Ergodic Theory for C-semigroups *

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Abstract: We define the ergodicity of C-semigroups in this paper, and then characterize the generators.

Key words: C-semigroups, generators, ergordicity.

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1. Introduction

The first ergodic theorems, which concerned the long run average behavior of individual trajectories on the phase space of a dynamical system, evolved in 1931-1932. In 1943, N.Dunford introduced resolvent techniques in ergodic theory and unified most of the known mean ergodic theorems. Following this, E.Hille and R.S.Phillips obtained a unified treatment of ergodic theory for C_0-semigroups based on the properties of the resolvent of the generator. Their work can be found in [7]. Further results along this line were obtained by Lin([9]) and Shaw([10]).

On the other hand, a significant generalization of strongly continuous semigroups that has recently received a great deal of attention is C-semigroups. An operator A generating a C-semigroup corresponds to the abstract Cauchy problem with respect to A

$$u'(t) = Au(t),\ u(0) = x \tag{$*$}$$

having a mild solution for every $x \in R(C)$. In this regard, it is also interesting to discuss the ergodic properties of C-semigroups. Shaw in [11] gave two notions of *Abel* limit and two notions of *Cesáro* limit, and obtained some properties of the ergodic limits. However, there was no definition for the ergodicity of C-semigroups. In this paper, we give definitions of *Abel*- and *Cesáro*-ergodicity of C-semigroups, which require that the operators Q_a and Q_c given in [11] have X as their domains, respectively, and then characterize the generators. In the case of $C = 1$, these definitions are the same as those of C_0-semigroups. Our results generalize those corresponding results for C_0-semigroups (see[7,9,10]).

We first clarify the relations between ergodicity of C-semigroups and that of mild solutions of $(*)$ (Theorem 2.7). Next, we investigate strong ergodicity in Section 3. As we know, $X = \ker A \oplus \overline{R(A)}$ characterizes the strong *Abel*-ergodicity of a C_0-semigroup (cf.[7]). For C-semigroups, we have a silmilar result: if $R(C)$ is dense in X, then $R(C) \subseteq$

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$\ker A \oplus \overline{R(A)}$ implies the strong *Abel*-ergodicity(Theorem 3.2). And Theorem 3.7 allows us to remove the hypothesis that $R(C)$ be dense, but an additional condition is added. Moreover, holomorphic C-semigroups are treated in Theorem 3.5.

The other major tool for C-semigroups is the Hille-Yosida space([2]), which connects C-semigroups with C_0-semigroups. When we consider the ergodicity in the Hille-Yosida space, we find this requires that the ergodic limit converge uniformly along the orbit in the original space X. This leads to the definition of orbital ergodicity (Definition 4.1). We show next that for C_0-semigroups, the orbital ergodicity and the ergodicity are equivalent (Theorem 4.2); and for C-semigroups, the orbital ergodicity and the ergodicity in the Hille-Yosida space are equivalent (Theorem 4.6). An orbitally ergodic C-semigroups will be constructed in Section 5 (Example 5.3).

Uniform ergodicity is discussed in Section 5. First, we give a sufficient condition that $X = \ker A \oplus R(A)$ (Theorem 5.1). Next, we establish a counterexample (Example 5.3); in which the generator of a uniformly *Abel*-ergodic C-semigroup has non-closed range, this indicates the essential difference between C_0-semigroups and C-semigroups, because the characterization of uniformly ergodic C_0-semigroup is $R(A)$ being closed. So we must seek for other techniques to investigate the uniform ergodicity of C-semigroups. By using the poles of the resolvent on the imaginary axis, we are able to obtain another characterization (Theorem 5.4).

In the following Section, we give some preliminaries, which contains the definitions and basic properties of C-semigroups and the Hille-Yosida space for A. The definitions and some remarks about ergodicity for C-semigroups are also contained. Moreover, some notations used throughout this paper are stated at the beginning of this section.

