1}{\xi_n(n_2p_n+s)}\left(\xi_n g_{n0} + \frac{n_2p_{n-1}}{n_2p_n+s}\sum_{i=0}^{n-1}\xi_i g_{i0}\left(\prod_{l=0}^{n-i-2}\frac{n_2p_{n-l-2}}{n_2p_{n-l-2}+s}\right)\right) \\
&= \frac{1}{\xi_n(n_2p_n+s)}\left(\xi_n g_{n0} + \sum_{i=0}^{n-1}\xi_i g_{i0}\left(\prod_{l=0}^{n-i-1}\frac{n_2p_{n-l-1}}{n_2p_{n-l-1}+s}\right)\right) \\
&= \frac{1}{\xi_n(n_2p_n+s)}\sum_{i=0}^{n}\xi_i g_{i0}\left(\prod_{l=0}^{n-i-1}\frac{n_2p_{n-l-1}}{n_2p_{n-l-1}+s}\right)
\end{aligned}$$

Since

$$p_i(\xi) = \frac{\sqrt{\xi+i+1}}{\sqrt{\xi+i}}\nu(\xi+i+1)$$

we have $\nu_{min} \le p_0 \le \infty$ and $\nu_{min} \le p_i \le \sqrt{2}\nu_{max}$ uniformly for $\xi \in [0,1[$. Since the function $\mathbb{R}_+ \ni r \to r/(r+s)$ is increasing for any $s>0$, we obtain

$$\begin{array}{ccl}
\frac{n_2\nu_{min}}{n_2\nu_{min}+s} & \le & \frac{n_2p_0}{n_2p_0+s} \le 1 \\
\vdots & & \vdots \\
\frac{n_2\nu_{min}}{n_2\nu_{min}+s} & \le & \frac{n_2p_i}{n_2p_i+s} \le \frac{\sqrt{2}n_2\nu_{max}}{\sqrt{2}n_2\nu_{max}+s} \\
\vdots & & \vdots
\end{array}$$

For any fixed $s>0$ we denote $\beta = \frac{\sqrt{2}n_2\nu_{max}}{\sqrt{2}n_2\nu_{max}+s} < 1$. Thus

$$\prod_{l=0}^{n-i-1}\frac{n_2p_{n-l-1}}{n_2p_{n-l-1}+s} \le \beta^{n-i} \tag{3.6}$$

for $i = 0, \ldots, n-1$, and for $i = n$, $\prod_{l=0}^{-1} \frac{n_2 p_{n-l-1}}{n_2 p_{n-l-1}+s} = 1$ by definition. Hence

$$\begin{aligned} \xi_n |f_{n0}| &= \frac{1}{n_2 p_n + s} \left| \sum_{i=0}^{n-1} \xi_i g_{i0} \left(\prod_{l=0}^{n-i-1} \frac{n_2 p_{n-l-1}}{n_2 p_{n-l-1} + s} \right) + \xi_n g_n \right| \\ &\leq \frac{1}{s} \sum_{i=0}^{n-1} \xi_{i0} |g_i| \beta^{n-i} + \xi_n |g_{n0}|. \end{aligned} \tag{3.7}$$

By (2.11) we have to prove that the series

$$\sum_{n=1}^{\infty} n^{k/2} \xi_n |f_{n0}| \tag{3.8}$$

is integrable over $[0, \xi[$. To do this, it is enough to prove the statement for any remainder of the series. Let us fix $N > 1$ (which will be determined later) and consider

$$\begin{aligned} \sum_{n=N}^{\infty} n^{k/2} \xi_n |f_{n0}| &\leq \frac{1}{s} \sum_{n=N}^{\infty} \left(\sum_{i=0}^{n-1} n^{k/2} \xi_i |g_{i0}| \beta^{n-i} + n^{k/2} \xi_n |g_{n0}| \right) \\ &= \frac{1}{s} \left(\sum_{i=0}^{\infty} \xi_i |g_{i0}| \beta^{-i} \left(\sum_{n=N+i}^{\infty} n^{k/2} \beta^n \right) + \sum_{i=N}^{\infty} i^{k/2} \xi_i |g_{i0}| \right), \end{aligned}$$

where the change of the order of summation is justified by the positivity of the terms.

