STABILITY OF SEMIGROUPS OF OPERATORS AND SPECTRAL SUBSPACES

RALPH DELAUBENFELS, VŨ QUỐC PHÓNG AND SHENGWANG WANG

Scientia Research Institute, Ohio University and Nanjing University

I. INTRODUCTION. An open problem is the spectral characterization of stable strongly continuous semigroups, where we say that a semigroup $\{e^{tA}\}_{t\geq 0}$ on a Banach space X is *stable* if

$$\lim_{t\to\infty} e^{tA}x = 0, \quad \text{for all } x \in X.$$

Intuitively, a bounded strongly continuous semigroup has generator A "contained in" the closed left half plane $\{z \in \mathbf{C} \,|\, Re(z) \leq 0\}$, and a stable strongly continuous semigroup has generator "contained in" the open left half plane $\{z \in \mathbf{C} \,|\, Re(z) < 0\}$. This intuition is realized in the following well-known sufficient condition for stability (note that the generator of a bounded strongly continuous semigroup has spectrum contained in the closed left half plane):

Proposition 1.0 ([H1], [H2], [A-B] and [L-Vu]). *Suppose A generates a bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ and $\sigma(A) \cap i\mathbf{R}$ is empty. Then $\{e^{tA}\}_{t\geq 0}$ is stable.*

This sufficient condition is far from necessary (see Proposition 1.2). A much weaker hypothesis, still consistent with the intuition of the generator being contained in the open left half plane, is to have the local spectral subspace $X^*(A^*, i\mathbf{R})$ (see Definition 2.7) be trivial.

Example 1.1. Let $\mathbf{C}^+$ be the closed right half plane, let $X \equiv L^p(\mathbf{C}^+, \mu)$ $(1 \leq p < \infty)$ and let

$$(Af)(z) \equiv -zf(z) \ \ (z \in \mathbf{C}^+, f \in X),$$

with maximal domain. Then the semigroup generated by A is given by

$$(e^{tA}f)(z) = e^{-tz}f(z) \ \ (z \in \mathbf{C}^+, f \in X);$$

1991 *Mathematics Subject Classification.* Primary 47D03, 47A11, Secondary 47B40.

Much of this work was done while the third author was visiting Ohio University and while the first author was visiting Nanjing University. The third author would like to thank the first two authors and Ohio University for their support and hospitality. The first author would like to thank the third author and Nanjing University for their support and hospitality.

thus we may characterize stability explicitly, as follows.

For any $f \in X$,

$$\lim_{t\to\infty} \|e^{tA}f\|^p = \int_{i\mathbf{R}} |f(z)|^p \, d\mu(z).$$

Thus

Proposition 1.2.

(1) *$\{e^{tA}\}_{t\geq 0}$ is stable if and only if $\mu(i\mathbf{R}) = 0$.*

(2) *$\sigma(A) \cap i\mathbf{R}$ is empty if and only if there exists an open set Ω containing $i\mathbf{R}$ such that*
$$\mu(\Omega) = 0.$$

For example, if μ is two-dimensional Lebesgue measure, then $\sigma(A) \cap i\mathbf{R} = i\mathbf{R}$, while $\mu(i\mathbf{R}) = 0$. Thus $\{e^{tA}\}_{t\geq 0}$ is stable but does not satisfy the hypotheses of Proposition 1.0.

The equivalence in Proposition 1.2 may be expressed in the language of local spectral subspaces. If A is as in Example 1.1, then for any closed $F \subseteq \mathbf{C}$, $X(A, F)$ is the set of all functions in X supported in F, and $X^*(A^*, F)$ is the set of all functions in X^* supported in F. Thus $X^*(A^*, i\mathbf{R})$ is trivial if and only if $X(A, i\mathbf{R})$ is trivial if and only if $\mu(i\mathbf{R}) = 0$.

Thus, for this very special case, we have the following characterizations of stability.

Proposition 1.3. *For A as in Example 1.1, the following are equivalent.*

(a) *$\{e^{tA}\}_{t\geq 0}$ is stable.*

(b) *$X(A, i\mathbf{R})$ is trivial.*

(c) *$X^*(A^*, i\mathbf{R})$ is trivial.*

For $\{e^{tA}\}_{t\geq 0}$ an arbitrary bounded strongly continuous linear semigroup generated by A on a Banach space X, we shall see that only one of the six implications, (c) implies (a), in Proposition 1.3, is true in general.