2.Preliminaries

Throughout this paper, X is a Banach space. All operators are linear. $B(X)$ is the space of bounded operators on X.If A is an operator on X, then $D(A)$, $R(A)$, $\ker A$, $\rho(A)$, $\sigma(A)$ denote, respectively, the domain, the range, the kernel, the resolvent set, and the spectrum of A. $C \in B(X)$ will be injective.

We start with the definitions and properties of C-semigroups.

Definition 2.1. A strongly continuous family $T(t) \in B(X)$ $(t \geq 0)$ is called a *C-semigroup* if $T(t+s)C = T(t)T(s)$ for $t, s \geq 0$ and $T(0) = C$.

The generator of $T(t)$, A, is defined by

$$Ax = C^{-1}[\lim_{t \searrow 0} \frac{1}{t}(T(t)x - Cx)]$$

with

$$D(A) = \{x \in X : \lim_{t \searrow 0} \frac{1}{t}(T(t)x - Cx) \text{ exists and is in } R(C)\}.$$

The complex number λ is in $\rho_C(A)$, the *C-resolvent* of A, if $(\lambda - A)$ is injective and $R(C) \subseteq R(\lambda - A)$. The set of all points in the complex plane which are not in the C-resolvent of A is denoted by $\sigma_C(A)$.

Lemma 2.2.([2]) Suppose A generates a C-semigroup $T(t)$ satisfying $\|T(t)\| \leq Me^{\omega t}$. Then

(a) A is a closed operator with $\overline{D(A)} \supseteq R(C)$;

(b) for all $x \in X, T(t)x = Cx + A\int_0^t T(s)xds$;

(c) for all $x \in D(A)$ and $t \geq 0$, $T(t)x \in D(A)$ with $AT(t)x = T(t)Ax$;

(d) $A = C^{-1}AC$, where $D(C^{-1}AC) = \{x \in X : Cx \in D(A) \text{ and } ACx \in R(C)\}$;

(e) $(\omega, \infty) \subseteq \rho_C(A)$, and for $r \in \mathbf{C}$ with Re $r > \omega$, $D((r - A)^{-1}) \supseteq R(C)$ with

$$(r - A)^{-1}Cx = \int_0^{+\infty} e^{-rt}T(t)xdt, \quad \forall x \in X.$$

Next, we need to introduce the Hille-Yosida space for an operator, for the details we refer to [2].

Definition 2.3. Suppose A has no eigenvalues in $(0, \infty)$. The *Hille-Yosida space for* A, Z_0, is the Banach space defined by

$$Z_0 = \{x \in X : \text{The Cauchy problem } u'(t) = Au(t) \text{ has a bounded uniformly continuous mild solution } u(\cdot, x)\}$$

with

$$\|x\|_{Z_0} = \sup\{\|u(t, x)\|; t \geq 0\} \text{ for } x \in Z_0.$$

The following lemma indicates the relations between the generators of C-semigroups and C_0-semigroups.

Lemma 2.4.([2]) Suppose A generates a bounded strongly uniformly continuous C-semigroup $T(t)$. Then the Hille-Yosida space for A, Z_0, is a Banach space and

$$Z_0 = \{x : t \to C^{-1}T(t)x \text{ is uniformly continuous and bounded}\}$$

with

$$\|x\|_{Z_0} = \sup_{t \geq 0} \|C^{-1}T(t)x\|.$$

Moreover, $R(C) \subset Z_0 \hookrightarrow X$, i.e., Z_0 is continuously embedded in X, and the restriction of A on Z_0, $A|_{Z_0}$, generates a contraction C_0-semigroup $S(t) = C^{-1}T(t)$ on Z_0.

Now we are in the position to give the definitions for ergodicity.

Definition 2.5. Let $T(t)$ be a C-semigroup with generator A.

(a) $T(t)$ is called *strongly(weakly, uniformly) Abel-ergodic* (*A-ergodic*, for short), if $(0, +\infty) \subset \rho_C(A)$ and $\lim_{\lambda \searrow 0} \lambda(\lambda - A)^{-1}C := P$ exists in the strong (weak, uniform) operator topology.

