COMMUTATOR CONDITIONS IMPLYING THE CONVERGENCE OF THE LIE–TROTTER PRODUCTS

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Abstract. In this paper we investigate commutator conditions for two strongly continuous semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ on a Banach space implying the convergence of the Lie–Trotter products $[T(\frac{t}{n})S(\frac{t}{n})]^n$. The results are then applied to various examples and, in particular, to the Ornstein–Uhlenbeck operator with degenerate Laplacian.

1. Introduction

In 1959 H. F. Trotter [13] obtained an explicit product formula for semigroups whose generator is the sum of two generators. Using this idea, more general product formulas were considered by P. R. Chernoff [4, Chap. 1, 2]. Following this approach one can prove the theorem below (cf. [6, Chap. III, Cor. 5.8]).

Theorem 1. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be strongly continuous semigroups on a Banach space E with generators $(A, D(A))$ and $(B, D(B))$, respectively, satisfying the stability condition*

$$\| \left[T(\tfrac{t}{n})S(\tfrac{t}{n})\right]^n \| \leq M e^{\omega t} \tag{1}$$

for all $t \geq 0, n \in \mathbb{N}$, and some constants $M \geq 1, \omega \in \mathbb{R}$.
Consider the sum $A + B$ on a subspace $D \subseteq D(A) \cap D(B)$ and assume that D and $(\lambda_0 - A - B)D$ are dense in E for some $\lambda_0 > \omega$.
Then the closure of $A+B$ exists and generates a strongly continuous semigroup $(U(t))_{t\geq 0}$ given by the so–called Lie–Trotter product formula

$$U(t)f = \lim_{n\to\infty} \left[T(\tfrac{t}{n})S(\tfrac{t}{n})\right]^n f \tag{2}$$

for all $f \in E$ and uniformly for t in compact intervals.

In [8] we gave an example of operators A and B such that the closure of the sum of A and B is a generator, but (1), and hence (2), is violated. Thus, to obtain the Lie–Trotter formula (2) a stability condition as in (1) is necessary.

On the other hand, in Theorem 1 one requires the range condition $\overline{(\lambda_0 - A - B)D} = E$, which also appears in generation results like the Lumer–Phillips theorem. In many applications this condition is hard to verify. The aim of this work is to replace the range condition by assumptions on the commutator of $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$, of A and B, or of their resolvents. The results will be applied to various examples in Section 4 and 5.

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2. Commutator conditions for semigroups and generators

To use commutator conditions which establish the convergence of the Lie–Trotter products, we modify the stability estimate (1).

Definition 2. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be strongly continuous semigroups on a Banach space E and $(F, |||\cdot|||)$ be a normed vector space which is embedded in E and invariant under both semigroups. The semigroup $(T(t))_{t\geq 0}$ is called* **exponentially bounded with respect to F** *if*

$$|||T(t)||| \leq Me^{\omega t}$$

for all $t \geq 0$ and some constants $M \geq 1,\ \omega \in \mathbb{R}$.
The semigroups $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are called **locally Trotter–stable with respect to F** *if there exist constants $t_0 > 0$ and $M_{t_0} \geq 1$ such that*

$$|||\left[T(\tfrac{t}{n})S(\tfrac{t}{n})\right]^n||| \leq M_{t_0} \tag{3}$$

for all $t \in [0, t_0]$ and $n \in \mathbb{N}$.

With this concept, we are able to state our main result.

Theorem 3. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be strongly continuous semigroups on a Banach space E and suppose that there exists a normed vector space $(F, |||\cdot|||)$ which is densely embedded in E and invariant under both semigroups such that the following conditions hold.*

- (a) *The two semigroups are exponentially bounded with respect to F and locally Trotter–stable with respect to both E and F.*
- (b) *The commutator condition*

$$\|T(t)S(t)f - S(t)T(t)f\| \leq t^{\alpha} M_1 |||f|||$$

is satisfied for all $f \in F, t \in [0, \delta]$ with $\delta > 0$, and for some constants $\alpha > 1$ and $M_1 \geq 0$.

Then the Lie–Trotter product formula holds, i.e.,

$$U(t)f := \lim_{n\to\infty} \left[T(\tfrac{t}{n})S(\tfrac{t}{n})\right]^n f$$

exists for all $f \in E$ and uniformly for t in compact intervals. Moreover, $(U(t))_{t\geq 0}$ is a strongly continuous semigroup with generator $(G, D(G))$ such that

$$G \supseteq \tfrac{d}{dt}\left[T(t)S(t)\right]_{|t=0} \supseteq A + B\ ,$$

where each operator is defined on its natural domain.

Proof. We denote the Lie–Trotter products by

$$P_n(t) := \left[T(\tfrac{t}{n})S(\tfrac{t}{n})\right]^n$$

for $n \in \mathbb{N}$ and $t \geq 0$. Let $t_0 > 0$ such that the stability condition (3) is fulfilled with respect to $(E, \|\cdot\|)$ and $(F, |||\cdot|||)$. Therefore, we obtain

$$\|P_n(t)\| = \|P_k(\tfrac{kt}{n})P_l(\tfrac{lt}{n})\| \leq M_{t_0}^2$$

for $t \in [0, \frac{3t_0}{2}]$ and $n = k + l$, $k, l \in \mathbb{N}$, such that $\frac{kt}{n}$, $\frac{lt}{n} \leq t_0$, $n \geq 2$. The analogous estimate also holds for $|||\cdot|||$. By induction we conclude that

$$\|P_n(t)\|, |||P_n(t)||| \leq M_T \tag{4}$$

for $(n,t) \in \mathbb{N} \times [0,T]$ with $T > 0$.
Take $t \in [0,T]$, $T > 0$, and $f \in F$. We choose $k, m, n \in \mathbb{N}$ with $0 < k \leq m$ such that $\frac{T}{m} \leq \delta$ and $m = k\,n$. Then, we obtain from the commutator condition (b) that