We show in this paper that, if $X^*(A^*, i\mathbf{R})$ is trivial, then $\{e^{tA}\}_{t\geq 0}$ is stable (Corollary 3.5). As indicated by Example 1.1, this is a significant strengthening of Proposition 1.0. When A^* is decomposable (see Definition 2.6), much more follows, and we obtain an equivalence (Theorem 3.7). We show that, when A^* is decomposable, then $X^*(A^*, i\mathbf{R})$ being trivial is equivalent to the existence of a dense subspace of X on which $\{e^{tA}\}_{t\geq 0}$ is exponentially decaying; this, in turn, is equivalent to there being no nontrivial complete trajectories for A^* of exponential type zero.

Interestingly, $X(A, i\mathbf{R})$ being trivial does *not* imply that $\{e^{tA}\}_{t\geq 0}$ is stable (see Example 3.6). Thus (b) implies (a) of Proposition 1.3 is not true in general.

Example 3.12 presents a stable semigroup $\{e^{tA}\}_{t\geq 0}$ with $\sigma(A) \subseteq i\mathbf{R}$. This means that (a) implies (b) and (a) implies (c) of Proposition 1.3 are false in general, since in Example 3.12 we have $X(A, i\mathbf{R}) = X$ and $X^*(A^*, i\mathbf{R}) = X^*$.

Propositions 3.1–3.4 give equivalences between trajectories and local spectral subspaces.

A generalization of Proposition 1.0 was obtained independently in [A-B] and [L-Vu], by completely different proofs (see also [Es-S-Z]), where we write $P\sigma$ for point spectrum:

Proposition 1.4. *Suppose A generates a bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$, $\sigma(A)\cap i\mathbf{R}$ is countable and $P\sigma(A^*)\cap i\mathbf{R}$ is empty. Then $\{e^{tA}\}_{t\geq 0}$ is stable.*

We will give a generalization of Proposition 1.4 in terms of local spectral subspaces for A^* in Section IV (Theorem 4.2), with a surprisingly short proof.

Note that, in Example 1.1, the hypotheses of Proposition 1.4 are satisfied if and only if (2) of Proposition 1.2 holds. Thus Proposition 1.4 is still far from an equivalence.

Sufficient conditions for stability of the bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$, in terms of the local spectrum of A, that also generalize Propositions 1.0 and 1.4, may be found in [B-N-R]. The proofs in [B-N-R] are completely different from ours.

In Section 5, when A^* has no eigenvalues in $(-\infty,0)$, we give a Hille-Yosida-type characterization of stable semigroups generated by A. In order that the bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ be stable, it is necessary and sufficient that

$$\{s^n\|(s+A^*)^{-n}x^*\| \mid s>0, n\in\mathbf{N}\}$$

be unbounded, for all nontrivial x^* in $\bigcap_{s>0,n\in\mathbf{N}} Im((s+A^*)^n)$.

II. PRELIMINARIES. Throughout this paper, all operators are linear, on a Banach space X. We will denote by $\mathcal{D}(A)$ the domain of the operator A, by $Im(A)$ the image of A, by $\rho(A)$ the resolvent set of A, by $\sigma(A)$ the spectrum of A, by $B(X)$ the space of all bounded operators from X to itself. See [C], [G] or [P] for basic material on strongly continuous semigroups; see [Er-W] or [Va] for basic material on local spectral theory and decomposable operators.

Definition 2.1. A *one-sided trajectory* is a mild solution of the (nonreversible) abstract Cauchy problem

$$\frac{d}{dt}u(t)=A(u(t))\ \ (t\geq 0),\quad u(0)=x.$$

This means that $u\in C([0,\infty),X)$, $\left(\int_0^t u(s)\,ds\right)\in\mathcal{D}(A)$, for all $t\geq 0$, with

$$u(t)=x+A\left(\int_0^t u(s)\,ds\right). \qquad (*)$$

A *complete trajectory* is a mild solution of the reversible abstract Cauchy problem

$$\frac{d}{dt}u(t) = A(u(t)) \ \ (t \in \mathbf{R}), \quad u(0) = x.$$

This means that $u \in C(\mathbf{R}, X)$, and satisfies (*) for all real t.

When A generates a strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$, it is straightforward to show that our definition of complete trajectory is equivalent to the definition given in [Vu2, (3.1)] and [Vu1].

Definition 2.2. Suppose $\omega \in \mathbf{R}$ ($\omega \geq 0$). The vector-valued function $\{w(t)\}_{t\geq 0}$ ($\{v(t)\}_{t\in\mathbf{R}}$) is *of exponential type* ω if, for any $\epsilon > 0$,

$$\lim_{t\to+\infty} e^{-(\omega+\epsilon)t}\|w(t)\| = 0 \quad \left(\lim_{t\to\pm\infty} e^{-(\omega+\epsilon)|t|}\|v(t)\| = 0\right).$$

Note that being of negative exponential type is equivalent to decaying exponentially.