Consider now the function $r \to r^{k/2} \beta^r$. This function is monotonically decreasing for $t > t_0 = -2(k \ln \beta)^{-1}$. Taking $N \geq \max\{t_0, -2(\ln \beta)^{-1}\}$ we have the estimate

$$\sum_{n=N+i}^{\infty} n^{k/2} \beta^n \leq \int_{N+i-1}^{\infty} r^{k/2} \beta^r \, dr.$$

To shorten notation, let $\gamma = \beta^{-1}$. Then

$$\int_{N+i-1}^{\infty} r^{k/2} \beta^r \, dr = \int_{N+i-1}^{\infty} r^{k/2} \gamma^{-r} \, dr = (\ln \gamma)^{-(1+k/2)} \int_{\ln \gamma (N+i-1)}^{\infty} z^{k/2} e^{-z} dz$$

and the last integral is the well-known incomplete Γ function. From [1], formula 6.5.32, it follows that there exists a constant M, independent of the lower limit of integration $z \geq 1$(but dependent on k) such that

$$\int_{z}^{\infty} z^{k/2} e^{-z} dz \leq M z^{k/2} e^{-z}. \tag{3.9}$$

Because of the definition of N, for any γ and $i \geq 0$, the lower limit of integration satisfies $(N+i-1)\ln\gamma \geq 1$, thus we can use the estimate (3.9) to obtain

$$\begin{aligned}\sum_{n=N+i}^{\infty} n^{k/2}\beta^n &\leq (\ln\gamma)^{-(1+k/2)} \int\limits_{\ln\gamma(N+i-1)}^{\infty} z^{k/2}e^{-z}dz \leq M(\ln\gamma)^{-1}(N+i-1)^{k/2}\beta^{N+i-1} \\ &= M_1(N+i-1)^{k/2}\beta^i. \end{aligned} \tag{3.10}$$

Since $i^{-k/2}(N+i-1)^{k/2} \leq (N+1)^{k/2}$ and also $(N-1)^{k/2} \leq (N+1)^{k/2}$, we obtain

$$\begin{aligned}\sum_{n=N}^{\infty} n^{k/2}\xi_n|f_{n0}| &\leq \frac{M_1}{s}\left(\xi_0|g_{00}|(N-1)^{k/2} + \sum_{i=1}^{\infty}\xi_i(N+i-1)^{k/2}|g_{i0}| + \sum_{i=N}^{\infty} i^{k/2}\xi_i|g_{i0}|\right) \\ &\leq \frac{M_2}{s}\left(\xi_0|g_{00}| + \sum_{i=1}^{\infty}\xi_i i^{k/2}|g_{i0}| + \sum_{i=1}^{\infty} i^{k/2}\xi_i|g_{i0}|\right) \\ &= \frac{2M_2}{s}|g_0|_k \end{aligned}$$

where $M_2 = M_1(N+1)^{k/2}$ and $g_0 = (g_{i0})_{i\in\mathbb{N}}$. For the initial part of the series (3.8) we have

$$\begin{aligned}\sum_{n=1}^{N-1} n^{k/2}\xi_n|f_{n0}| &\leq \frac{1}{s}\sum_{n=1}^{N-1}\left(\sum_{i=0}^{n-1} n^{k/2}\xi_i|g_{i0}|\beta^{n-i} + n^{k/2}\xi_n|g_n|\right) \\ &\leq M_3\left(|g_0|_k\sum_{n=1}^{N-1} n^{k+2} + |g_0|_k\right) \\ &\leq M_4|g_0|_k. \end{aligned}$$

Moreover

$$f_{00} = \frac{g_{00}}{n_2p_0 + s} = \frac{\xi_0 g_{00}}{s\xi_0 + \xi_1 n_2\nu_1}$$

thus

$$|f_{00}| \leq M_5\xi_0|g_{00}|. \tag{3.11}$$

Combining all the estimates we see that for any $s > 0$ there exists a constant M' (depending on s) such that

$$\|f_0\|_k \leq M'\|g_0\|_k \tag{3.12}$$

and $f_0 \in D(C)$ because of (3.11). Since (3.3) is diagonal, we immediately obtain

$$\begin{aligned}|\ f_1\ |_k &\leq \frac{\xi_0|g_{01}|}{s\xi_0 + \xi_1 n_2\nu_1} + \sum_{i=1}^{\infty}\frac{i^{k/2}\xi_i|f_{0i}|}{s\xi_i + \xi_{i+1}n_2\nu_{i+1}} \\ &\leq M''\ |\ g_1\ |_k, \end{aligned}$$

which, together with (3.12), gives

$$\|f\|_k \leq M\|g\|_k \tag{3.13}$$

for some constant M. This shows that the resolvent set of C^i_+ satisfies $\rho(C^i_+) \supset \{s \in \mathbb{C};\ Re\, s > 0\}$. However, the constant M could be larger than $1/s$ and the estimate (3.12) alone is not sufficient for the generation of a semigroup. To obtain the desired result we shall use the fact that C^i_+ is a resolvent positive operator for $s > 0$. To prove this let us recall that we have decomposed X_k into the direct sum $X_k = X_{k0} \oplus X_{k1}$, where any $g_0 \in X_{k1}$ satisfies $\int\limits_{S^2} g_0 d\boldsymbol{\omega} = 0$. Let P_0 and $P_1 = I - P_0$ be the complementary projections onto X_{k0} and X_{k1}, respectively. We may take $P_0 g = (4\pi)^{-1} \int\limits_{S^2} g d\boldsymbol{\omega}$ and $P_1 g = g - (4\pi)^{-1} \int\limits_{S^2} g d\boldsymbol{\omega}$.