$$\begin{aligned}
\|T(\tfrac{t}{m})S(\tfrac{jt}{m})f - S(\tfrac{jt}{m})T(\tfrac{t}{m})f\| &\leq \sum_{l=0}^{j-1} \|S(\tfrac{lt}{m})\left(T(\tfrac{t}{m})S(\tfrac{t}{m}) - S(\tfrac{t}{m})T(\tfrac{t}{m})\right) S(\tfrac{(j-1-l)t}{m})f\| \\
&\leq j\left(\tfrac{t}{m}\right)^{\alpha} M_2 |||f|||
\end{aligned}$$

for $j \in \mathbb{N}$ and some constant $M_2 \geq 0$. Hence, by forming a telescope sum, we have

$$\begin{aligned}
&\|\left[T(\tfrac{kt}{m})S(\tfrac{kt}{m})\right] f - \left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^k f\| \\
&\qquad = \sum_{j=1}^{k-1} \|T(\tfrac{tj}{m})\left(T(\tfrac{t}{m})S(\tfrac{tj}{m}) - S(\tfrac{tj}{m})T(\tfrac{t}{m})\right) S(\tfrac{t}{m}) \left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^{k-1-j} f\| \\
&\qquad \leq M_3 \sum_{j=1}^{k-1} j \cdot \left(\tfrac{t}{m}\right)^{\alpha} |||f||| \\
&\qquad = M_3\ \frac{t^{\alpha}(k-1)k}{2\,m^{\alpha}} |||f|||
\end{aligned}$$

for some suitable constant $M_3 \geq 0$. Note that

$$\left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^j = \left[T\left(\tfrac{tj/m}{j}\right) S\left(\tfrac{tj/m}{j}\right)\right]^j \leq M_T$$

for $j \in \mathbb{N}$ and $1 \leq j \leq m$. We now conclude that

$$\begin{aligned}
\|P_n(t)f - P_m(t)f\| &= \|\left[T(\tfrac{kt}{m})S(\tfrac{kt}{m})\right]^n f - \left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^{k\cdot n} f\| \\
&= \sum_{l=0}^{n-1} \|\left[T(\tfrac{kt}{m})S(\tfrac{kt}{m})\right]^l \left(T(\tfrac{kt}{m})S(\tfrac{kt}{m}) - \left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^k\right) \left[T(\tfrac{t}{m})S(\tfrac{t}{m})\right]^{k\cdot(n-1-l)} f\| \\
&\leq M_4\ \frac{t^{\alpha} n(k-1)k}{2\,m^{\alpha}} |||f||| \\
&\leq M_4\ \frac{t^{\alpha} k}{2\,m^{\alpha-1}} |||f||| \\
&\leq M_5 \frac{k}{2m^{\alpha-1}} |||f|||
\end{aligned} \tag{5}$$

for some $M_4, M_5 \geq 0$.
Thus, we have for arbitrary $i, j \in \mathbb{N}$ and $0 \leq t \leq T$ that

$$\begin{aligned} \|P_{2^i}(t)f - P_{2^j}(t)f\| &\leq \sum_{l=i}^{j-1} \|P_{2^l}(t)f - P_{2^{l+1}}(t)f\| \\ &\leq M_5 \sum_{l=i}^{j-1} \left(\frac{1}{2^{\alpha-1}}\right)^{l+1} |||f||| \,. \end{aligned}$$

The last expression converges to 0 as $i, j \to \infty$, since $\alpha > 1$. Hence, we can define

$$U(t)f := \lim_{i\to\infty} P_{2^i}(t)f \tag{6}$$

for $f \in F$ and uniformly for $t \in [0, T]$. Due to (4) we can extend $U(t)$ to a bounded linear operator on E.
For arbitrary $n \in \mathbb{N}$, we obtain from (5) that

$$\begin{aligned} \|P_n(t)f - U(t)f\| &\leq \|P_n(t)f - P_{n2^l}(t)f\| + \|P_{n2^l}(t)f - P_{2^l}(t)f\| + \|P_{2^l}(t)f - U(t)f\| \\ &\leq \sum_{k=0}^{l-1} \|P_{n2^k}(t)f - P_{n2^{k+1}}(t)f\| + \|P_{n2^l}(t)f - P_{2^l}(t)f\| + \|P_{2^l}(t)f - U(t)f\| \\ &\leq \frac{M_5}{n^{\alpha-1}} \sum_{k=0}^{l-1} \left(\frac{1}{2^{\alpha-1}}\right)^{k+1} |||f||| + M_5 \, \frac{n}{(n2^l)^{\alpha-1}} |||f||| + \|P_{2^l}(t)f - U(t)f\| \\ &\leq \left(\frac{2^{\alpha-1}M_5}{(2^{\alpha-1}-1)n^{\alpha-1}} + \frac{M_5 n^{2-\alpha}}{2^{l(\alpha-1)}}\right) |||f||| + \|P_{2^l}(t)f - U(t)f\| \end{aligned} \tag{7}$$

for $l \in \mathbb{N}$.
Fix $\epsilon > 0$. By the definition of the operators $U(t)$, there exists $l_0 \in \mathbb{N}$ such that

$$\|P_{2^l}(t)f - U(t)f\| \leq \epsilon$$

for $f \in F$ and $l \geq l_0$. Further, for $n \in \mathbb{N}$ with $\left(\frac{1}{n}\right)^{\alpha-1} \leq \frac{\epsilon(2^{\alpha-1}-1)}{2^{\alpha-1}M_5}$ and $l \geq \max\{l_0, l_1\}$ such that $2^{l_1(\alpha-1)} \geq \left(\frac{M_5 n^{2-\alpha}}{\epsilon}\right)$, we have

