LAPLACE TRANSFORMS OF VECTOR-VALUED FUNCTIONS WITH GROWTH ω AND SEMIGROUPS OF OPERATORS

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Abstract

For an arbitrary continuous, increasing function $\omega : [0, \infty) \mapsto \mathbf{C}$ of finite exponential type, we characterize Laplace-Stieltjes transforms of Banach-space-valued functions that are $O(\omega)$, and use this to establish a Hille-Yosida type theorem for strongly continuous semigroups that are $O(\omega)$. Corollaries include characterizing generators of strongly continuous semigroups that are $O(\exp(dt^{\frac{1}{d}}))$, for $d > 1$, in addition to the already known examples of exponential or polynomial growth.[1]
[2].

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1. Introduction

In this paper we deal with the Laplace-Stieltjes transforms of Banach-space-valued functions in a unified way, that includes the well-known polynomially bounded and exponentially bounded cases (see [7] and [1], respectively), and new cases of non-quasi-analytic growth including $O(\exp(dt^{\frac{1}{d}}))$ $(d > 1)$ growth.

In order to deal with our Laplace-Stieltjes transforms in a more general setting, we need a function $\omega : [0, \infty) \mapsto \mathbf{C}$ that satisfies the following conditions.

(a) $\omega(0) = 1,\ \omega(\cdot)$ is continuous and increasing on $[0, \infty)$.

(b) The number r_0 defined by

$$r_0 \equiv limsup_{t \to \infty} \frac{log\omega(t)}{t} \tag{*}$$

is finite. Note that $r_0 \geq 0$.

Once-integrated Laplace transforms of vector-valued functions have been shown to be important for studying abstract Cauchy problems of the first and second order (see [1] and [7]). We restrict ourselves in this paper to the first order abstract Cauchy problem

$$\frac{d}{dt}u(t,x) = Au(t,x) \quad (t \geq 0),\ u(0,x) = x. \tag{ACP}$$

As consequences of our Laplace transform results, we will characterize, in Section 3, $O(\omega(t))$ solutions of (ACP) and generators of $O(\omega(t))$ strongly continuous semigroups in terms of the resolvent of the generator. In other words, we will establish a Hille-Yosida type theorem for strongly continuous semigroups that are $O(\omega(t))$.

Our characterization of $O(\omega(t))$ Laplace-Stieltjes transforms is in terms of Laplace transforms of $t \mapsto t^n\omega(t)$, $n = 0, 1, 2, ...$ (Theorem 2.3). For $\omega(t) = (1 + t^k)$ or e^{rt}, since these Laplace transforms may be easily calculated, this immediately yields those well-known cases (Corollaries 2.4 and 2.5). But other types of growth are possible (see Example 3.6). For a large class of non-quasi-analytic ω; this means that

$$\int_0^\infty \frac{log\omega(t)}{1 + t^2}\, dt < \infty; \tag{**}$$

we also calculate the required Laplace transforms, to give a new explicit characterization of $O(\omega)$ Laplace-Stieltjes transforms (Theorem 2.12). This includes $\omega(t) = \exp(dt^{\frac{1}{d}})$, for $d > 1$ (Corollary 2.14).

For $d > 1$, we also give a sufficient condition, in terms of only the first power of the resolvent rather than all powers, for an operator A to generate a $O(\exp(dt^{\frac{1}{d}}))$ $(1-A)^{-2}$-regularized semigroup (Theorem 3.8), that is very close to a necessary condition (see Remark 3.9).

As a slight generalization of [6, Chapter 5], [7, Section 3] and [8, Section 1], we will introduce maximal continuously embedded subspaces where an operator generates a $O(\omega(t))$ strongly continuous semigroup.

In [15], the author studied the asymptotic behaviour of strongly continuous semigroups $\{T(t)\}_{t\geq 0}$ that satisfy $||T(t)|| \leq \omega(t)$. Here, in addition to (**), it is assumed that ω satisfies

$$\omega(t+s) \leq \omega(t)\omega(s) \tag{***}$$

and that $\omega_1(s) \equiv limsup_{t\to\infty}\frac{\omega(t+s)}{\omega(t)}$ is polynomially bounded. We will also slightly generalize this asymptotic result.

Throughout this paper, X is a Banach space and A is a (possibly unbounded) linear operator on X, with domain $D(A)$ and resolvent set $\rho(A)$. We will denote by $B(X)$ the algebra of all bounded linear operators on X.

2. Properties for Laplace-Stieltjes transform

In this section we present a new approach for Laplace-Stieltjes transforms that seems to be interesting and will be used in Section 3.

Definition 2.1 [1], [7]. *Suppose W is a Banach space and $r \in \mathbf{R}$. We say that $g : (r,\infty) \mapsto W$ is the Laplace-Stieltjes transform of G if $G : [0,\infty) \mapsto W$ is continuous, $G(0) = 0$, and*

$$g(\lambda) = \lambda \int_0^\infty e^{-\lambda t} G(t)\, dt \quad (\lambda > r).$$

We call G the determining function of g.

Let ω be as in Section 1.

Theorem 2.2. *Suppose $g : (r_0,\infty) \mapsto \mathbf{C}$, where r_0 is the constant defined in (*). Then the following are equivalent.*

(a) g is the Laplace-Stieltjes transform of a function $G : [0,\infty) \mapsto \mathbf{C}$ and there exists a constant $K_1 > 0$ so that

$$limsup_{h\to 0^+}\frac{1}{h}|G(t+h) - G(t)| \leq K_1\omega(t), \quad \forall t \geq 0.$$

(b) There exists a constant $K_2 > 0$ so that g is infinitely differentiable and

$$|g^{(n)}(\lambda)| \leq K_2 \int_0^\infty e^{-\lambda t} t^n \omega(t)\, dt, \quad \forall \lambda > r_0, n \in \mathbf{N}\cup\{0\}.$$

$K_1,\ K_2$ may be chosen so that $K_2 \le K_1 \le 2K_2$.

Proof. $(a) \to (b)$, with $K_2 = K_1$, is an immediate consequence of the definition of G and the equality in (a).