Let us take $g \geq 0$. Unfortunately, while $g_0 = P_0 g = (4\pi)^{-1} \int\limits_{S^2} g d\boldsymbol{\omega} \geq 0$, the complement g_1 is not necessarily nonnegative. However, as in (3.7) we have

$$f_{n0} = \frac{1}{\xi_n(n_2 p_n + s)} \left(\frac{1}{s} \sum_{i=0}^{n-1} \xi_i g_{i0} \prod_{l=0}^{n-i-1} \frac{n_2 p_{n-l-1}}{n_2 p_{n-l-1} + s} + \xi_n g_{n0} \right)$$

and from (3.3)

$$f_{n1} = \frac{1}{n_2 p_n + s} g_{n1} = \frac{1}{n_2 p_n + s} (g_n - g_{n0}).$$

Adding these two together we have

$$f_n = f_{n0} + f_{n1} = \frac{1}{\xi_n(n_2 p_n + s)} \left(\frac{1}{s} \sum_{i=0}^{n-1} \xi_i g_{i0} \prod_{l=0}^{n-i-1} \frac{n_2 p_{n-l-1}}{n_2 p_{n-l-1} + s} \right) + \frac{1}{n_2 p_n + s} g_n, \tag{3.14}$$

and since $g_n, g_{n0} \geq 0$, we obtain that $f_n \geq 0$ and C^i_+ is resolvent positive for $s > 0$.

Using again (3.14) with $g \geq 0$ we obtain that for $n = 0, 1 \ldots$

$$\xi_n f_n \geq \frac{\xi_n}{n_2 p_n + s} g_n \geq \frac{1}{s} \xi_n g_n$$

which yields

$$\|f\|_k \geq \frac{1}{s} \|g\|_k.$$

Since we observed earlier that $D(C^i_+)$ is dense in X_k, we can use Batty-Robinson-Arendt theorem (e.g. [2], Theorem 2.5) to claim that C^i_+ is the generator of a positive C_0-semigroup $(T^i_+(t))_{t \geq 0}$ satisfying $\omega(T^i_+) = s(C^i_+) = 0$.

The application of the Bounded Perturbation Theorem ends the proof of the theorem.■

4 An application to asymptotic analysis of kinetic equation with dominant elastic scattering

Let us consider the Cauchy problem for the kinetic equation

$$\partial_t f_\epsilon = S f_\epsilon + C^i f_\epsilon + \frac{1}{\epsilon} C^e f_\epsilon$$

$$f_\epsilon(0) \quad = \quad \overset{\circ}{f}, \tag{4.1}$$

where S is the streamining operator defined by $Su = \boldsymbol{v}\cdot\partial_{\boldsymbol{x}}u$ and C^i and C^e are inelastic and elastic scattering operators defined by (2.2) and (2.1), respectively. The presence of the large factor $1/\epsilon$ accounts for the fact that we analyze the case when the elastic collisions are dominant or, in other words, that the mean free path between elastic collisions is very small. A detailed explanation of the above scaling can be found in [14, 6]. Our aim here is to find the limit equation as $\epsilon \to 0$ and to prove that the solutions to the limit equation are limits of the solutions to (4.1) in L_1 norm.

In this section the problem becomes dependent on $\boldsymbol{x} \in \mathbb{R}^3$. Accordingly, we introduce the notation

$$\mathcal{X} = L_1(\mathbb{R}^3_x, X) = L_1(\mathbb{R}^6_{x,v}).$$

4.1 Formal asymptotic results

In this subsection we shall carry out the formal asymptotic analysis of the equation (4.1) so that it becomes clear what assumptions are needed for the availability of a rigorous asymptotic procedure. The properties of C^e are important even at this formal stage so that we recall the theorem from [5] which provides all the necessary information.

Theorem 4.1 *Let all the assumptions of Section 2 are satisfied. Then the operator C^e is a bounded operator in $\mathcal{X}$ with the following properties*

(i) For any $f \in \mathcal{X}$ and any non-decreasing function κ we have

$$\int\limits_{\mathbb{R}^6} (\kappa(f)C^e f)(\boldsymbol{x},\boldsymbol{v})d\boldsymbol{x}d\boldsymbol{v} \leq 0. \tag{4.2}$$

(ii) The null-space $N(C^e)$, the range $R(C^e)$ and the spectral projection $\mathbb{P}f$ onto $N(C^e)$ are given by

$$N(C^e) \quad = \quad \{f \in \mathcal{X};\ f \textit{ is independent on } \boldsymbol{\omega}\}. \tag{4.3}$$

$$R(C^e) \quad = \quad W = \{f \in \mathcal{X};\ \int\limits_{S^2} f(\boldsymbol{x}, v\boldsymbol{\omega})d\boldsymbol{\omega} = 0 \ a.e.\}. \tag{4.4}$$

$$(\mathbb{P}f)(\boldsymbol{x}, v) \quad = \quad \frac{1}{4\pi}\int\limits_{S^2} f(\boldsymbol{x}, v\boldsymbol{\omega})d\boldsymbol{\omega}. \tag{4.5}$$