$$\begin{aligned} \|P_n(t)f - U(t)f\| &\leq \left(\frac{2^{\alpha-1}M_5}{(2^{\alpha-1}-1)n^{\alpha-1}} + \frac{M_5 n^{2-\alpha}}{2^{l(\alpha-1)}}\right) |||f||| + \|P_{2^l}(t)f - U(t)f\| \\ &\leq 3\epsilon(\|f\| + |||f|||) \end{aligned}$$

for $f \in F$ and uniformly for $t \in [0, T]$. So the stability condition (4) implies that $P_n(t)$ converges strongly in E uniformly for $t \in [0, T]$.
To show the semigroup law for $(U(t))_{t\geq 0}$ and for rational numbers t, we use an idea from

the paper of Chernoff (cf. [4, Thm.2.5.1]). To that purpose, let $f \in E$ and $\epsilon > 0$. Then

$$\begin{aligned}
\|U(t)f - &U(\tfrac{t}{2})U(\tfrac{t}{2})f\| \\
&\le \|U(t)f - \left[T(\tfrac{t}{2n})\ S(\tfrac{t}{2n})\right]^{2n} f\| + \|\left[T(\tfrac{t}{2n})\ S(\tfrac{t}{2n})\right]^{2n} f - U(\tfrac{t}{2})U(\tfrac{t}{2})f\| \\
&\le \epsilon + \|\left[T(\tfrac{t}{2n})S(\tfrac{t}{2n})\right]^{n} \left(\left[T(\tfrac{t}{2n})S(\tfrac{t}{2n})\right]^{n} f - U(\tfrac{t}{2})\right) f\| \\
&\qquad + \|\left(\left[T(\tfrac{t}{2n})S(\tfrac{t}{2n})\right]^{n} - U(\tfrac{t}{2})\right) U(\tfrac{t}{2})f\| \\
&\le 3\epsilon
\end{aligned}$$

for sufficiently large n. This proves that $U(t) = U(\frac{t}{2})U(\frac{t}{2})$. In an analogous way, one can show $U(nt) = U(t)^n$ for $n \in \mathbb{N}$, from which one obtains the semigroup law for rational numbers.
By the uniform convergence of the Lie–Trotter products on compact intervals, we obtain that $(U(t))_{t\ge 0}$ is a strongly continuous semigroup. The assertion concerning the generator G follows from [4, Prop. 4.1]. □

In the following, we show how commutator conditions for the generators A and B imply condition (b) of Theorem 3. These conditions can be verified in many interesting applications (cf. Section 4 and 5).

Definition 4. *Let $(A, D(A))$ and $(B, D(B))$ be linear operators on a Banach space E. The operator*

$$C := AB - BA \qquad \text{with domain}$$

$$D(C) := D(AB) \cap D(BA) = \{f \in D(A) \cap D(B) : Bf \in D(A) \text{ and } Af \in D(B)\}$$

is called the **commutator** *of A and B.*

In the following we denote by $\|\cdot\|_C$ the **C–norm** defined by

$$\|f\|_C := \|f\| + \|Cf\|$$

for all $f \in D(C)$.
The next lemma allows to relate the commutator of $(T(t))_{t\ge 0}$ and $(S(t))_{t\ge 0}$ to the commutator C.

Lemma 5. *Let $(A, D(A))$ and $(B, D(B))$ be generators of strongly continuous semigroups $(T(t))_{t\ge 0}$ and $(S(t))_{t\ge 0}$ on a Banach space E, and let $(C, D(C))$ be the commutator of A and B. Suppose that there exists a subspace $F \subseteq D(C)$ which is dense in E and invariant under $(T(t))_{t\ge 0}$ and $(S(t))_{t\ge 0}$ such that for some $\delta > 0$the functions*

$$\begin{cases} s \mapsto AS(t)T(s)f\ ,\ \ s \mapsto CS(s)f \ \text{ are continuous and} \\ s \mapsto AS(s)f \ \text{ is differentiable} \end{cases}$$

in E for $s \in [0, \delta]$ and for all $f \in F$. Then the identity

$$T(t)S(t)f - S(t)T(t)f = \int_0^t T(s) \left(\int_0^t S(t-r)CS(r)T(t-s)f\,dr \right) ds \tag{8}$$

holds for all $f \in F$ and $t \in [0, \delta]$.

Proof. Let $\delta > 0$. We apply the fundamental theorem of calculus to the continuously differentiable functions

$$s \mapsto T(s)S(t)T(t-s)f \quad \text{and} \quad r \mapsto S(t-r)AS(r)f$$

for $0 \le r, s \le \delta$ and $f \in F$. Since the operator A is closed, we obtain by the differentiability of the second function that

$$\frac{d}{dr}S(t-r)AS(r)f = S(t-r)CS(r)f$$

for $0 \le r \le \delta$ and $f \in F$. Thus, we have

$$\begin{aligned}
T(t)S(t)f - S(t)T(t)f &= \int_0^t \frac{d}{ds}\left\{T(s)S(t)T(t-s)f\right\} ds \\
&= \int_0^t T(s)\left(AS(t) - S(t)A\right)T(t-s)f ds \\
&= \int_0^t T(s)\left(\int_0^t \frac{d}{dr}\left\{S(t-r)AS(r)T(t-s)f\right\} dr\right) ds \\
&= \int_0^t T(s)\left(\int_0^t S(t-r)\left(AB - BA\right)S(r)T(t-s)f dr\right) ds
\end{aligned}$$

for $f \in F$ and $t \in [0, \delta]$. $\square$

As an easy consequence of Lemma 5 we obtain the following.