(b) $T(t)$ is *strongly(weakly, uniformly) Cesáro-ergodic* (*C-ergodic*, for short) , if $\lim_{t \to \infty} \frac{1}{t}\int_0^t T(s)ds := P$ exists in the strong (weak, uniform) operator topology.

(c) $T(t)$ is *strongly(weakly, uniformly) ergodic* if $\lim_{t \to \infty} T(t) := P$ exists in the strong (weak, uniform) operator topology.

And we call P the ergodic limit.

The proof of the following lemma is a straightforward modification of the cited results in [6,7,13] and therefore omitted.
Lemma 2.6. Suppose $T(t)$ is a C-semigroup. Then

(a) the uniform (A-,C-)ergodicity of $T(t)$ implies its strong (A-,C-)ergodicity; and its strong (A-,C-)ergodicity implies its weak (A-,C-)ergodicity;

(b) if $T(t)$ is bounded, then the uniform (strong, weak) A-ergodicity of $T(t)$ is equivalent to its uniform (strong, weak) C-ergodicity;

(c) the uniform(strong,weak) ergodicity of $T(t)$ implies its uniform(strong,weak) C-ergodicity.

It appears that the ergodicity of $T(t)$ doesn't give the ergodicity of mild solutions of $(*)$, since the mild solutions are given by $C^{-1}T(t)$ (see [2]). However, when we consider the mild solutions of $(*)$, we usually take the initial values from $R(C)$, so the mild solution $C^{-1}T(t)x$ is in fact $T(t)y$, where $Cy = x$. Therefore, we have
Theorem 2.7. *$T(t)$ is an (A-,C-)ergodic C-semigroup with generator A if and only if for every initial value $x \in R(C)$, $(*)$ has a mild solution, which is an (A-,C-) ergodic function in the usual sense.*

3.Strong Ergodicity

We collect the basic properties of the A-ergodic limits in the following
Proposition 3.1. *Suppose $T(t)$ is a bounded C-semigroup generated by A, P is the ergodic limit. Then*

(a) $P^2 = CP = PC$;

(b) $C\ker A \subseteq R(P) \subseteq \ker A$, $C\ker P \subseteq \overline{R(A)} \subseteq \ker P$;

(c) $T(t)P = PC = CP, \forall t \geq 0$.

Proof. (a) For any $x \in X$,

$$\begin{aligned} P^2x &= \lim_{\lambda\searrow 0} \lambda(\lambda - A)^{-1}C(\lim_{\mu\searrow 0} \mu(\mu - A)^{-1}Cx) \\ &= \lim_{\lambda\searrow 0}\lim_{\mu\searrow 0} \lambda\mu(\lambda - A)^{-1}(\mu - A)^{-1}C^2x \\ &= \lim_{\lambda\searrow 0}\lim_{\mu\searrow 0} \frac{\lambda\mu}{\lambda - \mu}[(\mu - A)^{-1}C^2x - (\lambda - A)^{-1}C^2x] \\ &= PCx. \end{aligned}$$

Also, it is easy to see $PCx = CPx$.

(b) First note that if $x = x_1 + x_2$, where $x_1 \in \ker A$, and there exists $y \in D(A)$ such that $Ay = x_2$. For $\lambda > 0$,

$$\begin{aligned} \lambda(\lambda - A)^{-1}Cx &= \lambda(\lambda - A)^{-1}Cx_1 + \lambda(\lambda - A)^{-1}CAy \\ &= Cx_1 + \lambda^2(\lambda - A)^{-1}Cy - \lambda Cy \\ &\to Cx_1, \quad as \ \ \lambda \to 0, \end{aligned} \tag{1}$$

which implies $Px = Cx_1$, $Px_2 = 0$. Thus $C \ker A \subseteq R(P)$, $\overline{R(A)} \subseteq \ker P$.

Suppose next that $x \in R(P)$. Then there exists $y \in X$ such that $x = Py = \lim_{\lambda \searrow 0} \lambda(\lambda - A)^{-1}Cy$. Further, $\lambda A(\lambda - A)^{-1}Cy = \lambda^2(\lambda - A)^{-1}Cy - \lambda Cy \to 0$ as $\lambda \to 0$. It follows from the closedness of A that $x \in D(A)$ with $Ax = 0$. Therefore, $R(P) \subseteq \ker A$.