(iii) For $f \in W$ we have

$$\int\limits_{\mathbb{R}^6} (sign(f)C^e f)(\boldsymbol{x},\boldsymbol{v})d\boldsymbol{x}d\boldsymbol{v} \leq -4\pi\lambda_{min}\|f\|_{\mathcal{X}}. \tag{4.6}$$

Remark 4.1 Theorem 4.1 was proved under more general assumptions than that of Section 2. In the present context all the statements can be verified by direct computations as was done in [14]. In particular, we can derive the explicit expression for the inverse of $\mathbb{Q}C^e\mathbb{Q}$, where $\mathbb{Q} = I - \mathbb{P}$. In fact, from (2.1) we obtain immediately $C^e\mathbb{Q}f = -\lambda f$. If $f \in W$, then $-\lambda f \in W$, thus $\mathbb{Q}C^e\mathbb{Q}f = -\lambda f$. Consequently,

$$(\mathbb{Q}C^e\mathbb{Q})^{-1}f = -\lambda^{-1}f. \tag{4.7}$$

■

The asymptotic procedure we are going to use is fairly well-known, e.g. [20, 15]. We operate on both sides of (4.1) with the projections $\mathbb{P}$ and $\mathbb{Q}$ and, denoting $f_\epsilon = \mathbb{P}f_\epsilon + \mathbb{Q}f_\epsilon = v_\epsilon + w_\epsilon$, we obtain the following system

$$\begin{aligned} \partial_t v_\epsilon &= \mathbb{P}S\mathbb{P}v_\epsilon + \mathbb{P}S\mathbb{Q}w_\epsilon + \mathbb{P}C^i\mathbb{P}v_\epsilon + \mathbb{P}C^i\mathbb{Q}w_\epsilon \\ \partial_t w_\epsilon &= \mathbb{Q}S\mathbb{P}v_\epsilon + \mathbb{P}S\mathbb{Q}w_\epsilon + \mathbb{Q}C^i\mathbb{P}v_\epsilon + \mathbb{Q}C^i\mathbb{P}w_\epsilon + \frac{1}{\epsilon}\mathbb{Q}C^e\mathbb{Q}w_\epsilon, \end{aligned} \tag{4.8}$$

where we have already used the fact that $\mathbb{P}C^e \equiv 0$. The system has to be supplemented by the initial values

$$\begin{aligned} v_\epsilon(0) &= \mathbb{P}\overset{\circ}{f} = \overset{\circ}{v} \\ w_\epsilon(0) &= \mathbb{Q}\overset{\circ}{f} = \overset{\circ}{w}. \end{aligned}$$

Isotropic form of the scattering operators allows us to simplify system (4.8) even further. To do this we note that the operator C^i reduces $N(C^e)$ and $W = N(C^i)^\perp$, thus $\mathbb{P}C^i\mathbb{Q} = \mathbb{Q}C^i\mathbb{P} = 0$. Moreover, $S\mathbb{P}v_\epsilon$ is linear in $\boldsymbol{\omega}$, therefore $\mathbb{P}S\mathbb{P}v_\epsilon = 0$. Taking these into account we transform (4.8) into the following form

$$\begin{aligned} \partial_t v_\epsilon &= \mathbb{P}S\mathbb{Q}w_\epsilon + \mathbb{P}C^i\mathbb{P}v_\epsilon \\ \partial_t w_\epsilon &= \mathbb{Q}S\mathbb{P}v_\epsilon + \mathbb{Q}S\mathbb{Q}w_\epsilon + \mathbb{Q}C^i\mathbb{P}w_\epsilon + \frac{1}{\epsilon}\mathbb{Q}C^e\mathbb{Q}w_\epsilon. \end{aligned} \tag{4.9}$$

The asymptotic solution is sought in the form

$$\begin{aligned} f(t, t/\epsilon) &= \bar{f}(t) + \tilde{f}(t/\epsilon) = \rho(t) + \bar{w}_0(t) + \epsilon\bar{w}_1(t) + \ldots \\ &\quad + \tilde{\rho}_0(t/\epsilon) + \epsilon\tilde{\rho}_1(t/\epsilon) + \ldots + \tilde{w}_0(t/\epsilon) + \epsilon\tilde{w}_1(t/\epsilon) + \ldots, \end{aligned} \tag{4.10}$$