$(b) \to (a)$. First assume that g is real-valued. (b) implies that

$$h_{\pm}(\lambda) \equiv K_2 \int_0^{\infty} e^{-\lambda t}\omega(t)\,dt \pm g(\lambda), \quad \forall \lambda > r_0,$$

is completely monotone, that is

$$(-1)^n h_{\pm}^{(n)}(\lambda) = K_2 \int_0^{\infty} e^{-\lambda t} t^n \omega(t)\,dt \pm (-1)^n g^{(n)}(\lambda) \ge 0,$$
$$\forall n \in N \cup \{0\},\ \lambda > r_0.$$

By Bernstein's theorem [18, p. 156, Corollary 7], there exist nondecreasing functions $\alpha_{\pm}$ so that $\alpha_{\pm}(0) = 0$ and

$$h_{\pm}(\lambda) = \int_0^{\infty} e^{-\lambda t}\,d\alpha_{\pm}(t) = \lambda \int_0^{\infty} e^{-\lambda t}\alpha_{\pm}(t)\,dt, \quad \forall \lambda > r_0.$$

Therefore

$$\begin{aligned} g(\lambda) &= \lambda \int_0^{\infty} e^{-\lambda t}[\alpha_+(t) - K_2 \int_0^t \omega(s)\,ds]\,dt \\ &= \lambda \int_0^{\infty} e^{-\lambda t}[K_2 \int_0^t \omega(s)\,ds - \alpha_-(t)]\,dt. \end{aligned}$$

By the uniqueness of the Laplace transform, we have

$$\alpha_+(t) - K_2 \int_0^t \omega(s)\,ds = K_2 \int_0^t \omega(s)\,ds - \alpha_-(t).$$

Denote the preceding quantity by $G(t)$. Then g is the Laplace-Stieltjes transform of G and

$$limsup_{h\to 0^+} \frac{1}{h}|G(t+h) - G(t)| \le K_2\omega(t).$$

Now assume g is complex-valued. Write $g(\lambda) = g_1(\lambda) + ig_2(\lambda)$ where $g_1(\lambda)$ and $g_2(\lambda)$ are real-valued. Since $g_j(j = 1, 2)$ satisfies the conditions in (b), there exists $G_j : [0, \infty) \mapsto \mathbf{C}$ so that

$$\begin{aligned} g_j(\lambda) &= \lambda \int_0^{\infty} e^{-\lambda t} G_j(t)\,dt; \\ limsup_{h\to 0^+} &\frac{1}{h}|G_j(t+h) - G_j(t)| \le K_2\omega(t). \end{aligned}$$

Denote $G(t) \equiv G_1(t) + iG_2(t)$. Then

$$\begin{aligned} g(\lambda) &= \lambda \int_0^{\infty} e^{-\lambda t} G(t)\,dt \ \textit{and} \\ limsup_{h\to 0^+} &\frac{1}{h}|G(t+h) - G(t)| \le 2K_2\omega(t). \quad \square \end{aligned}$$

The following Theorem 2.3 is a vector-valued version of Theorem 2.2. In Theorem 2.3 and Corollaries 2.4 and 2.5, $K_1,\ K_2$ may be also chosen so that $K_2 \le K_1 \le 2K_2$.

Theorem 2.3. *Suppose W is a Banach space, r_0 is the constant defined in (*) and $g : (r_0, \infty) \mapsto W$. Then the following are equivalent.*

(a) g is the Laplace-Stieltjes transform of a function $G : [0, \infty) \mapsto W$ and there is a constant $K_1 > 0$ so that

$$limsup_{h \to 0^+} \frac{1}{h} ||G(t+h) - G(t)|| \leq K_1 \omega(t), \quad \forall t \geq 0.$$

(b) g is infinitely differentiable and there is a constant $K_2 > 0$ so that

$$||g^{(n)}(\lambda)|| \leq K_2 \int_0^\infty e^{-\lambda t} t^n \omega(t)\, dt, \quad \forall \lambda > r_0, n \in \mathbf{N} \cup \{0\}.$$

Proof. $(a) \to (b)$ is clear.

$(b) \to (a)$. From Theorem 2.2, for any $x^* \in W^*$, there exists a function $G_{x^*} : [0, \infty) \mapsto \mathbf{C}$ so that $\lambda \mapsto < g(\lambda), x^* >$ is the Laplace-Stieltjes transform of G_{x^*} and

$$limsup_{h \to 0^+} \frac{1}{h} |G_{x^*}(t+h) - G_{x^*}(t)| \leq 2K_2 ||x^*|| \omega(t), \quad \forall t \geq 0.$$

From [1, Theorem 1.1] and its proof, there is $G : [0, \infty) \mapsto W$ such that

$$G_{x^*}(t) = < G(t), x^* >, \quad \forall t \geq 0, \ x^* \in W^*.$$

$G(\cdot)$ is clearly the desired function. $\square$

Corollary 2.4 [7]. *Suppose W is a Banach space, $k \in N \cup \{0\}$ and $g : (0, \infty) \mapsto W$. Then the following are equivalent.*

(a) g is the Laplace-Stieltjes transform of a function $G : [0, \infty) \mapsto W$ and there is a constant $K_1 > 0$ so that

$$limsup_{h \to 0_+} \frac{1}{h} ||G(t+h) - G(t)|| \leq K_1 (1 + t^k), \quad \forall t \geq 0.$$

(b) g is infinitely differentiable and there is a constant $K_2 > 0$ so that

$$||g^{(n)}(\lambda)|| \leq K_2 [\frac{n!}{\lambda^{n+1}} + \frac{(n+k)!}{\lambda^{n+k+1}}], \quad \forall \lambda > 0, \ n \in N \cup \{0\}.$$

Proof. Set $\omega(t) \equiv 1 + t^k$. From

$$\begin{aligned} \int_0^\infty e^{-\lambda t} t^n \omega(t)\, dt &= \int_0^\infty e^{-\lambda t} [t^n + t^{n+k}]\, dt \\ &= \frac{n!}{\lambda^{n+1}} + \frac{(n+k)!}{\lambda^{n+K+1}} \end{aligned}$$

and Theorem 2.3, the corollary follows. $\square$

Corollary 2.5 [1]. *Suppose W is a Banach space, $r > 0$ and $g : (r,\infty) \mapsto W$. Then the following are equivalent.*