Corollary 6. *Let $(A, D(A))$ and $(B, D(B))$ be generators of strongly continuous semigroups $(T(t))_{t \ge 0}$ and $(S(t))_{t \ge 0}$ on a Banach space E and $(C, D(C))$ be the commutator of A and B. Suppose that*

(a) *there exists a subspace $F \subseteq D(C)$ which is dense in E and invariant under both semigroups*

(b) *for some $\delta > 0$the functions*

$$\begin{cases} s \mapsto AS(t)T(s)f\ ,\ \ s \mapsto CS(s)f \ \text{ are continuous and} \\ s \mapsto AS(s)f \ \text{ is differentiable} \end{cases}$$

in E for $s \in [0, \delta]$ and for all $f \in F$, and

(c) *the two semigroups are exponentially bounded with respect to F and locally Trotter–stable with respect to E and $(F, \|\cdot\|_C)$.*

Then the conclusion of Theorem 3 holds.

Proof. By the exponential boundedness of the semigroups and Lemma 5 we obtain the estimate

$$\begin{aligned}
\|T(t)S(t)f - S(t)T(t)f\| &= \|\int_0^t T(s)\left(\int_0^t S(t-r)CS(r)T(t-s)f dr\right) ds\| \\
&\le Mt^2\|f\|_C
\end{aligned}$$

for $f \in F$, all $t \in [0, \delta]$ and some constant $M \geq 0$. Thus, the assumption (b) of Theorem 3 is fulfilled. Since condition (a) of Theorem 3 was assumed explicitely, the assertion follows. $\square$

3. Commutator conditions for resolvent operators

In the following we translate the commutator condition from Theorem 3 for the semigroups into an equivalent expression for the powers of the corresponding resolvent operators denoted by $R(\lambda, A)$ and $R(\mu, B)$, respectively. In some special cases, such an equivalence can be obtained without using all powers of the resolvents.
First, we state the following result on the asymptotic behaviour of the Gamma function.

Lemma 7. *The Gamma function defined by*

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt \qquad \textit{for } \mathrm{Re} z > 0$$

satisfies

$$\lim_{n\to\infty} n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = 1 \qquad \textit{for } a, b \in \mathbb{R} \ .$$

Proof. By the Stirling Formula we can conclude that

$$\Gamma(x+1) = \sqrt{2\pi} x^{x+1/2} e^{-x} (1 + \phi(x))$$

for $x \in \mathbb{R}$, where ϕ is a function converging to zero as $\frac{1}{x}$ (see [1, p. 652]). Therefore, we have

$$n^{b-a} \frac{\Gamma(n+a)}{\Gamma(n+b)} = \left(\frac{n+a-1}{n+b-1}\right)^{n+a-1/2} \left(\frac{n}{n+b-1}\right)^{b-a} e^{b-a} \frac{1+\phi(n+a-1)}{1+\phi(n+b-1)}$$

for $a, b \in \mathbb{R}$ which converges to 1 as $n \to \infty$. $\square$

In the sequel, we assume without loss of generality that $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ are bounded semigroups. Note that we now use a commutator condition for the semigroups slightly different from the one in Theorem 3.

Theorem 8. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be bounded strongly continuous semigroups on a Banach space E with generators $(A, D(A))$ and $(B, D(B))$, respectively, and suppose that there exists a normed vector space $(F, ||| \cdot |||)$ in E. Furthermore, let $\alpha, \beta \geq 0$ and $M \geq 0$. Then the following statements are equivalent.*

(i) *The commutator condition for the semigroups*

$$\|T(t)S(s)f - S(s)T(t)f\| \leq t^\alpha s^\beta M |||f||| \tag{9}$$

is satisfied for all $f \in F$ and $s, t \geq 0$.

(ii) *The commutator condition for the corresponding resolvent operators*

$$\|R(\lambda, A)^n R(\mu, B)^n f - R(\mu, B)^n R(\lambda, A)^n f\| \leq \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)!^2 \lambda^{\alpha+n} \mu^{\beta+n}} M |||f|||$$

holds for all $f \in F$, $\lambda, \mu > 0$ and $n \in \mathbb{N}$.

Proof. (i) $\Rightarrow$ (ii). Let $f \in F$ and $\lambda, \mu > 0$. We use the integral representation of the resolvent (see [6, Chap. II, Thm. 1.10]) and obtain

$$\begin{aligned}
&\|R(\lambda, A)^n R(\mu, B)^n f - R(\mu, B)^n R(\lambda, A)^n f\| \\
&\qquad \le \frac{1}{(n-1)!^2} \int_0^\infty \int_0^\infty r^{n-1} s^{n-1} e^{-\lambda r} e^{-\mu s} \|T(r)S(s)f - S(s)T(r)f\| \, ds dr \\
&\qquad \le \frac{M}{(n-1)!^2} \int_0^\infty \int_0^\infty r^{n-1+\alpha} e^{-\lambda r} s^{n-1+\beta} e^{-\mu s} dr ds \, |||f||| \\
&\qquad = \frac{M}{(n-1)!^2 \lambda^{\alpha+n} \mu^{\beta+n}} \int_0^\infty r^{n-1+\alpha} e^{-r} dr \int_0^\infty s^{n-1+\beta} e^{-s} ds \, |||f||| \\
&\qquad = \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)!^2 \lambda^{\alpha+n} \mu^{\beta+n}} M |||f|||
\end{aligned}$$

for all $n \in \mathbb{N}$.