where

$$\rho, \tilde{\rho}_0, \tilde{\rho}_1 \ldots \in N(C^e),$$

and are called the *hydrodynamic* part of the expansion, whereas

$$\bar{w}_0, \bar{w}_1, \ldots, \tilde{w}_0, \tilde{w}_1, \ldots \in W$$

and are called the *kinetic* part of the expansion. Moreover, the terms depending on t are referred to as the *bulk* part of the asymptotic expansion and the terms depending on $\tau = t/\epsilon$

are known as the *initial layer*; they are to be determined independently of each other. Note that in accordance with the Chapman-Enskog (compressed) procedure ([20]) the hydrodynamic term of the bulk part of the expansion, ρ, is not expanded. The number of the terms in each expansion is determined in some sense *a posteriori* after having written the equations for the error. It follows that inserting the expansion (4.10) into (4.9) and equating the coefficients at equal powers of ϵ we obtain the following equations:

$$\begin{aligned} \partial_t \rho &= \mathbb{P}C^i\mathbb{P}\rho, & (4.11)\\ \mathbb{Q}C^e\mathbb{Q}\bar{w}_0 &= 0, \\ \mathbb{Q}C^e\mathbb{Q}\bar{w}_1 + \mathbb{Q}S\mathbb{P}\rho &= 0, \\ \partial_\tau \tilde{\rho}_0 &= 0, & (4.12)\\ \partial_\tau \tilde{w}_0 &= \mathbb{Q}C^e\mathbb{Q}\tilde{w}_0, & (4.13) \end{aligned}$$

which, as we shall see, define enough terms of the asymptotic expansion to obtain, at least formally, the convergence of the difference $f_\epsilon - f(t, t/\epsilon)$ to zero as $\epsilon \to 0$.

In fact, let us assume for a time being that all the equations above can be solved and that the solutions are sufficiently regular to make the manipulations to follow available. It can be proved that on this level of approximation the correct initial values for (4.11) and (4.13) are

$$\rho(0) = \overset{\circ}{v}, \qquad \tilde{w}_1(0) = \overset{\circ}{w}.$$

Note that the equation for $\bar{w}_1$ is algebraic and does not require any side condition and the solution to (4.12) is determined by the stipulated decay to zero as $\tau \to \infty$.

Thus we have $\bar{w}_0 = \tilde{\rho}_0 = 0$ and

$$\begin{aligned} \bar{w}_1 &= (\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho, \\ \tilde{w}_0 &= e^{\tau\mathbb{Q}C^e\mathbb{Q}} \overset{\circ}{w}. \end{aligned}$$

Hence, we take the pair $(\rho, \tilde{w}_0 + \epsilon\bar{w}_1)$ as the approximation of $f_\epsilon = (v_\epsilon, w_\epsilon)$; the error of this approximation is given by

$$\begin{aligned} y_\epsilon &= v_\epsilon - \rho \\ z_\epsilon &= w_\epsilon - \tilde{w}_0 - \epsilon\bar{w}_1. \end{aligned}$$

Assuming that the solution and the terms of the asymptotic expansion are regular enough (we require that they belong to the domains of all the operators involved in (4.9)) and taking into account Eqs. (4.11)-(4.13) we obtain the following system of equations for the error

$$\begin{aligned} \partial_t y_\epsilon - \mathbb{P}S\mathbb{Q}z_\epsilon - \mathbb{P}C^i\mathbb{P}y_\epsilon &= \epsilon\mathbb{P}S\mathbb{Q}\bar{w}_1 + \mathbb{P}S\mathbb{Q}\tilde{w}_0 & (4.14)\\ \partial_t z_\epsilon - \mathbb{Q}S\mathbb{P}y_\epsilon - \mathbb{Q}S\mathbb{Q}z_\epsilon - \mathbb{Q}C^i\mathbb{Q}z_\epsilon - \frac{1}{\epsilon}\mathbb{Q}C^e\mathbb{Q}z &= \mathbb{Q}S\mathbb{Q}\bar{w}_1 + \epsilon\mathbb{Q}S\mathbb{Q}\tilde{w}_0 \\ &\quad +\epsilon\mathbb{Q}C^i\mathbb{Q}\bar{w}_1 + \mathbb{Q}C^i\mathbb{Q}\tilde{w}_0 - \epsilon\partial_t\bar{w}_1, \end{aligned}$$

with the initial conditions

$$y_\epsilon(0) = z_\epsilon(0) = 0.$$

The next step can be summarized as follows: due to the assumed regularity of all the terms, the error (y_ϵ, z_ϵ) is a classical solution of (4.14). Defining

$$e_\epsilon = y_\epsilon + z_\epsilon, \tag{4.15}$$

we see that e_ϵ is a classical solution to

$$\begin{aligned} \partial_t e_\epsilon - Se_\epsilon - C^i e_\epsilon - \frac{1}{\epsilon}C^e e_\epsilon &= \epsilon(S\mathbb{Q}\bar{w}_1 + \mathbb{Q}C^i\mathbb{Q}\bar{w}_1 - \partial_t\bar{w}_1) + S\mathbb{Q}\tilde{w}_0 + \mathbb{Q}C^i\mathbb{Q}\tilde{w}_0. \\ e_\epsilon(0) &= 0, \end{aligned} \tag{4.16}$$