Finally, let $x \in \ker P$, then $A(\lambda - A)^{-1}Cx = \lambda(\lambda - A)^{-1}Cx - Cx \to -Cx$ as $\lambda \to 0$; this proves $C \ker P \subseteq \overline{R(A)}$.

(c) For $x \in X$, by Lemma 2.6.(c),

$$\begin{aligned} T(t)Px &= \lim_{s\to\infty} \frac{1}{s}\int_0^s T(t)T(r)xdr \\ &= \lim_{s\to\infty} \frac{1}{s}\int_0^s T(t+r)Cxdr \\ &= \lim_{s\to\infty} \frac{t+s}{s} \cdot \frac{1}{t+s}[\int_0^{t+s} T(r)Cxdr - \int_0^t T(r)Cxdr] \\ &= PCx = CPx. \end{aligned}$$

This is (c). $\square$

In the case of $R(C)$ dense, we have the following strongly ergodic theorem for C-semigroups.

Theorem 3.2. *Suppose $T(t)$ is a bounded C-semigroup with generator A, and $\overline{R(C)} = X$. Then the following assertions are equivalent.*

(a) *$T(t)$ is strongly A-ergodic;*

(b) *$T(t)$ is weakly A-ergodic;*

(c) *for every $x \in X$, $\{\lambda(\lambda - A)^{-1}Cx : 0 < \lambda < 1\}$ is weakly sequentially compact, i.e., there exists a sequence λ_n, $\lambda_n \to 0$, and some $y \in X$ such that weak-$\lim_{n\to\infty} \lambda_n(\lambda_n - A)^{-1}Cx = y$;*

(d) $R(C) \subseteq \ker A \oplus \overline{R(A)}$.

Proof. (a)⇒(b) and (b)⇒(c) are obvious.

(c)⇒(d). For $x \in X$, there exist $\lambda_n \to 0$ and $y \in X$, such that $y_n = \lambda_n(\lambda_n - A)^{-1}Cx \to y$ weakly, hence

$$\begin{aligned} Ay_n &= \lambda_n A(\lambda_n - A)^{-1}Cx = -\lambda_n Cx + \lambda_n^2(\lambda_n - A)^{-1}Cx \\ &= -\lambda_n Cx + \lambda_n y_n \to 0 \text{ as } \lambda_n \to 0, \end{aligned}$$

since y_n is bounded. The closedness of A implies that A is weakly closed, so that $y \in D(A)$ and $Ay = 0$.

Moreover,

$$Cx - y = weak - \lim_{n\to\infty}(Cx - y_n) = weak - \lim_{n\to\infty}(-\frac{Ay_n}{\lambda_n}),$$

hence $Cx - y \in \overline{(R(A))_w}$, the weak closure of $R(A)$. From [12, p.158], we know $\overline{(R(A))_w} = \overline{R(A)}$, which leads to $Cx = y + (Cx - y) \in \ker A + \overline{R(A)}$. And from (1) we know that the sum is a direct sum.

(d)$\Rightarrow$(a). Since $T(t)$ is bounded, by Lemma 2.2, $R(C) \subseteq R(\lambda - A)$ and $\lambda(\lambda - A)^{-1}C$ is uniformly bounded for $\lambda > 0$. Again by (1), for $x = x_1 + x_2$, where $x_1 \in \ker A, x_2 \in R(A)$, we have $\lambda(\lambda - A)^{-1}Cx \to Cx_1$ as $\lambda \to 0$.

Moreover, by the assumption, $R(C) \subseteq \ker A \oplus \overline{R(A)}$ and $R(C)$ is dense, we have $\ker A + R(A)$ is dense in X; since the family of operators $\{\lambda(\lambda - A)^{-1}C\}_{\lambda>0}$ is uniformly bounded, the limit $\lim\limits_{\lambda\searrow 0} \lambda(\lambda - A)^{-1}Cx$ exists for every $x \in X$. □

From the proof of the above theorem, one can obtain

Theorem 3.3. *Suppose $T(t)$ is a bounded C-semigroup and $\ker A + R(A)$ is dense in X. Then $T(t)$ is strongly Abel-ergodic.*

Remark 3.4. By Lemma 2.6.(b), in the case of $T(t)$ bounded, the A-ergodicity and C-ergodicity are equivalent, so we can replace A-ergodic by C-ergodic in Theorem 3.2 and 3.3.