Proposition 2.3 ([Vu2, Theorem 3.2]). *Suppose $\{e^{tA}\}_{t\geq 0}$ is a bounded strongly continuous semigroup generated by A. Then the following are equivalent.*

(a) *$\{e^{tA}\}_{t\geq 0}$ is stable.*

(b) *There exist no nontrivial complete bounded trajectories for A^*.*

Definition 2.4. If A is a closed operator on X, and $x \in X$, we write $\rho(x, A)$ for the set of those points $\lambda \in \mathbf{C}$ for which there exist an open neighborhood U of λ and a $\mathcal{D}(A)$-valued analytic function r on U such that

$$(\lambda - A)r(\lambda) = x \quad \text{for all } \lambda \in U. \tag{**}$$

$\rho(x, A)$ and its complement, denoted by $\sigma(x, A)$, are called the *local resolvent set* and the *local spectrum* of A at x, respectively.

Definition 2.5. The closed operator A has the *single-valued extension property* (SVEP) if, whenever Ω is an open subset of the complex plane, $f : \Omega \to \mathcal{D}(A)$ is analytic, and

$$(\lambda - A)f(\lambda) = 0, \quad \text{for all } \lambda \in \Omega,$$

then $f \equiv 0$.

Note that SVEP is equivalent to saying that the function r in Definition 2.4 is unique. The SVEP of A implies the existence of an analytic function

$$\lambda \mapsto R(\lambda, A, x)$$

that satisfies (**) of Definition 2.4 on $\rho(x, A)$.

Definitions 2.6. The closed operator A is said to have the *spectral decomposition property* (SDP) if for every open cover $\{G_i\}_{i=0}^{n}$ ($n \in \mathbf{N}$) of $\sigma(T)$ with G_i ($i =$

$1, 2, ..., n$) relatively compact and G_0 a neighborhood of ∞, there exists a system of invariant subspaces $\{X_i\}_{i=0}^n$ of T such that

(1) $X_i \subseteq \mathcal{D}(A)$, for $i = 1, 2, ..., n$;

(2) $\sigma(A|_{X_i}) \subseteq G_i$, for $i = 0, 1, ..., n$; and

(3) $X = \sum_{i=0}^n X_i$.

Y is called a *spectral maximal space* of A if for every Z which is invariant for A, the condition $\sigma(A|_Z) \subseteq \sigma(A|_Y)$ implies $Z \subseteq Y$.

The operator A is said to be *decomposable* if it has SDP, and the subspaces X_i in (1) may be chosen to be spectral maximal ([Er-W, Definition II.5.21]).

If $A \in B(X)$, then A has SDP if and only if A is decomposable ([Er-W, Corollary II.6.5]). In general, A is decomposable if and only if A has SDP and $X(A, \emptyset)$ is trivial ([Er-W, Theorem II.6.4]).

Definitions 2.7. Let Ω be an arbitrary subset of $\mathbf{C}$, A a closed operator. Define the *local spectral subspace corresponding to* Ω:

$$X(A, \Omega) \equiv \{x \mid \sigma(x, A) \subseteq \Omega\}.$$

We introduce the following:

$$\Lambda(A, \Omega) \equiv \{x \mid \sigma(x, A) \subseteq \Omega \text{ and is compact }\}.$$

For F a compact subset of $\mathbf{C}$, a closely related subspace, denoted $\Xi(A, F)$, is introduced in [Er-W, Theorem 4.34]. The following proposition will enable us to extend this definition to the case when $X(A, F)$ is not closed.

Proposition 2.8. *Assume A is closed and has the SVEP. If $\sigma(x, A)$ is compact, then there is a unique decomposition*

$$x = x_0 + x_1,$$

where $\sigma(x_0, A) = \sigma(x, A)$, $\sigma(x_1, A) = \emptyset$ and $R(z, A, x_0) \to 0$ as $z \to \infty$.

Proof: Define

$$x_0 \equiv \frac{1}{2\pi i} \int_{\Gamma_1} R(z, A, x)\, dz,$$

where Γ_1 is any contour surrounding $\sigma(x, A)$, and

$$x_1 \equiv x - x_0.$$

For $\mu \in \rho(x, A)$, define

$$R(\mu) \equiv \frac{1}{2\pi i} \int_{\Gamma_2} \frac{R(z, A, x)}{\mu - z}\, dz,$$

where Γ_2 is any contour surrounding $\sigma(x, A)$ whose winding number with respect to μ is zero. Then

$$\begin{aligned}(\mu - A)R(\mu) &= \frac{1}{2\pi i}\int_{\Gamma_2} \frac{((\mu - z) + (z - A))R(z, A, x)}{\mu - z}\, dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_2} R(z, A, x)\, dz + \frac{1}{2\pi i}\left(\int_{\Gamma_2} \frac{1}{\mu - z}\, dz\right) x = x_0.\end{aligned}$$

Thus $R(\mu) = R(\mu, A, x_0)$, for $\mu \in \rho(x, A)$. It is clear from the definition that $R(z, A, x_0) \to 0$ as $z \to \infty$.