(ii) $\Rightarrow$ (i). Recall the Post–Widder Inversion Formula

$$T(t)f = \lim_{n\to\infty} \left[\tfrac{n}{t} R(\tfrac{n}{t}, A)\right]^n f$$

is valid for $f \in E$ and uniformly for t in compact intervals ([6, Chap. III, Cor. 5.5]). Let $f \in F$ and $s, t \ge 0$. Applying Lemma 7 we conclude that

$$\begin{aligned}
\|T(t)S(s)f - S(s)T(t)f\| &\le \lim_{n\to\infty} \frac{n^{2n}}{t^n s^n} \|R(\tfrac{n}{t}, A)^n R(\tfrac{n}{s}, B)^n f - R(\tfrac{n}{s}, B)^n R(\tfrac{n}{t}, A)^n f\| \\
&\le \frac{n^{2n}}{t^n s^n} \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{(n-1)!^2 \left(\frac{n}{t}\right)^{\alpha+n} \left(\frac{n}{s}\right)^{\beta+n}} M |||f||| \\
&= n^{-(\alpha+\beta)} t^\alpha s^\beta \frac{\Gamma(n+\alpha)\Gamma(n+\beta)}{\Gamma(n)^2} M |||f||| \\
&\longrightarrow t^\alpha s^\beta M |||f|||
\end{aligned}$$

as $n \to \infty$. $\square$

We now present some cases where a commutator condition for the resolvent operator alone suffices.

Proposition 9. *Let $(T(t))_{t\ge0}$ and $(S(t))_{t\ge0}$ be bounded strongly continuous semigroups on a Banach space E with generators $(A, D(A))$ and $(B, D(B))$, respectively. Let $\alpha, \beta \ge 0$ such that $\alpha + \beta \ge 2$. Then the following statements are equivalent.*

(i) *The commutator condition for the semigroups*

$$\|T(t)S(s)f - S(s)T(t)f\| \le t^\alpha s^\beta M_1 \|f\|$$

is satisfied for all $f \in E$, $s, t \ge 0$, and some $M_1 \ge 0$.

(ii) *The commutator condition for the resolvent operators*

$$\|R(\lambda, A)R(\mu, B)f - R(\mu, B)R(\lambda, A)f\| \le \frac{M_2}{\lambda^{\alpha+1}\mu^{\beta+1}} \|f\|$$

holds for all $f \in E$, $\lambda, \mu > 0$, $n \in \mathbb{N}$, and some $M_2 \ge 0$.

Moreover, if $\alpha + \beta > 2$, then the semigroups $(T(t))_{t\ge0}$ and $(S(t))_{t\ge0}$ commute.

Proof. The implication (i) $\Rightarrow$ (ii) is proved analogously as in Theorem 8.
(ii) $\Rightarrow$ (i). For simplicity, we define

$$\mathcal{R}_A(\tfrac{n}{t}) := \tfrac{n}{t}R(\tfrac{n}{t}, A) \ , \mathcal{R}_B(\tfrac{n}{t}) := \tfrac{n}{t}R(\tfrac{n}{t}, B)$$

and the commutator

$$\left[\mathcal{R}_A(\tfrac{n}{t}), \mathcal{R}_B(\tfrac{n}{t})\right] := \tfrac{n}{t}R(\tfrac{n}{t}, A)\tfrac{n}{t}R(\tfrac{n}{t}, B) - \tfrac{n}{t}R(\tfrac{n}{t}, B)\tfrac{n}{t}R(\tfrac{n}{t}, A)$$

for all $n \in \mathbb{N}$ and $t \geq 0$. We note that

$$\| \left(\mathcal{R}_A(\tfrac{n}{t})\right)^j \|, \| \left(\mathcal{R}_B(\tfrac{n}{s})\right)^j \| \leq M$$

for all $j, n \in \mathbb{N}, 1 \leq j \leq n$, for $t, s \geq 0$ and some $M \geq 0$, since the semigroups are bounded. Moreover, by assumption, we have

$$\| \left[\mathcal{R}_A(\tfrac{n}{t}), \mathcal{R}_B(\tfrac{n}{t})\right] \| \leq M_2 \left(\tfrac{t}{n}\right)^{\alpha+\beta}$$

for all $n \in \mathbb{N}, \ t \geq 0$ and some $M_2 \geq 0$. Let $f \in E$ and $t, s \geq 0$. We obtain

$$\begin{aligned} &\|T(t)S(s)f - S(s)T(t)f\| \\ &\quad = \lim_{n\to\infty} \| \left(\mathcal{R}_A(\tfrac{n}{t})\right)^n \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n f - \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n \left(\mathcal{R}_A(\tfrac{n}{t})\right)^n f\| \end{aligned} \tag{10}$$

and

$$\begin{aligned} &\left(\mathcal{R}_A(\tfrac{n}{t})\right)^n \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n f - \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n \left(\mathcal{R}_A(\tfrac{n}{t})\right)^n f \\ &\quad = \mathcal{R}_A(\tfrac{n}{t}) \left[\left(\mathcal{R}_A(\tfrac{n}{t})\right)^{n-1}, \left(\mathcal{R}_B(\tfrac{n}{s})\right)^{n-1}\right] \mathcal{R}_B(\tfrac{n}{s})f \\ &\qquad + \left[\mathcal{R}_A(\tfrac{n}{t}), \left(\mathcal{R}_B(\tfrac{n}{s})\right)^{n-1}\right] \left(\mathcal{R}_A(\tfrac{n}{t})\right)^{n-1} \mathcal{R}_B(\tfrac{n}{s})f \\ &\qquad + \left(\mathcal{R}_B(\tfrac{n}{s})\right)^{n-1} \left[\left(\mathcal{R}_A(\tfrac{n}{t})\right)^n, \mathcal{R}_B(\tfrac{n}{s})\right] f \end{aligned}$$