If $T(t)$ is holomorphic, we will see that the three definitions of ergodicity are equivalent.

Theorem 3.5. *Suppose $T(t)$ is a bounded holomorphic C-semigroup, and $R(C)$ is dense in X. Then the following assertions are equivalent.*

(a) *$T(t)$ is strongly A-ergodic;*

(b) *$T(t)$ is strongly C-ergodic;*

(c) *$T(t)$ is strongly ergodic, that is, $\lim\limits_{t\to\infty} T(t)x$ exists for every $x \in X$.*

Proof. It suffices to show (a)$\Rightarrow$(c). By [3,Theorem 6.1], for bounded holomorphic C-semigroup, we have $\lim\limits_{t\to\infty} T(t)x = 0$ for every $x \in \overline{R(A)}$. And for $x \in \ker A$, $T(t)x \equiv Cx$. Therefore, for any $x \in \ker A \oplus \overline{R(A)}$, $\lim\limits_{t\to\infty} T(t)x$ exists. If (a) holds, by Theorem 3.2, $R(C) \subset \ker A \oplus \overline{R(A)}$, so that (c) holds since $R(C)$ is dense. □

By applying Theorem 3.3, we also have

Theorem 3.6. *Suppose $T(t)$ is a bounded holomorphic C-semigroup, and suppose $\ker A + R(A)$ is dense in X. Then $T(t)$ is strongly ergodic, i.e., $\lim\limits_{t\to\infty} T(t)x$ exists for every $x \in X$.*

We end this section with an ergodic theorem in the general case.

Theorem 3.7. *Let $T(t)$ be a bounded C-semigroup with generator A. Then $T(t)$ is strongly A-ergodic if and only if for every $x \in X$, we have $Cx = y + z$, where $y \in \ker A$, $z \in \overline{R(A)}$ such that $\lambda(\lambda - A)^{-1}z \to 0$ as $\lambda \to 0$. And this decomposition is unique.*

Proof. Necessity. For $x \in X$, let $y = Px = \lim\limits_{\lambda\searrow 0} \lambda(\lambda - A)^{-1}Cx$; by Proposition 3.1, $Px \in \ker A$, thus $(\lambda - A)Px = \lambda Px$; this implies $Px \in R(\lambda - A)$ and $\lambda(\lambda - A)^{-1}Px = Px$ for $\lambda > 0$. Next, choose $z = Cx - y$, then $z \in \bigcap\limits_{\lambda>0} R(\lambda - A)$. From the proof (c)$\Rightarrow$(d) of Theorem 3.2, we have $z = Cx - y \in \overline{R(A)}$. Hence,

$$\lambda(\lambda - A)^{-1}z = \lambda(\lambda - A)^{-1}Cx - \lambda(\lambda - A)^{-1}Px = \lambda(\lambda - A)^{-1}Cx - Px \to 0.$$

Moreover, since $\ker A$ and $\overline{R(A)}$ have only the zero element in common, this decomposition is unique.

Sufficiency. The assumption $y \in \ker A$ implies that $y \in R(\lambda - A)$ and $\lambda(\lambda - A)^{-1}y = y$. Thus

$$\lambda(\lambda - A)^{-1}Cx = \lambda(\lambda - A)^{-1}y + \lambda(\lambda - A)^{-1}z \to y.$$

This proves our claim. □

4.Orbital Ergodicity

In this section, we introduce the notions of orbital ergodicity and ergodicity in Z_0, the Hille-Yosida space of A. Next we will clarify the relations between them.