(a) g is the Laplace-Stieltjes transform of a function $G : [0,\infty) \mapsto W$ and there is a constant $K_1 > 0$ so that

$$limsup_{h\to 0+}\frac{1}{h}||G(t+h) - G(t)|| \leq K_1 e^{rt}, \quad \forall t \geq 0.$$

(b) g is infinitely differentiable and there is a constant $K_2 > 0$ so that

$$||g^{(n)}(\lambda)|| \leq K_2 n!(\lambda - r)^{-(n+1)}, \quad \forall \lambda > r, n \in \mathbf{N} \cup \{0\}.$$

Proof. Set $\omega(t) \equiv e^{rt}$. From

$$\begin{aligned}\int_0^\infty e^{-\lambda t} t^n \omega(t) &= \int_0^\infty e^{-(\lambda - r)} t^n \, dt \\ &= n!(\lambda - r)^{-(n+1)}, \quad \forall \lambda > r, n \in \mathbf{N} \cup \{0\},\end{aligned}$$

and Theorem 2.3, the corollary follows. $\square$

To study the third special case, we first make some preparations.

Let $\{M_k\}$ $(k \in N \cup \{0\})$ be a sequence of positive numbers with $M_0 = 1$. Denote $m_k \equiv \frac{M_k}{M_{k-1}}$ $(k \in N)$. The following two conditions can be found in [4, 14].

(M.1) *Logrithmic convexity* : $M_k^2 \leq M_{k-1}M_{k+1}$, or equivalently, $\{m_k\}$ increases.
(M.2) *Non-quasi-analyticity* : $\sum_{k=1}^\infty \frac{M_{k-1}}{M_k} < \infty$, or equivalently, $\sum_{k=1}^\infty \frac{1}{m_k} < \infty$.

A simple example that satisfies (M.1) but not (M.2) is $\{k!\}$ $(k \in N \cup \{0\})$. Another well-known sequence that satisfies (M.1) is $N_{0,d} = 1$, $N_{k,d} = (\frac{k}{e})^{kd}$ $(k \in N)$, where $d > 0$ (see [3]). This sequence satisfies (M.2) if and only if $d > 1$. Denote $n_{k,d} = \frac{N_{k,d}}{N_{k-1,d}}$ $(k \in N)$. If $d = 1$, we will write $N_k = N_{k,d}$, $n_k = n_{k,d}$ and $M_k^* = \frac{M_k}{N_k}$. We define the respective associate functions for $\{M_k\}$, $\{N_k\}$ and $\{M_k^*\}$ on $[0,\infty)$ in the following way.

$$\begin{aligned} &M(t) \equiv sup\{log\frac{t^k}{M_k} : k \in N \cup \{0\}\}, \ \forall t > 0, \ M(0) = 1; \\ &N(t) \equiv sup\{log\frac{t^k}{N_k} : k \in N \cup \{0\}\}, \ \forall t > 0, \ N(0) = 1; \\ &M^*(t) \equiv sup\{log\frac{t^k}{M_k^*} : k \in N \cup \{0\}\}, \ \forall t > 0, \ M^*(0) = 1.\end{aligned}$$

The following lemma is an immediate consequence of the definition of $M(\cdot)$.

Lemma 2.6 [14]. *Assume $\{M_k\}$ satisfies (M.1) and $lim_{k\to\infty} m_k = \infty$. Then*

$$e^{M(t)} = \frac{t^k}{M_k} \quad \forall m_k \leq t < m_{k+1}, \ k \in N.$$

Lemma 2.7. *The function $N(t)$ satisfies*

$$e^{N(t)} \le e^t \text{ and } lim_{t\to\infty}\frac{e^{N(t)}}{e^t} = 1.$$

Proof. Regard k as a continuous variable, differentiate $\frac{t^k}{N_k}$ in k and set the resulting derivative equal to zero to find $k = t$. Here we note that $N_k = (\frac{k}{e})^k$. Thus $sup_{k\in[0,\infty)}\frac{t^k}{N_k} = e^t$. This implies that $e^{N(t)} \le e^t$ and that

$$e^{N(t)} = g(t) \equiv max\{\frac{t^{[t]}}{(\frac{[t]}{e})^{[t]}}, \frac{t^{[t]+1}}{(\frac{[t]+1}{e})^{[t]+1}}\}$$

A direct calculation gives the desired equality. □

It is easy to see that (M.1) and (M.2) imply that $\frac{m_k}{k} \to \infty$. From $n_k < k$, we have $\frac{m_k}{n_k} \to \infty$. In the following, instead of (M.1) and (M.2), we introduce the following condition.

(M.3) $\{\frac{m_k}{n_k}\}$ increases and $lim_{t\to\infty}\frac{m_k}{n_k} \to \infty$. The former is equivalent to $\{M_k^*\}$ being logarithmically convex.

When (M.3) is true, $M^*(t)$ is finite for all $t \ge 0$ by Lemma 2.6. Clearly, (M.3) implies (M.1). But the following example shows that (M.3) does not imply (M.2). Let

$$M_0 = 1; \; M_k \equiv N_k\, \Pi_{j=1}^k log(1+j), \quad \forall k \in N.$$

Then $m_k = n_k log(1+k)$ and hence (M.3) holds. The following calculation

$$\sum_{k=1}^{\infty}\frac{1}{m_k} = \sum_{k=1}^{\infty}\frac{1}{n_k log(1+k)}$$
$$\ge \sum_{k=1}^{\infty}\frac{1}{k log(1+k)} = \infty$$

shows that (M.2) is not true. Here we note that $n_k < k$.