The semigroup solving this equation is contractive in $\mathcal{X}$, [3], thus using the Duhamel formula we obtain the estimate

$$\|e_\epsilon(t)\| \leq \epsilon \int_0^t \|S\mathbb{Q}\bar{w}_1(s) + \mathbb{Q}C^i\mathbb{Q}\bar{w}_1(s) - \partial_s\bar{w}_1(s)\|ds + \int_0^t \|S\mathbb{Q}\tilde{w}_0(s/\epsilon) + \mathbb{Q}C^i\mathbb{Q}\tilde{w}_0(s/\epsilon)\|ds. \tag{4.17}$$

From the above inequality we see that if all the terms in the first integral exist and are locally bounded in t on $[0, \infty[$, then the contribution of this integral is of order of ϵ. As far as the second integral is concerned, the initial layer is assumed to be exponentially decaying, that is, to be of order of $e^{-\omega t/\epsilon}$ for some $\omega > 0$. If this property is preserved after having operated on $\tilde{w}_0$ with the operators $S\mathbb{Q}$ and $\mathbb{Q}C^i\mathbb{Q}$, then upon integration we obtain that also the contribution of this term is of order of ϵ, thus $\|e_\epsilon\| = O(\epsilon)$ and the convergence is proved.

Hence, we see that to complete our considerations we have to prove that all the terms of the asymptotic expansion and the solution to (4.1) are sufficiently regular for the all the discussed steps to be permissible. Checking this is the subject of the next subsection.

4.2 Regularity of the terms of the asymptotic expansion

The fact that the operator $K_\epsilon = S + C^i + \frac{1}{\epsilon}C^e$ generates a semigroup of contractions has been proved in [3]. Precisely speaking, [3] contains a proof of generation theorem for $S + C^i$, but since C^e is a bounded operator, the result extends to the present case by the Bounded Perturbation Theorem. Moreover, since C^e is a dissipative operator (in fact, it is conservative), the factor $1/\epsilon$ doesn't affect the contractivity of the full semigroup. Under the Maxwell molecule assumption the domain of the generator $D(K_\epsilon)$ is equal to $D(S) \cap D(C^i)$, [3], thus, if $\mathring{u} \in D(K_\epsilon)$, then we can write

$$\partial_t f_\epsilon = K_\epsilon f_\epsilon = Sf_\epsilon + C^i f_\epsilon + \frac{1}{\epsilon}C^e f_\epsilon.$$

We prove the following lemma.

Lemma 4.1 *If $\mathring{f} \in D(S) \cap D(C^i)$, then $\mathbb{P}f_\epsilon, \mathbb{Q}f_\epsilon \in D(S) \cap D(C^i)$ and therefore (4.1) and (4.8) are equivalent.*

Proof. Firstly, we prove that $\mathbb{P}u \in D(S)$. Let us recall that

$$D(S) = \{f \in \mathcal{X}, \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} f \in \mathcal{X}\},$$

where the gradient of f is taken in the distributional sense. Clearly, if $f \in \mathcal{X}$, then $\mathbb{P}f \in \mathcal{X}$. Let ϕ be any test function. Then

$$\langle \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} \mathbb{P}f, \phi \rangle = - \int_{\mathbb{R}^6_{x,v}} \left(\int_{S^2} f(\boldsymbol{x}, v\boldsymbol{\omega}) d\boldsymbol{\omega} \right) \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} \phi d\boldsymbol{v} d\boldsymbol{x}.$$

Since $(\boldsymbol{\omega}', \boldsymbol{\omega}, v, \boldsymbol{x}) \to f(\boldsymbol{x}, v\boldsymbol{\omega})(v\boldsymbol{\omega}' \cdot \partial_{\boldsymbol{x}} \phi(\boldsymbol{x}, v\boldsymbol{\omega}'))$ is integrable on $S^2 \times S^2 \times \mathbb{R}^6_{x,v}$ we can change the order of integration and using the integrability of $\boldsymbol{v} \cdot \partial_{\boldsymbol{x}} f$ over $\mathbb{R}^6_{x,v}$, we can rewrite the distributional derivative as

$$\langle \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} \mathbb{P}f, \phi \rangle = \int_{\mathbb{R}^6_{x,v}} \left(\int_{S^2} \boldsymbol{v} \cdot \partial_{\boldsymbol{x}} f d\boldsymbol{\omega} \right) d\boldsymbol{x} d\boldsymbol{v},$$

hence, for any $f \in D(S)$ we have $S\mathbb{P}f = \mathbb{P}Sf$ and therefore $\mathbb{P}f \in D(S)$. Since $\mathbb{Q}f = f - \mathbb{P}f$, we have immediately that also $\mathbb{Q}f \in D(S)$.