Definition 4.1. We call $T(t)$ (a C_0-semigroup or C-semigroup) *orbitally A-ergodic*, if $T(t)X \subset R(\lambda - A)$ for all $t \geq 0, \lambda > 0$, and $\lim\limits_{\lambda \searrow 0} \lambda(\lambda - A)^{-1}T(t)x$ exists uniformly in t.

$T(t)$ is *orbitally C-ergodic*, if for $x \in X$, $\lim\limits_{s \to \infty} \frac{1}{s} \int_0^s T(t + \tau)x d\tau$ exists uniformly in t.

$T(t)$ is *orbitally ergodic*, if for $x \in X$, $\lim\limits_{s \to \infty} T(t + s)x$ exists uniformly in t.

It is remarked that if $T(t)$ is bounded, then ortitally A-ergodicity and C-ergodicity are equivalent.

The following result for the orbital ergodicity of C_0-semigroup is easy to verify.

Theorem 4.2. *If $T(t)$ is a bounded C_0-semigroup, then $T(t)$ is orbitally (A-,C-)ergodic if and only if $T(t)$ is (A-,C-)ergodic.*

In order to relate orbital ergodicity of C-semigroups with their ergodicity, we now define the ergodicity in the Hille-Yosida space, Z_0.

Definition 4.3. Suppose $T(t)$ is a bounded uniformly continuous C-semigroup with generator A on X. We call $T(t)$ is

(a) *uniformly(strongly,weakly) A-ergodic from X to Z_0*, if $\lim\limits_{\lambda \searrow 0} \lambda(\lambda - A|_{Z_0})^{-1}C$ exists in the uniform(strong,weak) operator topology in Z_0.

(b) *uniformly(strongly,weakly) C-ergodic from X to Z_0*, if $\lim\limits_{t \to \infty} \frac{1}{t} \int_0^t T(s)ds$ exists in the uniform(strong,weak) operator topology in Z_0.

(c) *uniformly(strongly,weakly) ergodic from X to Z_0*, if $\lim\limits_{t \to \infty} T(t)$ exists in the uniform(strong,weak) operator topology in Z_0.

And we know (a)$\Longleftrightarrow$(b), (c)$\Rightarrow$(b).

The following result is an analogue of Theorem 3.2, and the proof is also analogous. But note that here we do not assume that $R(C)$ is dense.

Theorem 4.4. *Suppose $T(t)$ is a bounded strongly uniformly continuous C-semigroup. The following conditions are equivalent.*

(a) *$T(t)$ is strongly A-ergodic from X to Z_0;*

(b) *$T(t)$ is weakly A-ergodic from X to Z_0;*

(c) *for every $x \in X$ $\{\lambda(\lambda - A|_{Z_0})^{-1}Cx : 0 < \lambda < 1\}$ is weakly sequentially compact in Z_0;*

(d) $R(C) \subseteq \ker(A|_{Z_0}) + \overline{R(A|_{Z_0})_0}$; *here* $\overline{(\cdot)_0}$ *denotes the closure in* Z_0.

Remark 4.5. (a) It is easy to show that $\ker(A|_{Z_0}) = \ker A$.

(b) Since $Z_0 \hookrightarrow X$, $T(t)$ being strongly A-ergodic from X to Z_0 implies that it is strongly A-ergodic in X.

Theorem 4.6. *Suppose $T(t)$ is a bounded strongly uniformly continuous C-semigroup. Then $T(t)$ is strongly A-ergodic from X to Z_0 if and only if $T(t)$ is orbitally A-ergodic.*

Proof. By Lemma 2.4,

$$\begin{aligned} &\|\frac{1}{s}\int_0^s T(\tau)x d\tau - Px\|_0 \\ = &\sup_{t\geq 0}\|\frac{1}{s}C^{-1}T(t)\int_0^s T(\tau)x d\tau - C^{-1}T(t)Px\| \\ = &\sup_{t\geq 0}\|\frac{1}{s}\int_0^s T(t+\tau)d\tau - Px\|, \end{aligned}$$

here Proposition 2.1 is applied. □

So Theorem 4.4.(d) gives a characterization of orbital A-ergodicity.