Lemma 2.8. *Assume (M.3) holds. Then $M(\cdot)$, $N(\cdot)$ and $M^*(\cdot)$ satisfy*

$$e^{M(\lambda t)} \le e^{N(\alpha t)} e^{M^*(\frac{\lambda}{\alpha})}, \quad \forall \lambda > 0,\ \alpha > 0,\ t > 0.$$

Proof. The relation follows from

$$log\frac{(\lambda t)^k}{M_k} = log\frac{(\alpha t)^k}{N_k} + log\frac{(\frac{\lambda}{\alpha})^k}{M_k^*}, \quad \forall k \in N \cup \{0\},$$

and the definitions of $M(\cdot)$, $N(\cdot)$ and $M^*(\cdot)$. □

Lemma 2.9. *Assume (M.3) holds. Then*

$$lim_{t\to\infty}\frac{M(t)}{t}=0.$$

Proof. Lemmas 2.8 and 2.7 imply that

$$\frac{M(\lambda t)}{t}\leq\frac{N(\alpha t)}{t}+\frac{M^*(\frac{\lambda}{\alpha})}{t}\leq\alpha+\frac{M^*(\frac{\lambda}{\alpha})}{t}.$$

Set $\lambda=1$ to obtain

$$0\leq limsup_{t\to\infty}\frac{M(t)}{t}\leq\alpha.$$

α being arbitrarily fixed, the lemma follows. □

Lemma 2.10. *Assume (M.3) holds. Then for any $\lambda>0$, there exists a function $\alpha_\lambda(\cdot)$ on $[0,\infty)$ so that*

(a) $\alpha_\lambda(\cdot)$ is positive, continuous and decreasing;
(b) $lim_{t\to\infty}\alpha_\lambda(t)=0$;
(c) there exists $K>0$ such that

$$e^{t\alpha_\lambda(t)}e^{M^*(\frac{\lambda}{\alpha_\lambda(t)})}\leq Ke^{M(\lambda t)},\quad\forall t\geq 0.$$

K may be chosen so that $K\leq e$.

Proof. Given $\lambda>0$. Define

$$\alpha_\lambda(t)=[\lambda\ n_k m_k(\frac{1}{m_k}-\frac{1}{m_{k+1}})][(n_{k+1}-n_k)+\lambda t(\frac{n_k}{m_k}-\frac{n_{k+1}}{m_{k+1}})]^{-1},$$
$$\forall m_k\leq\lambda t<m_{k+1},\quad k\in N;$$

$$\alpha_\lambda(t)=\alpha_\lambda(m_1),\quad\forall 0\leq\lambda t<m_1.$$

The fact that $\alpha_\lambda(t)$ satisfies (a) and (b) is an immediate consequence of its definition. We now prove (c). For $m_k\leq\lambda t<m_{k+1}$, a few steps of calculation give us

$$n_k\leq t\alpha_\lambda(t)\leq n_{k+1};$$

$$\frac{\lambda}{\alpha_\lambda(t)}=[(\frac{m_{k+1}}{n_{k+1}}-\frac{m_k}{n_k})(\lambda t-m_k)](m_{k+1}-m_k)^{-1}+\frac{m_k}{n_k}.$$

The latter implies that

$$\frac{m_k}{n_k}\leq\frac{\lambda}{\alpha_\lambda(t)}\leq\frac{m_{k+1}}{n_{k+1}},\quad\forall m_k\leq\lambda t<m_{k+1}.$$

This, together with Lemmas 2.6 and 2.7, shows that there is $K > 0$ so that

$$e^{t\alpha_\lambda(t)}e^{M^*(\frac{\lambda}{\alpha_\lambda(t)})} \leq Ke^{N(t\alpha_\lambda(t))}e^{M^*(\frac{\lambda}{\alpha_\lambda(t)})}$$
$$= K\frac{[t\alpha_\lambda(t)]^k}{N_k}\frac{[\frac{\lambda}{\alpha_\lambda(t)}]^k}{M^*} = Ke^{M(\lambda t)}.$$

We now show that K may be chosen so that $K \leq e$. Assume $k \leq t < k+1$, then from the proof of Lemma 2.7,

$$ee^{N(t)} \geq e\frac{t^k}{(\frac{k}{e})^k} = \frac{e^{k+1}t^k}{k^k} \geq e^{k+1} \geq e^t.$$

The lemma follows. □

We now state and prove our third special case.

Theorem 2.11. *Assume (M.3) holds and $g : (0, \infty) \mapsto \mathbf{C}$. Then the following are equivalent.*

(a) g is the Laplace-Stieltjes transform of a function $G : [0, \infty) \mapsto \mathbf{C}$ and there is a constant $K_1 > 0$ so that

$$limsup_{h\to 0^+}\frac{1}{h}|G(t+h) - G(t)| \leq K_1e^{M(t)}, \quad \forall t > 0.$$

(b) g is infinitely differentiable and there is a constant $K_2 > 0$ so that, for every $\alpha > 0$,

$$|g^{(n)}(\lambda)| \leq K_2 n!(\lambda - \alpha)^{-(n+1)}e^{M^*(\frac{1}{\alpha})}, \quad \forall \lambda > \alpha, n \in \mathbf{N} \cup \{0\}.$$

K_1, K_2 may be chosen so that $K_2 \leq K_1 \leq 2eK_2$.

Proof. $(a) \to (b)$. Set $\omega(t) \equiv e^{M(t)}$. Then $r_0 = lim_{t\to\infty}\frac{log\omega(t)}{t} = 0$ by Lemma 2.9. Theorem 2.2 implies that (a) is equivalent to

(a') g is infinitely differentiable and

$$|g^{(n)}(\lambda)| \leq K_2\int_0^\infty e^{-\lambda t}t^n e^{M(t)}\, dt, \quad \forall \lambda > 0.$$

where K_2 may be chosen so that $K_2 \leq K_1 \leq 2K_2$ by Theorem 2.2. Given $\alpha > 0$, let $\lambda > \alpha$. From Lemmas 2.7 and 2.8,

$$\int_0^\infty e^{-\lambda t}t^n e^{M(t)}\, dt$$
$$= \int_0^\infty e^{-(\lambda-\alpha)t}t^n e^{-\alpha t}e^{M(t)}\, dt$$
$$\leq \int_0^\infty e^{-(\lambda-\alpha)t}t^n e^{M^*(\frac{1}{\alpha})}\, dt$$
$$= n!(\lambda - \alpha)^{-(n+1)}e^{M^*(\frac{1}{\alpha})}.$$

(b) follows.