Fix $\mu \in \rho(x, A)$. Let Γ_2 be as above, and choose a contour Γ_3 that surrounds μ, whose winding number with respect to any point in $\sigma(x, A)$ is zero. Then

$$\begin{aligned}R(\mu, A, x) - R(\mu, A, x_0) &= \frac{1}{2\pi i}\int_{\Gamma_3} \frac{R(z, A, x)}{z - \mu}\, dz - \frac{1}{2\pi i}\int_{\Gamma_2} \frac{R(z, A, x)}{\mu - z}\, dz \\ &= \frac{1}{2\pi i}\int_{\Gamma_3 + \Gamma_2} \frac{R(z, A, x)}{z - \mu}\, dz.\end{aligned}$$

Thus

$$R(\mu, A, x_1) = R(\mu, A, x) - R(\mu, A, x_0) \;\; (\mu \in \rho(x, A))$$

can be extended analytically to the entire complex plane, so that $\sigma(x_1, A) = \emptyset$.

All that remains to show is uniqueness. Assume that $x = x_0' + x_1'$ is another such decomposition. Then since $(x_0 - x_0') = (x_1 - x_1')$, we know that $R(\mu, A, (x_0 - x_0'))$ can be extended analytically to all of $\mathbf{C}$ and goes to 0 as $\mu \to \infty$. By Liouville's theorem,

$$R(\mu, A, (x_0 - x_0')) \equiv 0,$$

so that by SVEP $(x_0 - x_0') = 0$, as desired. □

Definition 2.9. When A has the SVEP, define, for Ω an arbitrary subset of $\mathbf{C}$, x_0 and x from Proposition 2.8,

$$\Xi(A, \Omega) \equiv \{x_0 \,|\, \sigma(x, A) \subseteq \Omega \text{ is compact }\}.$$

It is not hard to show that this definition is equivalent to that given in [Er-W, Theorem 4.34], when $X(A, \Omega)$ is closed.

Proposition 2.10. *If A is decomposable and $\Omega \subseteq \mathbf{C}$, then*

$$\Xi(A, \Omega) = \Lambda(A, \Omega).$$

Proof: Since A is decomposable, $x_1 = 0$ in the decomposition of Proposition 2.8. □

Lemma 2.11. *Suppose u is a nontrivial bounded complete trajectory for B. Then*

(1) *$\sigma(u(0), B)$ is a nonempty subset of $i\mathbf{R}$.*

(2) *If λ_0 is an isolated point of $\sigma(u(0), B)$, then λ_0 is an eigenvalue of B.*

Proof: Let $Sp(u)$ denote the spectrum of the function u, as in [Vu1]. Define

$$R_\lambda = \begin{cases} \int_0^\infty e^{-\lambda t} u(t)dt, & Re\lambda > 0, \\ -\int_{-\infty}^0 e^{-\lambda t} u(t)dt, & Re\lambda < 0. \end{cases}$$

By [d2, Lemma 2.10], $(\lambda - B)R_\lambda = u(0)$, when $Re(\lambda) \neq 0$. Let Λ be the regular set for the function u and denote the analytic continuation of R_λ to Λ by $\overline{R}_\lambda$. Then we still have

$$(\lambda - B)\overline{R}_\lambda = u(0), \quad \text{for all } \lambda \in \Lambda.$$

This implies that $\Lambda \subset \rho(u(0), B)$, but since $\overline{R}_\lambda$ is the maximal continuation of R_λ and $R(\lambda, B, u(0))$ is a continuation of R_λ, it follows $\overline{R}_\lambda = R(\lambda, B, u(0))$ and $\Lambda = \rho(u(0), B)$; hence $Sp(u(0)) = \sigma(u(0), B)$.

In particular, this implies assertion (1).