Many C-semigroups are orbitally A-ergodic; for instance, given an A-ergodic C_0-semigroup $S(t)$, then $CS(t)$ is an orbitally A-ergodic C-semigroup. Another example will be given in the next section.

5.Uniform Ergodicity

We first give a sufficient condition for uniform A-ergodicity.

Theorem 5.1. *Suppose $T(t)$ is a bounded C-semigroup, and $X = \ker A \oplus R(A)$. Then $T(t)$ is uniformly A-ergodic.*

Proof. Since A is closed, also $\ker A$ is closed. Then by $X = \ker A \oplus R(A)$ and [12,p.217], $R(A)$ is closed.

By the open mapping theorem(cf.[12,p.213]), there is some $m > 0$ such that each $x \in R(A)$ is equal to Ay for some $y \in D(A)$ with $\|y\| \leq m\|x\|$.

Next, we claim that $R(A)$ is invariant under $R_\lambda \equiv \lambda(\lambda - A)^{-1}C$. In fact, for $x = Ay \in R(A)$,

$$R_\lambda x = \lambda(\lambda - A)^{-1}Cx = \lambda(\lambda - A)^{-1}CAy = AR_\lambda y \in R(A).$$

Moreover, given $x \in R(A)$, we can choose $y \in D(A)$ satisfying $Ay = x$ and $\|y\| \leq m\|x\|$, hence

$$\begin{aligned} \|R_\lambda x\| &= \|R_\lambda Ay\| = \|AR_\lambda y\| \leq m\|AR_\lambda\|\,\|x\| \\ &\leq \lambda[\|C\| + \sup_{\lambda>0}\|\lambda(\lambda - A)^{-1}C\|]\,\|x\| \\ &= \lambda M\|x\|, \end{aligned}$$

where $M = \|C\| + \sup\limits_{\lambda>0} \|\lambda(\lambda - A)^{-1}C\|$, which implies

$$\|R_\lambda|_{R(A)}\| \leq \lambda M. \tag{2}$$

On the other hand, by (1),

$$R_\lambda|_{\ker A} = C|_{\ker A}. \tag{3}$$

And by [12,p.247], the projection Q of X onto $\ker A$ along $R(A)$ is continuous. Let $P = CQ$. For any $x \in X$, $x = Qx + (1-Q)x$, note that $R(Q) = \ker A$, $\ker Q = R(A)$, together with (2) and (3), we have

$$\begin{aligned} \|R_\lambda x - Px\| &= \|R_\lambda Qx - CQx + R_\lambda(1-Q)x\| \\ &= \|R_\lambda(1-Q)x\| \leq \lambda M\|(1-Q)x\| \end{aligned}$$

so that $\|\lambda(\lambda - A)^{-1}C - P\| \to 0$ as $\lambda \to 0$, proving our result. □

Corollary 5.2. *If $0 \in \rho(A)$, then $T(t)$ is uniformly A-ergodic.*

This is an immediate consequence of Theorem 5.1; one can also derive it directly from $\frac{1}{t}\int_0^t T(s)ds = \frac{1}{t}(A^{-1}T(t) - A^{-1}C) \to 0$.

It is well known that the characterization of a uniformly A-ergodic C_0-semigroup is that $R(A)$ is closed (cf.[13]). However, it is not true for C-semigroups. We illustrate this in the following example, which was suggested by the authors in [8]. And we will see the C-semigroup we give here is orbitally A-ergodic.

Example 5.3. Suppose X is a complex Banach space and $C_o \in B(X)$ is injective with $\overline{R(C_o)} \neq X$. Let $A = \begin{pmatrix} i & C_o^{-1} \\ 0 & -i \end{pmatrix}$ with $D(A) = X \times R(C_o)$, and $C = \begin{pmatrix} C_o & 0 \\ 0 & C_o \end{pmatrix}$.