$(b) \to (a)$. Given $\alpha > 0$, denote

$$h_{\pm,\alpha}(\lambda) = K_2(\lambda-\alpha)^{-1}e^{M^*(\frac{1}{\alpha})} \pm g(\lambda) \quad \forall\lambda > \alpha.$$

Assume g is real-valued. From the inequality in (b), $h_{\pm,\alpha}(\cdot)$ are completely monotone. By Bernstein's theorem again, there exist increasing functions $\beta_{\pm,\alpha}(t)$ so that

$$h_{\pm,\alpha}(\lambda) = \int_0^\infty e^{-\lambda t}\, d\beta_{\pm,\alpha}(t) = \lambda \int_0^\infty e^{-\lambda t}\beta_{\pm,\alpha}(t)\, dt.$$

Then

$$\begin{aligned} g(\lambda) &= \lambda \int_0^\infty e^{-\lambda t}[\beta_{+,\alpha}(t) - \frac{K_2}{\alpha}e^{M^*(\frac{1}{\alpha})}(e^{\alpha t}-1)]\, dt \\ &= \lambda \int_0^\infty e^{-\lambda t}[\frac{K_2}{\alpha}e^{M^*(\frac{1}{\alpha})}(e^{\alpha t}-1) - \beta_{-,\alpha}(t)]\, dt. \end{aligned}$$

By the uniqueness of the Laplace transform, we have

$$\begin{aligned} &\beta_{+,\alpha}(t) - \frac{K_2}{\alpha}e^{M^*(\frac{1}{\alpha})}(e^{\alpha t}-1) \\ &\qquad = \frac{K_2}{\alpha}e^{M^*(\frac{1}{\alpha})}(e^{\alpha t}-1) - \beta_{-,\alpha}(t), \end{aligned}$$

which is independent of α, since g is. Denote it by $G(t)$. Then

$$\begin{aligned} g(\lambda) &= \lambda \int_0^\infty e^{-\lambda t}G(t)\, dt; \\ limsup_{h\to 0^+}\frac{1}{h}|G(t+h) - G(t)| &\le K_2 e^{M^*(\frac{1}{\alpha})}e^{\alpha t}, \quad \forall\alpha > 0. \end{aligned}$$

α being independent of t, we may choose $\alpha = \alpha_1(t)$, where $\alpha_1(t)$ is the function defined in Lemma 2.10 corresponding to $\lambda = 1$. The application of Lemma 2.10 gives (a) and from the process of the arguments of Theorem 2.2, Lemma 2.10, K_1 may be chosen so that $K_1 \le 2eK_2$. $\square$

The following is a vector-valued version of Theorem 2.11.

Theorem 2.12. *Assume $\{M_k\}$ satisfies (M.3), W is a Banach space, and $g : (0,\infty) \mapsto W$. Then the following are equivalent.*

(a) g is the Laplace-Stieltjes transform of a function $G : [0,\infty) \mapsto W$ and there is a constant $K_1 > 0$ so that

$$limsup_{h\to 0^+}\frac{1}{h}||G(t+h) - G(t)|| \le K_1 e^{M(t)}, \quad \forall t > 0.$$

(b) g is infinitely differentiable and there is a constant $K_2 > 0$ so that, for every $\alpha > 0$,

$$||g^{(n)}(\lambda)|| \leq K_2 n!(\lambda - \alpha)^{-(n+1)} e^{M^*(\frac{1}{\alpha})},$$
$$\forall \lambda > \alpha,\ n \in N \cup \{0\}.$$

$K_1,\ K_2$ may be chosen so that $K_1 \leq 2eK_2$.

We now consider a special case of Theorem 2.12. Let $\{N_{d,k}\}$ be the sequence given previously and define

$$N_d(t) \equiv sup_{k \in N \cup \{0\}}[log \frac{t^k}{N_{d,k}}].$$

When $d > 1$, write $N^*_{d,k} = \frac{N_{d,k}}{N_k}$. Then $N^*_{d,k} = N_{d-1,k}$. Define

$$N^*_d(t) \equiv sup_{k \in N \cup \{0\}}[log \frac{t^k}{N^*_{d,k}}] = N_{d-1}(t).$$

The following lemma is an immediate generalization of Lemma 2.7.

Lemma 2.13. *Given $d > 0$. Then*

$$e^{N_d(t)} \leq exp(dt^{\frac{1}{d}}) \text{ and } lim_{t \to \infty} e^{N_d(t)} exp(-dt^{\frac{1}{d}}) = 1.$$

With the help of Lemma 2.13, we may prove

Corollary 2.14. *Suppose W is a Banach space, $d > 1$ and $g : (0, \infty) \mapsto W$. Then the following are equivalent.*

g is the Laplace-Stieltjes transform of a function $G : [0, \infty) \mapsto W$ and there is a constant $K_1 > 0$ so that

$$limsup_{h \to 0^+} \frac{1}{h}||G(t+h) - G(t)|| \leq K_1 exp(dt^{\frac{1}{d}}).$$

(b) g is infinitely differentiable and there exists $K_2 > 0$ so that, for every $\alpha > 0$,

$$||g^{(n)}(\lambda)|| \leq K_2 n!(\lambda - \alpha)^{-(n+1)} exp((d-1)\alpha^{-\frac{1}{d-1}}),$$
$$\forall \lambda > \alpha, n \in N \cup \{0\}.$$

Recall that $\omega(\cdot)$ is a continuous and increasing function defined on $[0, \infty)$ satisfying $\omega(0) = 1$ (see Section 1).

Theorem 2.15. *Assume $\omega(\cdot)$ is the function given above. Then the following are equivalent.*

(a) $\omega(\cdot)$ satisfies (**).

(b) There exist a sequence $\{m_k\}_{k\in N}$ of positive numbers satisfying

$$1 = m_1 \le \frac{m_2}{n_2} \le \frac{m_3}{n_3} \le \cdots; \quad \sum_{k=1}^{\infty} \frac{1}{m_k} < \infty;$$
$$\omega(t) \le e^{M(t)},$$

where $M(\cdot)$ is the function associated with $\{M_k\}_{k\in N\cup\{0\}}$.