Assume that λ_0 is an isolated point of $\sigma(u(0), B)$. Then λ_0 is an isolated point of $Sp(u)$. Let $\mathcal{U}$ be a compact neighborhood of λ_0 which does not contain other elements of $Sp(u)$ and $\phi(t) \in L^1(\mathbf{R})$ be such that $supp(\hat{\phi}) \subset \mathcal{U}$. Put

$$y(t) = (u(0) * \phi)(t).$$

Then $y(t)$ is again a bounded, nontrivial, complete trajectory and $Sp(y) = \{\lambda_0\}$. It follows that there is $x_0 \in X$ such that $y(t) = e^{\lambda_0 t} x_0$ for all $t \in \mathbf{R}$. This implies $Bx_0 = \lambda_0 x_0$, giving us assertion (2). □

Lemma 2.12. *Suppose A has SDP and Ω is an open subset of the complex plane. Then*

$$[\Xi(A, \Omega)]^\perp = X^*(A^*, \Omega^c).$$

Proof: For $n \in \mathbf{N}$, let $\Omega_n \equiv \Omega \cap \{z \in \mathbf{C} \,|\, |z| < n\}$.

By [Er-W, Theorem 9.8(II)(ii)],

$$[\Xi(A, \Omega_n)]^\perp = X^*(A^*, \Omega_n^c),$$

for all $n \in \mathbf{N}$. Therefore,

$$\begin{aligned} [\Xi(A, \Omega)]^\perp &= [\Xi(A, \cup_n \Omega_n)]^\perp = \left[\bigcup_n \Xi(A, \Omega_n)\right]^\perp \\ &= \bigcap_n [\Xi(A, \Omega_n)]^\perp = \bigcap_n X^*(A^*, \Omega_n^c) = X^*(A^*, \cap_n \Omega_n^c) = X^*(A^*, \Omega^c). \end{aligned}$$

□

Similarly, we need to extend [Er-W, Theorem 9.8(I)(ii)].

Lemma 2.13. *Suppose A has SDP and Ω is an open subset of the complex plane. Then*

$$[X(A,\Omega)]^{\perp} = \text{ the weak}^* \text{ closure of } \Xi(A^*,\Omega^c).$$

Proof: For any $n \in N$, let $\Omega_n \equiv \Omega \bigcup \{z \in \mathbf{C} \,|\, |z| > n\}$. By [Er-W, Theorem 9.8(I)(ii)],

$$[X(A,\Omega_n)]^{\perp} = \Xi(A^*,\Omega_n^c),$$

for all $n \in \mathbf{N}$. Therefore, denoting, for any $W \subseteq X^*$, by $\overline{W}$ the weak* closure of W,

$$\begin{aligned}
[X(A,\Omega)]^{\perp} &= [X(A,\cap_n \Omega_n)]^{\perp} = \left[\bigcap_n X(A,\Omega_n)\right]^{\perp} \\
&= \overline{\left[\text{span}\left(\bigcup_n [X(A,\Omega_n)]^{\perp}\right)\right]} = \overline{\bigcup_n \Xi(A^*,\Omega_n^c)} = \overline{\Xi(A^*,\cup_n \Omega_n^c)} \\
&= \overline{\Xi(A^*,\Omega^c)}.
\end{aligned}$$

□

Definition 2.14. Suppose A has no eigenvalues in $(0,\infty)$. The following are from [d1, Chapter V].

The *Hille-Yosida space* for A, $Z_0(A)$, is the set of all $x \in X$ for which the abstract Cauchy problem has a bounded, uniformly continuous, one-sided trajectory. This trajectory is then unique; we will denote it by $t \mapsto u(t,x)$. The norm on $Z_0(A)$ is defined by

$$\|x\|_{Z_0(A)} \equiv \sup_{t\geq 0} \|u(t,x)\|.$$

The *weak Hille-Yosida space* for A, $Y(A)$, is (see [d2, Theorem 5.10]) the set of all $x \in \bigcap_{s>0,n\in\mathbf{N}} Im((s-A)^n)$ such that

$$\{\|s^n(s-A)^{-n}x\| \,|\, s>0, n\in\mathbf{N}\}$$

is bounded. The norm on $Y(A)$ is

$$\|x\|_{Y(A)} \equiv \sup\{\|s^n(s-A)^{-n}x\| \,|\, s>0, n\in\mathbf{N}\}.$$

Proposition 2.15 ([d2, Theorem 5.5]). *If A is as in Definition 2.14, then $Z_0(A)$ is the closure, in $Y(A)$, of $\mathcal{D}\left(A|_{Y(A)}\right)$.*

III. LOCAL ANALOGUES OF EMPTY PERIPHERAL SPECTRUM.

Propositions 3.1–3.4 contain relationships between local spectral subspaces and trajectories that may be of some independent interest. Corollaries 3.5 and 3.10 contain simple sufficient conditions for stability of the semigroup $\{e^{tA}\}_{t\geq 0}$. When A and A^* are decomposable, we obtain more than stability: exponential decay of $t \mapsto e^{tA}x$, for all x in a dense subspace (Theorem 3.7 and Corollary 3.14). Theorem 3.7 contains equivalences, between $X^*(A^*, i\mathbf{R})$ being trivial and the asymptotic behaviour of $\{e^{tA}\}_{t\geq 0}$ just alluded to.