It is not hard to show that A generates a bounded C-semigroup

$$T(t) = \begin{pmatrix} e^{it}C_o & \frac{1}{2i}(e^{it} - e^{-it}) \\ 0 & e^{-it}C_o \end{pmatrix}$$

and

$$(\lambda - A)^{-1}C \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} (\lambda - i)^{-1}C_o x_1 + (\lambda - i)^{-1}(\lambda + i)^{-1}x_2 \\ (\lambda + i)^{-1}C_o x_2 \end{pmatrix}.$$

Obviously $\lambda(\lambda - A)^{-1}C \to 0$ as $\lambda \to 0$. But $D(A) = X \times R(C_o)$, $R(A) = X \times R(C_o)$. So $R(A)$ may not be closed even if $R(C_o)$ is dense in X.

Next we show $T(t)$ is orbitally A-ergodic. In fact,

$$\begin{aligned} \frac{1}{s}\int_0^s T(t+r)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} dr &= \frac{1}{s}\int_0^s \begin{pmatrix} e^{i(t+r)}C_o x_1 + \frac{1}{2i}(e^{i(t+r)} - e^{-i(t+r)})x_2 \\ e^{-i(t+r)}C_o x_2 \end{pmatrix} dr \\ &= \frac{1}{2is}\begin{pmatrix} 2(e^{is} - 1)e^{it}C_o x_1 - i[(e^{is} - 1)e^{it} - (e^{-is} - 1)e^{-it}]x_2 \\ 2(1 - e^{-is})e^{-it}C_o x_2 \end{pmatrix} \end{aligned}$$

converges to 0 as $s \to \infty$ uniformly for $t \geq 0$.

At last, we discuss the relations between strong C-ergodicity and uniform C-ergodicity by using the poles of the C-resolvent on the imaginary axis.

Theorem 5.4. *Assume that $\|T(t)\| = O(t)(t \to \infty)$ and $\sigma_C(A) \bigcap i\mathbf{R}$ consists of poles of the C-resolvent only. Then the following are equivalent.*

(a) *$T(t)$ is strongly C-ergodic;*

(b) *every point in $\sigma_C(A) \bigcap i\mathbf{R}$ is a pole of order 1;*

(c) *$T(t)$ is uniformly C-ergodic; i.e., $\frac{1}{t}\int_0^t T(s)ds$ converges in the operator norm.*

Proof. If λ_0 is a pole of $(\lambda - A)^{-1}C$, then $\lambda \mapsto (\lambda - A)^{-1}C$ can be expanded as a Laurent series $(\lambda - A)^{-1}C = \sum_{n=-k}^{+\infty}(\lambda - \lambda_0)^n B_n$ for $0 < |\lambda - \lambda_0| < \delta$ and some $\delta > 0$. The coefficients B_n are bounded linear operators, given by

$$B_n = \frac{1}{2\pi i}\int_\Gamma (z - \lambda_0)^{-(n+1)}(z - A)^{-1}Cdz, \ n \in \mathbf{Z} \tag{4}$$

where $\Gamma = \{z \in \mathbf{C} : |z - \lambda_0| = \frac{\delta}{2}\}$. Write $B_{-1} = P$, from (4) one deduces that $B_{-(n+1)} = (\lambda - A)^n P$ for $n \geq 0$.

(a)$\Rightarrow$(b). If $i\eta$ is a pole of order $k \geq 2$, then $B_{-k} \neq 0$ while $B_{-(k+1)} = 0$. So that there exists $x \in D(A^k)$ such that $(A - i\eta)^k x \neq 0$ but $(A - i\eta)^{k+1}x = 0$. Hence

$$e^{-i\eta t}T(t)x = Cx + t(A - i\eta)Cx + \frac{t^2}{2!}(A - i\eta)^2 Cx + \cdots + \frac{t^{k-1}}{(k-1)!}(A - i\eta)^{k-1}Cx.$$

This implies that $T(t)x$ is not strongly C-ergodic.

(b)$\Rightarrow$(c). Since 0 is a pole of order 1 or regular point of the resolvent, so

$$(\lambda - A)^{-1}C = \frac{P}{\lambda} + \sum_{n=1}^{\infty}\lambda^n B_n,$$

for $|\lambda| < \delta$ and some $\delta > 0$, it follows that $\lambda(\lambda - A)^{-1}C \to P$ as $\lambda \to 0$. □

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