Proof. (a) $\to$ (b). We may assume that $\omega(t)$ is strictly increasing and $\omega(t) \to \infty$, as $t \to \infty$. Otherwise, instead of $\omega(t)$, we may choose $(1+t)\omega(t)$. For every $k \in N$, define $a_k \in [1,\infty)$ and $l_k \in N$ so that

$$\omega(a_k) = e^{l_k}; \quad \frac{a_k}{n_k} \ge \frac{a_{k-1}}{n_{k-1}}; \quad l_k \ge k.$$

Then we have

$$\frac{k}{a_k} \le \frac{l_k}{a_k} = l_k \int_{a_k}^{\infty} \frac{1}{t^2}\,dt \le \int_{a_k}^{\infty} \frac{log\omega(t)}{t^2}\,dt.$$

Since $a_k \to \infty$, it follows that $\frac{k}{a_k} \to 0$. Thus the following calculation

$$\sum_{j=1}^{k} \frac{1}{a_j} - \frac{k}{a_{k+1}} = \sum_{j=1}^{k} j\Big(\frac{1}{a_j} - \frac{1}{a_{j+1}}\Big)$$
$$\le \sum_{j=1}^{k} l_j \int_{a_j}^{a_{j+1}} \frac{1}{t^2}\,dt \le \int_1^{\infty} \frac{log\omega(t)}{t^2}\,dt$$

implies that

$$\sum_{j=1}^{\infty} \frac{1}{a_j} \le \int_1^{\infty} \frac{log\omega(t)}{t^2}\,dt < \infty.$$

Let $m_k \equiv \frac{a_k}{e}$ $(k \in N)$. Then

$$m_1 \le \frac{m_2}{n_2} \le \frac{m_3}{n_3} \le \cdots; \quad \sum_{k=1}^{\infty} \frac{1}{m_k} < \infty.$$

Define $M_0 \equiv 1$, $M_k \equiv \Pi_{j=1}^{k} m_j$ for $k \in N$. Given $t \in [0,\infty)$, there exists a unique $k \in N$ such that $a_k \le t < a_{k+1}$. From

$$e^{M(t)} = sup_{j\ge 0} \frac{t^j}{M_j} \ge \frac{t^{l_{k+1}}}{M_{l_{k+1}}} \ge \frac{a_1 \cdots a_{l_{k+1}}}{M_{l_{k+1}}},$$

we have

$$\frac{\omega(t)}{e^{M(t)}} \leq \Pi_{j=1}^{l_{k+1}} \frac{em_j}{a_j} = 1.$$

(b) $\to$ (a) is clear, since $\sum_{k=1}^{\infty} \frac{1}{m_k} < \infty$ is equivalent to $\int_0^\infty \frac{M(t)}{1+t^2}\, dt < \infty$. $\square$

3. A Hille-Yosida type theorem for strongly continuous semigroups with growth ω

In this section we apply the results established in the previous section to produce Hille-Yosida type theorems for once integrated semigroups, when the generating operators are not densely defined, and strongly continuous semigroups, when the generating operators are densely defined (Theorem 3.2 and its corollaries). In particular, we will produce a Hille-Yosida type theorem for strongly continuous semigroups with non-quasi-analytic growth, which seems to be first appearing here. This includes $O(\exp(dt^{\frac{1}{d}}))$ growth (Corollary 3.5). Theorem 3.8 involves only the resolvent, rather than all powers of the resolvent.

Definition 3.1 [2]. *Assume A is closed. The exponentially bounded strongly continuous family of operators $\{S(t)\}_{t\geq 0}$ on X is a once integrated semigroup generated by A if $S(0) = 0$ and there is $r \in R$ such that $(r, \infty) \subseteq \rho(A)$ and*

$$(\lambda - A)^{-1}x = \lambda \int_0^\infty e^{-\lambda t} S(t)x\, dt \quad (x \in X,\ \lambda > r).$$

Recall that $\omega(\cdot)$ is the function defined in Section 1. The following theorem is a direct consequence of Theorem 2.3.

Theorem 3.2. *Suppose A is closed. Then the following are equivalent.*

(a) A is the generator of a once integrated semigroup $\{S(t)\}_{t\geq 0}$ and there is a constant $K_1 > 0$ so that

$$limsup_{h\to 0^+} \frac{1}{h}||S(t+h) - S(t)|| \leq K_1\omega(t) \quad \forall t \geq 0.$$

(b) $(r_0, \infty) \subseteq \rho(A)$ and there is a constant $K_2 > 0$ so that

$$||(\lambda - A)^{-n}|| \leq \frac{K_2}{(n-1)!} \int_0^\infty e^{-\lambda t} t^n \omega(t)\, dt, \quad \forall \lambda > r_0,\ n \in N.$$

If A is densely defined, then (a) and hence (b) are equivalent to

(c) A is the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ so that

$$||T(t)|| \leq K_1\omega(t), \quad \forall t \geq 0.$$

Proof. To use Theorem 2.3 to prove the theorem, it suffices to note that

$$(\frac{d}{d\lambda})^{n-1}[(\lambda - A)^{-1}] = (-1)^{n-1}(n-1)!(\lambda - A)^{-n}, \quad \forall \lambda > r_0,\ n \in N. \quad \square$$

Setting $\omega(t) = (1 + t^k)$ and e^{rt} gives [6, Theorem 2.9] and [1, Theorem 4.1], respectively. The following theorem and its corollaries characterize strongly continuous semigroups of other types of growth, and follow directly from Theorem 2.12.

Theorem 3.3. *Suppose A is closed and $\{M_k\}$ satisfies (M.3). Then the following are equivalent.*

(a) A is the generator of a once-integrated semigroup $\{S(t)\}_{t\geq 0}$ and there is a constant $K_1 > 0$ so that

$$limsup_{h\to 0^+}\frac{1}{h}||S(t+h)-S(t)|| \leq K_1 e^{M(t)}, \quad \forall t \geq 0.$$

(b) $(0,\infty) \subseteq \rho(A)$ and there is a constant $K_2 > 0$ so that, for any $\alpha > 0$,

$$||(\lambda-\alpha)^n(\lambda-A)^{-n}|| \leq K_2 e^{M^*(\frac{1}{\alpha})}, \quad \forall \lambda > \alpha, \ n \in N.$$

If A is densely defined, then (a) and hence (b) are equivalent to

(c) A is the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ so that

$$||T(t)|| \leq K_1 e^{M(t)}, \quad \forall t \geq 0.$$

$K_1,\ K_2$ may be chosen so that $K_2 \leq K_1 \leq 2eK_2$.