Proposition 3.1. *Suppose $\omega \in \mathbf{R}$, and $X_\omega \equiv X(A, \{z \in \mathbf{C} \,|\, Re(z) \leq \omega\})$ is trivial. Then there exist no nontrivial one-sided trajectories of exponential type ω for A.*

Proof: Suppose, for the sake of contradiction, there exists a nontrivial one-sided trajectory, u, of exponential type ω. Then for any $s \geq 0$,

$$u(s) \in X_\omega.$$

This may be seen by constructing a local resolvent $z \mapsto R(u(s), z)$ in $Re(z) > \omega$,

$$R(u(s), z) \equiv \int_0^\infty e^{-zt} u(t+s)\, dt.$$

See [d2, Lemma 2.10] for the (short) proof that

$$(z - A)R(u(s), z) = u(s) \ \ (Re(z) > \omega).$$

Since u is nontrivial, there exists $s \geq 0$ such that $u(s)$ is nontrivial, contradicting our hypotheses. $\square$

To obtain an equivalence, we need an adjoint operator that is decomposable.

Proposition 3.2. *Suppose $\omega \in \mathbf{R}$, $\mathcal{D}(A)$ is dense and A^* is decomposable. Then the following are equivalent.*

(a) $X_\omega^* \equiv X^*(A^*, \{z \in \mathbf{C} \,|\, Re(z) \leq \omega\})$ *is trivial.*

(b) *There exist no nontrivial one-sided trajectories of exponential type ω for A^*.*

Proof: (a) implies (b) was just shown. To show that (b) implies (a), suppose (a) fails.

We will show that there exists compact

$$\Omega \subseteq \{z \in \mathbf{C} \,|\, Re(z) \leq \omega\}$$

such that $X^*(A^*, \Omega)$ is nontrivial.

Since A^* is decomposable, $\sigma(A^*|_{X_\omega^*}) \subseteq \{z \in \mathbf{C} \,|\, Re(z) \leq \omega\}$. If there is complex $z \in \sigma(A^*|_{X_\omega^*})$ such that $Re(z) < \omega$, then the decomposability of A^* quickly produces the desired Ω. So assume that

$$\sigma(A^*|_{X_\omega^*}) \subseteq \omega + i\mathbf{R}.$$

For $n \in \mathbf{N}$, denote

$$S_{\omega,n} \equiv \{z \in \mathbf{C} \,|\, |z - \omega| \geq n\}.$$

Then there exists $n_0 \in \mathbf{N}$ such that

$$X_\omega^* \not\subseteq X^*(A^*, S_{\omega,n_0}); \tag{*}$$

otherwise we would have

$$X_\omega^* \subseteq \bigcap_{n=1}^{\infty} X^*(A^*, S_{\omega,n}) = X^*\left(A^*, \cap_{n=1}^{\infty} S_{\omega,n}\right) = X^*(A^*, \phi) = \{\vec{0}\},$$

since A^* is decomposable. Denote now, for $k \in \mathbf{N}$,

$$G_{\omega,k} \equiv \{z \,|\, |Re(z) - \omega| \leq \frac{1}{k}, |Im(z)| \leq n_0 + 1\}.$$

From [Er-W, Theorem 6.2],

$$X_\omega^* \subseteq X^*(A^*, S_{\omega,n_0}) + X^*(A^*, G_{\omega,k}), \quad \text{for all } k \in \mathbf{N}.$$

This, together with (*), implies that

$$X^*(A^*, G_{\omega,k}) \neq \{\vec{0}\}, \quad \text{for all } k \in \mathbf{N}.$$

By the weak* compactness of the unit balls of $X^*(A^*, G_{\omega,k})$ ([Er-W, Theorem 9.3]), we may now conclude that

$$X^*(A^*, \omega + i[-(n_0+1), (n_0+1)]) = \bigcap_{k=1}^{\infty} X^*(A^*, G_{\omega,k}) \neq \{\vec{0}\},$$

so that we may choose $\Omega \equiv \omega + i[-(n_0+1), (n_0+1)]$.

Since Ω is compact and A^* is decomposable, it follows that $A^*|_{X^*(A^*,\Omega)}$ is bounded. Since $\Omega \subseteq \{z \in \mathbf{C} \,|\, Re(z) \leq \omega\}$, a short argument with the Riesz-Dunford functional calculus now shows that

$$\{e^{tA^*|_{X^*(A^*,\Omega)}}\}_{t\geq 0}$$

is of exponential type ω. Thus, for any $x^* \in X^*(A^*, \Omega)$, $\{e^{tA^*|_{X^*(A^*,\Omega)}} x^*\}_{t\geq 0}$ is a one-sided trajectory of exponential type ω, so that (b) fails. $\square$

Identically, we have the following.