for all $n \in \mathbb{N}$. Moreover, we can estimate

$$\begin{aligned} \| \left[\left(\mathcal{R}_A(\tfrac{n}{t})\right)^n, \left(\mathcal{R}_B(\tfrac{n}{s})\right)\right] f\| &\leq \sum_{j=0}^{n-1} \| \left(\mathcal{R}_A(\tfrac{n}{t})\right)^{n-1-j} \left[\mathcal{R}_A(\tfrac{n}{t}), \mathcal{R}_B(\tfrac{n}{s})\right] \left(\mathcal{R}_A(\tfrac{n}{t})\right)^j f\| \\ &\leq nM^2 M_2 \left(\tfrac{t}{n}\right)^\alpha \left(\tfrac{s}{n}\right)^\beta \|f\| \ , \\ \| \left[\mathcal{R}_A(\tfrac{n}{t}), \left(\mathcal{R}_B(\tfrac{n}{s})\right)^{n-1}\right] f\| &\leq (n-1)M^2 M_2 \left(\tfrac{t}{n}\right)^\alpha \left(\tfrac{s}{n}\right)^\beta \|f\| \ , \end{aligned}$$

and by induction

$$\| \left[\left(\mathcal{R}_A(\tfrac{n}{t})\right)^j, \left(\mathcal{R}_B(\tfrac{n}{s})\right)^j\right] f\| \leq j^2 M^2 M_2 \left(\tfrac{t}{n}\right)^\alpha \left(\tfrac{s}{n}\right)^\beta \|f\|$$

for each $j, n \in \mathbb{N}, \ 1 \leq j \leq n$. Therefore, equation (10) yields

$$\begin{aligned} &\|T(t)S(s)f - S(s)T(t)f\| \\ &\quad = \lim_{n\to\infty} \| \left(\mathcal{R}_A(\tfrac{n}{t})\right)^n \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n f - \left(\mathcal{R}_B(\tfrac{n}{s})\right)^n \left(\mathcal{R}_A(\tfrac{n}{t})\right)^n f\| \\ &\quad \leq \lim_{n\to\infty} \frac{n^2 t^\alpha s^\beta}{n^{\alpha+\beta}} M^2 M_2 \|f\| \\ &\quad \leq t^\alpha s^\beta M_1 \|f\| \end{aligned} \tag{11}$$

for some constant $M_1 \geq 0$.
Using the estimate (11), it follows that the semigroups commute if
$\alpha + \beta > 2$. $\square$

A similar commutator condition for the resolvent operators in the context of nonautonomous evolution equations was studied in [9].

4. Applications to semigroups from quantum mechanics

First, we apply our results to groups arising in quantum mechanics. Let $(T(t))_{t\in\mathbb{R}}$ and $(S(t))_{t\in\mathbb{R}}$ be a pair of unitary groups on a complex Hilbert space H satisfying the so–called Weyl relation (see [10, p. 274])

$$T(t)S(s) = e^{ist}S(s)T(t) \text{ for all } s,t \in \mathbb{R} . \tag{12}$$

These semigroups are evidently exponentially bounded and locally Trotter–stable with respect to $E = F = H$, and we obtain condition (b) of Theorem 3 from the estimate

$$\|T(t)S(t)f - S(t)T(t)f\| = \| \left(e^{it^2} - 1\right) S(t)T(t)f\| \leq t^2 \sum_{k=1}^{\infty} \frac{(t^2)^{k-1}}{k!}\|f\| \leq t^2 \|f\|$$

for $f \in H$ and $t \geq 0$. Therefore, the Lie–Trotter product formula holds.

Based on this relation, P.E.T. Jørgensen, R.T Moore [7, Chap. 11] and H. Suzuki [12] treated generalized Weyl relations leading to the following result.

Corollary 10. *Let $(T(t))_{t\geq0}$, $(S(t))_{t\geq0}$ and $(V(t))_{t\geq0}$ be strongly continuous semigroups on a Banach space E with generators $(A, D(A)), (B, D(B))$ and $(C, D(C))$, respectively. Moreover, suppose that there exists a subspace $F \subseteq D(C)$ which is dense in E and invariant under the semigroups $(T(t))_{t\geq0}$ and $(S(t))_{t\geq0}$ such that the following conditions hold.*

(a) *The two semigroups are exponentially bounded with respect to F and locally Trotter–stable with respect to both E and $(F, \|\cdot\|_C)$.*
(b) *The generalized Weyl relation, i.e.*

$$T(t)S(t)f = V(t^2)S(t)T(t)f$$

holds for $f \in E$ and all $t \geq 0$.

Then the conclusion of Theorem 3 holds.

Proof. We can estimate the commutator of $(T(t))_{t\geq0}$ and $(S(t))_{t\geq0}$ as

$$\begin{aligned}\|T(t)S(t)f - S(t)T(t)f\| &= \| \left(V(t^2) - Id\right) [T(t)S(t)]\, f\| \\ &= \| \int_0^{t^2} V(s)C\, [T(t)S(t)]\, f ds\| \\ &\leq t^2 M \|f\|_C\end{aligned}$$

for $f \in F$, $t \in [0, \delta]$ with $\delta > 0$ and some constant $M > 0$. So Theorem 3 implies the assertion. $\square$