The following corollary offers a Hille-Yosida type theorem for once integrated semigroups and strongly continuous semigroups with non-quasi-analytic growth.

Corollary 3.4. *Suppose A is closed. Then the following are equivalent.*

(a) A is the generator of a once-integrated semigroup $\{S(t)\}_{t\geq 0}$ and there exists a continuous increasing function $\omega(\cdot)$ defined on $[0,\infty)$ that satisfies $\omega(0) = 1$ and (**), such that

$$limsup_{h\to 0^+}\frac{1}{h}||S(t+h)-S(t)|| \leq \omega(t),$$

(b) There is a sequence $\{M_k\}$ ($k \in N \cup \{0\}$) satisfying (M.2) and (M.3) such that (a) and hence (b) of Theorem 3.3 is true.

If A is densely defined, then (a) and hence (b) are equivalent to

(c) A is the generator of a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ and there exists a continuous increasing function $\omega(\cdot)$ defined on $[0,\infty)$ that satisfies $\omega(0) = 1$ and (**), such that

$$||T(t)|| \leq \omega(t), \quad \forall t \geq 0.$$

Proof. (a) $\to$ (b) is an immediate consequence of Theorems 2.15 and 3.3. (b) $\to$ (a) follows by setting $\omega(t) \equiv e^{M(t)}$ for all $t \geq 0$. $\square$

Similarly, Corollary 2.14 gives us the following.

Corollary 3.5. *Suppose A is closed and $d > 1$. Then the following are equivalent.*

(a) A is the generator of a once-integrated semigroup $\{S(t)\}_{t\geq 0}$ and there is a constant $K_1 > 0$ so that

$$limsup_{h\to 0^+}\frac{1}{h}||S(t+h)-S(t)|| \leq K_1 exp(dt^{\frac{1}{d}}), \quad \forall t \geq 0.$$

(b) $(0,\infty) \subseteq \rho(A)$ *and there is a constant* $K_2 > 0$ *so that, for any* $\alpha > 0$,

$$||(\lambda-\alpha)^n(\lambda-A)^{-n}|| \le K_2 exp((d-1)\alpha^{-\frac{1}{d-1}}), \quad \forall \lambda > \alpha, \ n \in N.$$

If A *is densely defined, then (a) and hence (b) are equivalent to*

(c) A *is the generator of a strongly continuous semigroup* $\{T(t)\}_{t\ge 0}$ *such that*

$$||T(t)|| \le K_1 exp(dt^{\frac{1}{d}}), \quad \forall t \ge 0.$$

Example 3.6. Weighted translation provides examples of strongly continuous semigroups with growth as in Corollary 3.5 that grow more rapidly than any polynomial.

Let $g : [0,\infty) \to (0,\infty)$, $1 \le p < \infty$, and define

$$L^p([0,\infty), g\,ds) \equiv \{f : [0,\infty) \to \mathbf{C} \mid \int_0^\infty |f(s)|^p g(s)\,ds < \infty\}$$

with norm

$$\|f\|^p \equiv \int_0^\infty |f(s)|^p g(s)\,ds.$$

Let A be the generator of right translation on $L^p([0,\infty), g\,dt)$; that is,

$$\left(e^{tA}f\right)(s) \equiv f(s-t) \ \ (s,t \ge 0),$$

where $f(s) \equiv 0$ for $s < 0$.

Then it is not hard to show that

$$\|e^{tA}\|^p = \sup_{s\ge 0}\left[\frac{g(t+s)}{g(s)}\right] \quad (t \ge 0).$$

If we choose, for $0 < r < 1$, $g(s) \equiv \exp(s^r)$, then, for any positive t, $\left[\frac{g(t+s)}{g(s)}\right]$ is decreasing as a function of s, thus

$$\|e^{tA}\| = \exp\left(\frac{t^r}{p}\right) \quad (t \ge 0).$$

Definition 3.7. If $C \in B(X)$ is injective, then a C-regularized semigroup generated by A is a strongly continuous family $\{W(t)\}_{t\ge 0} \subseteq B(X)$ such that $W(0) = C, W(t)W(s) = CW(t+s)$, for all $s,t \ge 0$, and

$$Ax = C^{-1}\left[\lim_{t\to 0}\frac{1}{t}(W(t)x - Cx)\right],$$

with maximal domain. Note that, when $C = I$, a C-regularized semigroup is a strongly continuous semigroup. See [6] for basic information.

Here is a sufficient condition for generating a $O(\exp(dt^{\frac{1}{d}}))$ $(1-A)^{-2}$-regularized semigroup, that involves only the resolvent rather than all its powers.

Theorem 3.8. *Suppose $d > 1$ and*

$$\{z \in \mathbf{C} \mid Re(z) > 0\} \subseteq \rho(A)$$

with

$$\|(z-A)^{-1}\| = O\left((Re(z)-\alpha)^{-1}\exp\left((d-1)\alpha^{-\frac{1}{d-1}}\right)\right), \tag{1}$$

whenever $Re(z) > \alpha > 0$. Then A generates a norm-continuous $(1-A)^{-2}$-regularized semigroup $\{W(t)\}_{t\geq 0}$ such that

$$\|W(t)\| = O\left(t^{1-\frac{1}{d}}\exp\left((d+1)t^{\frac{1}{d}}\right)\right). \tag{2}$$

Proof: As in the proof of [7, Theorem 4.10], define, for $0 < \epsilon < \frac{1}{2}$,

$$W(t) \equiv \int_{2\epsilon-i\infty}^{2\epsilon+i\infty} e^{zt}(z-A)^{-1}\frac{dz}{2\pi i(1-z)^2}.$$

As in that proof, this is a $(1-A)^{-2}$-regularized semigroup generated by A. For $0 < \epsilon < \frac{1}{4}$, there is a constant M so that (choosing $\alpha = \epsilon, z = 2\epsilon + iy$), by (1),

$$\|W(t)\| \leq M\int_{\mathbf{R}} e^{2\epsilon t}\epsilon^{-1}\exp\left((d-1)\epsilon^{-\frac{1}{d-1}}\right)\frac{dy}{(\frac{1}{2}+|y|)^2}; \tag{3}$$