Proposition 3.3. *Suppose $\omega \geq 0$, and $X(A, \{z \in \mathbf{C} \,|\, |Re(z)| \leq \omega\})$ is trivial. Then there exist no nontrivial complete trajectories of exponential type ω for A.*

Proposition 3.4. *Suppose $\omega \geq 0$, $\mathcal{D}(A)$ is dense and A^* is decomposable. Then the following are equivalent.*

(a) *$X^*(A^*, \{z \in \mathbf{C} \,|\, |Re(z)| \leq \omega\})$ is trivial.*

(b) *There exist no nontrivial complete trajectories of exponential type ω for A^*.*

The following is an immediate corollary of Propositions 3.3 and 2.3.

Corollary 3.5. *If A generates a bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ and $X^*(A^*, i\mathbf{R})$ is trivial, then $\{e^{tA}\}_{t\geq 0}$ is stable.*

Example 3.6. This example will show that Corollary 3.5 is false with A^* replaced by A.

Let X be the closed subspace of $L^2(i\mathbf{R})$

$$\{f \in L^2(i\mathbf{R}) \mid f \text{ extends analytically to the right half plane }\}.$$

Define

$$(Af)(z) \equiv -zf(z) \ \ (Re(z) \geq 0, f \in X),$$

with maximal domain, so that A generates the strongly continuous semigroup

$$(e^{tA}f)(z) \equiv e^{-tz}f(z) \ \ (Re(z) \geq 0, f \in X).$$

It is not hard to see that

(1) $\{e^{tA}\}_{t\geq 0}$ is not stable; in fact, for all $t \geq 0, e^{tA}$ is an isometry.

(2) $X(A, i\mathbf{R})$ is trivial; in fact, for any F properly contained in $\{z \in \mathbf{C} \mid Re(z) \geq 0\}$, $X(A, F)$ is trivial.

Assertions (1) and (2) show that we have the desired counterexample.

In the following, note that (d) implies (c) holds whenever A generates a strongly continuous semigroup.

Theorem 3.7. *Suppose A generates a bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ and A and A^* are decomposable. Then the following are equivalent.*

(a) $X^*(A^*, i\mathbf{R})$ *is trivial.*

(b) $\Lambda(A^*, i\mathbf{R})$ *is trivial.*

(c) *There exist no nontrivial complete trajectories of exponential type zero for A^*.*

(d) *There exists a dense subspace E, of X, such that, for all $x \in E$, $\{e^{tA}x\}_{t\geq 0}$ decays exponentially.*

(e) $\Lambda(A, \{z \in \mathbf{C} \mid Re(z) < 0\})$ *is dense.*

(f) $X(A, \{z \in \mathbf{C} \mid Re(z) < 0\})$ *is dense.*

Proof: The equivalence of (a) and (c) is Proposition 3.4, with $\omega = 0$. The proof of Proposition 3.2(b) implies (a) implies that (a) and (b) are equivalent.

(a) if and only if (e). Since A generates a bounded strongly continuous semigroup, $\sigma(A^*) = \sigma(A) \subseteq \{z \mid Re(z) \leq 0\}$, so that $X^*(A^*, \{z \mid Re(z) \geq 0\}) = X^*(A^*, i\mathbf{R})$. Thus, by Proposition 2.10 and Lemma 2.12,

$$\begin{aligned}[\Lambda(A, \{z \in \mathbf{C} \mid Re(z) < 0\})]^\perp &= [\Xi(A, \{z \in \mathbf{C} \mid Re(z) < 0\})]^\perp \\ &= X^*(A^*, \{z \mid Re(z) \geq 0\}) = X^*(A^*, i\mathbf{R}),\end{aligned}$$

and the equivalence follows.
The equivalence of (b) and (f) is the same as the equivalence of (a) and (e), with Lemma 2.12 replaced by Lemma 2.13.
(e) implies (d). For $x \in \Lambda(A, \{z \in \mathbf{C} \mid Re(z) < 0\})$, note that, since A has SDP, $A|_{X(A,\sigma(x,A))}$ is bounded. Since $\sigma(x, A) \subseteq \{z \in \mathbf{C} \mid Re(z) < 0\}$, $\{e^{tA|_{X(A,\sigma(x,A))}}\}_{t\geq 0}$, and hence

$$\{e^{tA}x\}_{t\geq 0} = \{e^{tA|_{X(A,\sigma(x,A))}}x\}_{t\geq 0}$$

decays exponentially.
(d) implies (c). Suppose v is a complete trajectory of exponential type zero for A^*. We must show that $v(0) = 0$.