The fact that $\mathbb{P}f \in D(C^i)$ is straightforward, as $\mathbb{P}$ commutes with C^i. ■

The question of the sufficient regularity of the bulk part of expansion is addressed in the next lemma. We denote by $L_1(\Omega, X, d\mu)$ the space of X-valued functions integrable over Ω with respect to the measure μ.

Lemma 4.2 *Let* $\overset{\circ}{v} = \mathbb{P} \overset{\circ}{f} \in L_1(\mathbb{R}^3_v, W^2_1(\mathbb{R}^3_x), (v^{-1} + v^2) d\boldsymbol{v})$. *Then for each interval* $[0, T]$, $0 < T < +\infty$, *there exists a constant* M_T *such that*

$$\max_{t \in [0,\infty[} \{ \|S\mathbb{Q}\bar{w}_1(t)\|_{\mathcal{X}} + \|\mathbb{Q}C^i\mathbb{Q}\bar{w}_1(t)\|_{\mathcal{X}} + \|\partial_t \bar{w}_1(t)\|_{\mathcal{X}} \} \leq M_T \tag{4.18}$$

Proof. Let us fix $0 < T < \infty$. First we consider

$$S\mathbb{Q}\bar{w}_1(t) = S(\mathbb{Q}C^e\mathbb{Q})^{-1}\mathbb{Q}S\mathbb{P}\rho(t).$$

Due to (4.7), it is enough to analyze $t \to S^2\rho(t)$. Note that since the differentiation with respect to $\boldsymbol{x}$ commutes with all the operators, the required $\boldsymbol{x}$-regularity of the initial value translates into $\boldsymbol{x}$-regularity of the solution ρ. Using Theorem 3.1 we see that $\mathbb{P}C^i\mathbb{P}$ generates a semigroup in $X_2 = L_1(\mathbb{R}^3_v, (1 + v^2) d\boldsymbol{v})$ and therefore we have for all $t \in [0, T]$

$$\|S^2\rho(t)\|_{\mathcal{X}} \leq M_1 \| \overset{\circ}{v} \|_{L_1(\mathbb{R}^3_v, W^2_1(\mathbb{R}^3_x), (1+v^2) d\boldsymbol{v})},$$

for some constant M. Next we consider

$$\mathbb{Q}C^i\mathbb{Q}\bar{w}_1(t) = -\lambda^{-1}C^iS\rho(t).$$

Using (2.10) and Theorem 3.1 we obtain

$$\|\mathbb{Q}C^i\mathbb{Q}\bar{w}_1(t)\|_{\mathcal{X}} \leq M_2' \left\| (1+v^{-1})v\|\rho(t)\|_{W_1^1(\mathbb{R}^3_x)} \right\|_{L_1(\mathbb{R}^3_v)} \leq M_2 \| \overset{\circ}{v} \|_{L_1(\mathbb{R}^3_v, W_1^2(\mathbb{R}^3_x),(1+v)d\boldsymbol{v})},$$

uniformly on $[0,T]$. To estimate the last term

$$\partial_t \bar{w}_1(t) = -\lambda \mathbb{Q}S\mathbb{P}\partial_t\rho(t) = -\lambda\mathbb{Q}SC^i\rho(t)$$

we shall use an extension of Theorem 3.1. It follows (e.g. [12], p. 115-116) that if $\mathbb{P}C^i\mathbb{P}$ generates a semigroup in X_k, then its restriction to $D(C^i)$ generates a semigroup with the same properties. Thus, if $\|C^i \overset{\circ}{v}\|_{X_1}$ is finite, then on $[0,T]$

$$\|C^i\rho(t)\|_{X_1} \leq M'\|C^i \overset{\circ}{v}\|_{X_1} \tag{4.19}$$

The condition $\|C^i \overset{\circ}{v}\|_{X_1} < \infty$ is equivalent to (2.11) (with $k=1$), and this condition is weaker than that imposed on the initial value. Therefore, by (4.19) we have

$$\|\partial_t\bar{w}_1(t)\|_{\mathcal{X}} \leq M_3'\|SC^i\rho(t)\|_{\mathcal{X}} \leq M_3' \left\| C^i\|\rho(t)\|_{W_1^1(\mathbb{R}^3_x)} \right\|_{X_1} \leq M_3 \| \overset{\circ}{v} \|_{L_1(\mathbb{R}^3_v, W_1^2(\mathbb{R}^3_x),(v^{-1}+v)d\boldsymbol{v})}$$

and combining all three estimates we complete the proof of the lemma. ■

The regularity of the initial layer is dealt with in the next lemma.