Define

$$K \equiv M\int_{\mathbf{R}}\frac{dy}{(\frac{1}{2}+|y|)^2},$$

and in (3), for $t > 4^{\frac{d}{d-1}}$, choose

$$\epsilon \equiv t^{-\frac{(d-1)}{d}},$$

then we get from (3)

$$\|W(t)\| \leq Kt^{\frac{d-1}{d}}\exp\left((d+1)t^{\frac{1}{d}}\right).$$

□

Remark 3.9. If A generated a $(1-A)^{-2}$-regularized semigroup such that

$$\|W(t)\| = O\left(\exp\left(dt^{\frac{1}{d}}\right)\right),$$

then we would have

$$\|(z-A)^{-1}(1-A)^{-2}\| = O\left((Re(z)-\alpha)^{-1}\exp\left((d-1)\alpha^{-\frac{1}{d-1}}\right)\right),$$

the same as (1) with $(z-A)^{-1}$ replaced by $(z-A)^{-1}(1-A)^{-2}$.

Thus Theorem 3.8 is almost an equivalence involving only the resolvent (in the right half plane) rather than all powers of the resolvent.

In [6, Chapter 5], [7], [8], [12] and [13], maximal continuously embedded subspaces, where a closed operator generates a bounded strongly continuous group, semigroup, or a Lipschitz continuous once-integrated semigroup, were constructed. In the following, we establish analogous constructions for strongly continuous semigroups and locally Lipschitz continuous once-integrated semigroups with growth ω.

The following concepts can be found in [6].

A *mild solution* of (ACP) is u so that $t \mapsto u(t,x) \in C([0,\infty), X)$, $\int_0^t u(s,x)\,ds \in D(A)$, for all $t \geq 0$, and

$$u(t,x) = A\int_0^t u(s,x)\,ds + x \quad (t \geq 0).$$

A *weak mild solution* of (ACP) is v so that $t \mapsto v(t,x)$ is locally Lipschitz continuous, $\int_0^t v(s,x)\,ds \in D(A)$, for all $t \geq 0$, and

$$v(t,x) = A\int_0^t v(s,x)\,ds + tx \quad (t \geq 0).$$

Definition 3.10. *Let $Z(A,\omega)$ be the set of all $x \in X$ for which (ACP) has a mild solution $u(\cdot,x)$ so that*

$$t \to [\omega(t)]^{-1}||u(t,x)||$$

is bounded and uniformly continuous on $[0,\infty)$. Define

$$||x||_{Z(A,\omega)} = sup_{t\geq 0}[\omega(t)]^{-1}||u(t,x)||.$$

Let $Y(A,\omega)$ be the set of all $x \in X$ for which (ACP) has a mild solution $v(\cdot,x)$ so that

$$esssup\{\omega(t)^{-1}|\frac{d}{dt} < v(t,x), x^* > | : t \geq 0, x^* \in X^*,\ ||x^*|| \leq 1\} < \infty.$$

We recall that $\omega_1(s) \equiv limsup_{t\to\infty}\frac{\omega(t+s)}{\omega(t)}$ is polynomially bounded (see Section 1).

The following theorem is an analogue of [6, Theorem 5.5], [7, Theorem 3.6], [8, Proposition 1.8] and [15, Theorem 7]. We omit its proof.

Theorem 3.11. *Suppose $m \in N \cup \{0\}$ and $\lambda - A$ is injective for $\lambda > r$. Then the following are true.*

(a) $Y(A,\omega)$ and $Z(A,\omega)$ are Banach spaces continuously embedded in X.

(b) $A|_{Z(A,\omega)}$ generates a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ such that

$$||T(t)|| \leq \omega(t), \quad \forall t \geq 0.$$

(c) $A|_{Z(A,\omega)}$ generates a once-integrated semigroup $\{S(t)\}_{t\geq 0}$ such that

$$limsup_{h\to 0^+}\frac{1}{h}||S(t+h)-S(t)|| \leq \omega(t), \quad \forall t \geq 0.$$

(d) $Z(A,\omega)$ equals the closure of $D(A|_{Z(A,\omega)})$ in $Y(A,\omega)$.

(e) $x \in Y(A,\omega)$ if and only if $x \in Im((\lambda - A)^{-n})$ for all $n \in N$, $\lambda > 0$ and there is a constant $K > 0$ such that

$$||(\lambda - A)^{-n}x|| \leq \frac{K}{(n-1)!}\int_0^\infty e^{-\lambda t}t^n\omega(t)\,dt.$$

(f) $Z(A,\omega)$ is maximal unique; that is, if W is a Banach space continuously embedded in X so that $A|_W$ generates a strongly continuous semigroup that is $O(\omega(t))$, then W is continuously embedded in $Z(A,\omega)$.

(g) Assume $\omega(\cdot)$ satisfies $(**), (***)$ and $\omega_1(s)$ is polynomially bounded and

$$P\sigma((A|_{Z(A,\omega)})^*) \cap i\mathbf{R} = \emptyset.$$

Then for every $x \in Z(A,\omega)$, such that $\sigma(A,x) \cap i\mathbf{R}$ is countable, we have

$$lim_{t\to\infty}\frac{1}{\omega_1(t)}||T(t)x|| = 0.$$

Letting $\omega(t) = 1 + t^k$, e^{rt} and $e^{M(t)}$, respectively, we will have three special cases of (a)—(f) of Theorem 3.11.

Remark 3.12. It is surprising that if $\omega(\cdot)$ satisfies (***), (b) of Theorem 3.11 can be improved as

$$||T(t)||_{Z(A,\omega)} \leq \omega(t), \quad \forall t \geq 0.$$

Given $x \in Z(A,\omega)$. We have

$$\begin{aligned}||[\omega(s)]^{-1}T(s)T(t)x|| &= ||[\omega(s)]^{-1}T(s+t)x|| \\ &\leq [\omega(s)]^{-1}\omega(s+t)||x||_{Z(A,\omega)} \leq \omega(t)||x||_{Z(A,\omega)}, \quad \forall s \geq 0,\end{aligned}$$

as desired. $\square$

The results of this section are easily extended to regularized semigroups (see [6]). We omit the details.

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