5. Applications to Ornstein-Uhlenbeck operators

In this section we consider the (finite–dimensional) Ornstein–Uhlenbeck operator which has been studied e.g., in [5]. Let $E := C_0(\mathbb{R}^d)$ or $L^p(\mathbb{R}^d)$, $1 \le p < \infty$. For any symmetric, positive semi–definite matrix $A := (a_{ij})$ and a matrix $B := (b_{ij}) \in \mathcal{L}(\mathbb{R}^d)$, *the Ornstein–Uhlenbeck operator* is defined by

$$\mathcal{O}f := \sum_{i,j=1}^{d} a_{ij} D_{ij} f + \sum_{i,j=1}^{d} b_{ij} x_j D_i f = < \nabla, A\nabla f > + < Bx, \nabla f > \tag{13}$$

for each $f \in \mathcal{S}(\mathbb{R}^d)$, the Schwartz space of rapidly decreasing functions, $x \in \mathbb{R}^d$, and $\nabla := (\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d})$. Without loss of generality, we write A as $\operatorname{diag}(a_1, \dots, a_d)$ with $a_k > 0$ for $1 \le k \le j$, $a_{j+1} = \cdots = a_d = 0$ and define the operators $\mathcal{A}$ and $\mathcal{B}$ as the closures of

$$\mathcal{A}f := < \nabla, A\nabla f > \quad \text{and} \quad \mathcal{B}f := < Bx, \nabla f >$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. One simply checks that $\mathcal{A}$ and $\mathcal{B}$ are closable on $\mathcal{S}(\mathbb{R}^d)$ (cf. [14, Chap. II.6]). The operator $(\mathcal{A}, D(\mathcal{A}))$ generates a strongly continuous semigroup $(\mathcal{T}(t))_{t\ge0}$ given by

$$\mathcal{T}(t)f(x) = \frac{1}{(4\pi t \prod_{i=1}^{j} a_i)^{j/2}} \int_{\mathbb{R}^j} e^{-\sum_{i=1}^{j} \frac{|x_i - s_i|^2}{4a_i t}} f(s_1, \cdots, s_j, x_{j+1}, \cdots, x_d) ds_1 \cdots ds_j \tag{14}$$

for $t > 0, x \in \mathbb{R}^d$, and $f \in E$ (see [3]). Furthermore, the operator $(\mathcal{B}, D(\mathcal{B}))$ generates the strongly continuous semigroup $(\mathcal{S}(t))_{t\ge0}$ given by

$$\mathcal{S}(t)f(x) = f(e^{tB}x) \tag{15}$$

for all $f \in E$ and $x \in \mathbb{R}^d$ (see [6, Chap. II, Sec. 3.28]).
These semigroups and their generators have the following useful properties.

Lemma 11. *Let $(\mathcal{T}(t))_{t\ge0}$, $(\mathcal{S}(t))_{t\ge0}$ and their generators $(\mathcal{A}, D(\mathcal{A}))$and $(\mathcal{B}, D(\mathcal{B}))$be as above. Then the following properties hold.*

(a) *The semigroups $(\mathcal{T}(t))_{t\ge0}$and $(\mathcal{S}(t))_{t\ge0}$ are locally Trotter–stable with respect to E.*

(b) *The commutator of $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$ is given by*

$$\mathcal{A}\mathcal{B}f - \mathcal{B}\mathcal{A}f = 2 < BA\nabla, \nabla f >$$

for all $f \in \mathcal{S}(\mathbb{R}^d)$.

(c) *The functions*

$$\begin{cases} s \mapsto AS(t)T(s)f \ , s \mapsto CS(s)f \ \text{are continuous and} \\ s \mapsto AS(s)f \ \text{is differentiable} \end{cases}$$

in E for all $f \in F$.

(d) *The operator $< BA\nabla, \nabla >$ commutes with $(\mathcal{T}(t))_{t\ge0}$ on $\mathcal{S}(\mathbb{R}^d)$ and*

$$< BA\nabla, \nabla \mathcal{S}(t)f > = \mathcal{S}(t) < e^{tB} BA e^{tB^\star} \nabla, \nabla f >$$

for $f \in \mathcal{S}(\mathbb{R}^d)$, where $B^\star$ denotes the transpose of the matrix B.

Proof. Clearly, the Schwarz space $\mathcal{S}(\mathbb{R}^d)$ is invariant under both semigroups and both generators $\mathcal{A}$ and $\mathcal{B}$. Moreover, $(\mathcal{T}(t))_{t\geq 0}$ is a semigroup of contractions and

$$\|\mathcal{S}(t)f\|_{C_0} \leq \|f\|_{C_0}\ ,\ \ \|\mathcal{S}(t)f\|_{L^p} \leq e^{t\omega}\|f\|_{L^p}$$

for $f \in E$ and $\omega := \frac{tr(B)}{p}$. It follows that the semigroups are locally Trotter–stable with respect to E.
To determine the commutator of $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$ on $\mathcal{S}(\mathbb{R}^d)$, we note that

$$\frac{\partial}{\partial x_k} < Bx, \nabla f >=< Be_k, \nabla f > + < Bx, \frac{\partial}{\partial x_k}\nabla f >$$

for $1 \leq k \leq d,\ f \in \mathcal{S}(\mathbb{R}^d)$, and the k-th unit vector e_k. Hence, we have

$$\frac{\partial^2}{\partial^2 x_k} < Bx, \nabla f >= 2 < Be_k, \frac{\partial}{\partial x_k}\nabla f > + < Bx, \nabla\frac{\partial^2}{\partial x_k^2} f > \tag{16}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$. Therefore, we obtain from identity (16) that

$$\begin{aligned}\mathcal{A}\mathcal{B}f - \mathcal{B}\mathcal{A}f &= 2\sum_{k=1}^{d} < Be_k a_k \frac{\partial}{\partial x_k}, \nabla f > + \sum_{k=1}^{d} < Bx, a_k \frac{\partial^2}{\partial x_k^2}\nabla f > \\ &\quad - < Bx, \nabla \sum_{k=1}^{d} a_k \frac{\partial^2}{\partial x_k^2} f > \\ &= 2 < BA\nabla, \nabla f >\end{aligned}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$.
Clearly, the operators $\mathcal{A}$ and $\mathcal{B}$ are continuous for the usual topology on the space $\mathcal{S}(\mathbb{R}^d)$ induced by the family of seminorms