Lemma 4.3 *Let* $\overset{\circ}{w} = \mathbb{Q}\overset{\circ}{f} \in L_1(\mathbb{R}^3_v, W_1^2(\mathbb{R}^3_x),(v^{-1}+v)d\boldsymbol{v})$. *Then there exist a positive constant* L *such that for any* $t \geq 0$

$$\|S\mathbb{Q}\tilde{w}_0(t/\epsilon)\|_{\mathcal{X}} + \|\mathbb{Q}C^i\mathbb{Q}\tilde{w}_0(t/\epsilon)\|_{\mathcal{X}} \leq Le^{-\lambda_{min}t/\epsilon}. \tag{4.20}$$

Proof. Let us consider first the term $\mathbb{Q}C^i\mathbb{Q}\tilde{w}_0(t/\epsilon)$. Since C^i_- is bounded, we see that we have to check only the behaviour of $C^i_+\tilde{w}_0(t/\epsilon)$. Moreover, due to the explicit representation $\tilde{w}_0(t/\epsilon) = e^{-\lambda t/\epsilon}\overset{\circ}{w}$, the multiplication by α_+ commutes with the semigroup, hence the only troublesome part might be $K_+\tilde{w}_0(t/\epsilon)$. We have however

$$\begin{aligned}\|K_+\tilde{w}_0(t/\epsilon)\|_X &= 4\pi\left(\int_{S^2}\int_0^\infty \nu(\sqrt{v^2+1})e^{-\sqrt{v^2+1}v^{-1}\lambda(v)t/\epsilon}\overset{\circ}{w}(\boldsymbol{x},v\boldsymbol{\omega})d\boldsymbol{v}\right)\\ &\leq L'e^{-\lambda_{min}t/\epsilon}\|\overset{\circ}{w}\|_{D(C^i)}\end{aligned}$$

for some constant L'. The estimate in $\mathcal{X}$ is obtained by integrating the above inequality with respect to $\boldsymbol{x}$.

The estimate of $S\mathbb{Q}\tilde{w}_0(t/\epsilon)$ is straightforward as S commutes with the semigroup generated by $\mathbb{Q}C^e\mathbb{Q}$ and therefore the regularity of the initial data carries forward to the solution. The lemma is proved. ■

Combining all the considerations of this section we see that we can prove the following theorem

Theorem 4.2 *Assume that* $\mathbb{P}\overset{\circ}{f} \in L_1(\mathbb{R}^3_v, W^2_1(\mathbb{R}^3_x), (v^{-1}+v^2)d\boldsymbol{v})$ *and* $\mathbb{Q}\overset{\circ}{f} \in L_1(\mathbb{R}^3_v, W^2_1(\mathbb{R}^3_x), (v^{-1}+v)d\boldsymbol{v})$*. Let* f_ϵ *be the solution of (4.1) with the initial data* $\overset{\circ}{f}$*, and* ρ *be the solution to (4.11) with the initial value* $\mathbb{P}\overset{\circ}{f}$*. Then for each interval* $[0,T]$*,* $0 < T < +\infty$*, there exists a constant* K *depending only on the initial data and the coefficients of the equation, such that*

$$\|f_\epsilon(t) - \rho(t) - e^{-\lambda t/\epsilon}\mathbb{Q}\overset{\circ}{f}\|_{\mathcal{X}} \leq K\epsilon \tag{4.21}$$

uniformly on $[0,T]$.

Proof. For the proof we note that the assumptions on the initial data adopted here are stronger than that of Lemma 4.1 so that all the lemmas are available. Therefore all the steps of Subsection 4.1 are justified and by Lemmas 4.2 and 4.3 we have

$$\begin{aligned}
\|e_\epsilon(t)\|_{\mathcal{X}} &\leq \epsilon\int_0^t \|S\mathbb{Q}\bar{w}_1(s) + \mathbb{Q}C^i\mathbb{Q}\bar{w}_1(s) - \partial_s\bar{w}_1(s)\|_{\mathcal{X}}ds \\
&\quad + \int_0^t \|S\mathbb{Q}\tilde{w}_0(s/\epsilon) + \mathbb{Q}C^i\mathbb{Q}\tilde{w}_0(s/\epsilon)\|_{\mathcal{X}}ds \\
&\leq \epsilon M_T T + L\int_0^t e^{-\lambda_{min}s/\epsilon}ds = \epsilon M_T T + \epsilon\int_0^{t/\epsilon} e^{-\lambda_{min}r}dr \leq K\epsilon
\end{aligned}$$

The only difference now is that in (4.15) we had $e_\epsilon = f_\epsilon - \rho - \tilde{w}_0 - \epsilon\bar{w}_1$, whereas in (4.21) the last term is missing. However, the estimates of Lemma 4.2 can be carried also for $\bar{w}_1$ alone, showing that it is locally bounded on $[0,T]$. Since it is multiplied by ϵ, it can be moved to the right-hand side of the inequality without changing it. Hence the theorem is proved. ■

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