$$p_{\alpha,n} = \sup_{x\in\mathbb{R}^d} |x^n D^\alpha f(x)|\ .$$

By a straightforward computation, using [11, Thm. 7.2 and Thm. 7.4], we obtain that the statements in (c) are fulfilled with respect to the usual topology on $\mathcal{S}(\mathbb{R}^d)$. Since this topology is finer than the topology on E the assertion (c) is proved.
$(\mathcal{T}(t))_{t\geq 0}$ is a convolution semigroup and the convolution commutes with differential operators. Therefore, the commutator of $(\mathcal{A}, D(\mathcal{A}))$ and $(\mathcal{B}, D(\mathcal{B}))$ commutes with $(\mathcal{T}(t))_{t\geq 0}$.
For the semigroup $(\mathcal{S}(t))_{t\geq 0}$ we have the following commutator relation

$$\nabla\mathcal{S}(t)f = \sum_{k=1}^{d} \mathcal{S}(t) < e^{tB}e_k, \nabla f > e_k\ ,$$

and therefore

$$< BA\nabla, \nabla\mathcal{S}(t)f >= \mathcal{S}(t) < e^{tB}BAe^{tB^\star}\nabla, \nabla f >$$

for $f \in \mathcal{S}(\mathbb{R}^d)$. □

We now can apply Cor 6. The fact that $\mathcal{A} + \mathcal{B}$ is a generator seems to be known (cf. [5]).

Proposition 12. *Let $(\mathcal{T}(t))_{t\geq 0}$, $(\mathcal{S}(t))_{t\geq 0}$ be the strongly continuous semigroups on E given by (14) and (15) and generated by $(\mathcal{A}, D(\mathcal{A}))$and $(\mathcal{B}, D(\mathcal{B}))$, respectively. Then the conclusion of Theorem 3 holds.*

Proof. By Lemma 11 and Lemma 5 we obtain the estimate

$$\begin{aligned}
\|\mathcal{T}(t)\mathcal{S}(t)f - \mathcal{S}(t)\mathcal{T}(t)f\| &= \left\| \int_0^t \mathcal{T}(s) \left(\int_0^t \mathcal{S}(t-r)\left(\mathcal{A}\mathcal{B} - \mathcal{B}\mathcal{A}\right)\mathcal{S}(r)\mathcal{T}(t-s) f\, dr \right) ds \right\| \\
&\leq 2 \left\| \int_0^t \mathcal{T}(s)\mathcal{S}(t)\mathcal{T}(t-s) \left(\int_0^t < e^{rB}BAe^{rB^\star}\nabla, \nabla f > dr \right) ds \right\| \\
&\leq t^2 M \sup_{0\leq r\leq t} \| < e^{rB}BAe^{rB^\star}\nabla, \nabla f > \| \\
&\leq t^2 M \|AB\| e^{2t\|B\|} |||f|||
\end{aligned}$$

for $f \in \mathcal{S}(\mathbb{R}^d)$ and the norm

$$|||f||| := \|f\| + \sum_{1\leq i,j\leq d} \left\| \frac{\partial^2}{\partial_i \partial_j} f \right\| .$$

In the next step, we show the exponential boundedness with respect to $(\mathcal{S}(\mathbb{R}^d), |||\cdot|||)$. It is clear that $(\mathcal{T}(t))_{t\geq 0}$is contractive on $\mathcal{S}(\mathbb{R}^d)$ with respect to the norm $|||\cdot|||$. On the other hand, we have

$$\|\frac{\partial^2}{\partial_{ij}}\mathcal{S}(t)f\| = \|\mathcal{S}(t) < e^{tB}e_{ij}e^{tB^\star}\nabla, \nabla f > \| \leq e^{2t\omega}\|\mathcal{S}(t)\| \cdot |||f|||$$

with

$$e_{ij} := \begin{array}{cc} \begin{array}{ccccc} & & \downarrow i & & \end{array} & \\ \begin{pmatrix} 0 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 1 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & \cdots & 0 & \cdots & 0 \end{pmatrix} & \begin{array}{c} \\ \\ \underset{j}{\leftarrow} \\ \\ \\ \end{array} \end{array} .$$

Therefore, we can estimate

$$|||\mathcal{S}(t)f||| \leq d^2 e^{2t\omega}\|\mathcal{S}(t)\| \cdot |||f|||$$

for $f \in \mathcal{S}(\mathbb{R}^d)$.
Finally, this implies the local Trotter–stability

$$|||\left[\mathcal{T}(\tfrac{t}{n})\mathcal{S}(\tfrac{t}{n})\right]^n||| \leq e^{2t\omega}|||f|||$$

with respect to $(\mathcal{S}(\mathbb{R}^d), |||\cdot|||)$. We can now apply Theorem 3 concluding the proof. ☐

As a concrete example we mention a semigroup appearing in mathematical finance ([2]).

Example 13. On the space $E := C_0(\mathbb{R}^2)$ or $L^p(\mathbb{R}^2)$ $(1 \leq p < \infty)$ the Cauchy problem

$$\left\{ \begin{array}{rcl} \frac{d}{dt}u(t,x,y) &=& \frac{\partial^2}{\partial x^2}u(t,x,y) + x\frac{\partial}{\partial y}u(t,x,y)\ , \quad t \geq 0\ , \\ u(0,x,y) &=& f(x,y)\ , \end{array} \right. \tag{17}$$

is investigated in [2]. Taking

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$

the Cauchy problem (17) can be reformulated as an Ornstein–Uhlenbeck operator $\mathcal{O}$ defined in (13). Therefore, all assumptions of Proposition 12 are satisfied and $\mathcal{O}$ generates a semigroup given as the limit of the Lie–Trotter products.

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