Note that the restriction of A^* to $\overline{\mathcal{D}(A^*)}$ generates a strongly continuous semigroup, which for convenience we will write as e^{tA^*}. By applying $(\lambda - A^*)^{-1}$, for $\lambda \in \rho(A^*)$, if necessary, we may assume that $v(t) \in \mathcal{D}(A^*)$, for all $t \in \mathbf{R}$.

For $x \in E, t \geq 0$,

$$| < x, v(0) > | = | < x, e^{tA^*}v(-t) > | = | < e^{tA}x, v(-t) > | \leq \|e^{tA}x\|\|v(-t)\|.$$

Since $\{e^{tA}x\}_{t\geq 0}$ decays exponentially and v is of exponential type zero, the product

$$\|e^{tA}x\|\|v(-t)\|$$

converges to zero as $t \to +\infty$. Thus $< x, v(0) >= 0$, for all $x \in E$. Since E is dense, this implies that $v(0) = 0$, as desired. $\square$

Remark 3.8. Theorem 3.7(c) if and only if (d) is analogous to Proposition 2.3.

Remark 3.9. It is an immediate consequence of [B-N-R, Corollary 2.7] that, if A generates a bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ and satisfies (f) of Theorem 3.7, then $\{e^{tA}\}_{t\geq 0}$ is stable. When A and A^* are decomposable, Theorem 3.7(f) implies (d) is strengthening that conclusion.

When A^* is not decomposable, it is not clear what the relationships between (a), (b), (e) and (f) of Theorem 3.7 are.

The following is an immediate corollary of Theorem 3.7(b) implies (d).

Corollary 3.10. *Suppose A generates a bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$, $\Lambda(A^*, i\mathbf{R})$ is trivial, and A and A^* are decomposable. Then $\{e^{tA}\}_{t\geq 0}$ is stable.*

Note that Example 1.1 indicates to what extent Corollary 3.10 is stronger than Proposition 1.0.

Example 3.12. For A as in Example 1.1, the converses of Corollaries 3.5 and 3.10 are true (see Proposition 1.3). In general, the converses of Corollaries 3.5 and 3.10 are false, even when A or A^* is decomposable, as is shown by the following example, which may be found (for general g) in [d1, 2.15–2.17]; see also [B-N-R, Example 3.1].

Suppose $g : \mathbf{R} \to [0, \infty)$ is continuous and nondecreasing. Let $X \equiv L^p(\mathbf{R}, gdx)$ $(1 \leq p < \infty)$, and let A be $\frac{d}{dx}$, the generator of left translation

$$(e^{tA}f)(x) \equiv f(x+t) \ (x \in \mathbf{R}, t \in \mathbf{R}).$$

$\{e^{tA}\}_{t \geq 0}$ is a semigroup of contractions.

It's not hard to show that e^{tA} is stable if and only if $g(-\infty) = 0$.

For $\alpha > 0$, define

$$g_\alpha(x) \equiv (1-x)^{-\alpha} 1_{(-\infty,0]}(x) + 1_{[0,\infty)}(x).$$

Then

$$\|e^{tA}\|^p = \sup_{x \in \mathbf{R}} \frac{g(x-t)}{g(x)} = (1+|t|)^\alpha \quad \text{for all } t \leq 0.$$

This implies that $\sigma(A) = \sigma(A^*) \subseteq i\mathbf{R}$, so that $X(A, i\mathbf{R}) = X$ and $X^*(A^*, i\mathbf{R}) = X^*$, and A has a $BC^n(i\mathbf{R})$ functional calculus, for n any integer greater than $2 + \frac{\alpha}{p}$; hence both A and A^* are decomposable.

But $\{e^{tA}\}_{t \geq 0}$ is stable; thus we have a counterexample to the converse of Corollary 3.5.

Since A^* is decomposable, $\Lambda(A^*, i\mathbf{R})$ is dense in X^*. Thus we also have a counterexample to Corollary 3.10.

In this example, e^{tA} is more than stable; it has the property that, for $k < \frac{\alpha}{p}$,

$$\lim_{t \to \infty} (1+t)^k \|e^{tA}\| = 0.$$

(Compare with Theorem 3.7(d)). This may be seen by a quick calculation, as follows.

For $t \geq 0$, $\epsilon > 0$, f of compact support,

$$\begin{aligned}(1+t)^{\alpha-\epsilon}\|e^{tA}f\|^p &= (1+t)^{\alpha-\epsilon} \int_{\mathbf{R}} |f(y)|^p g(y-t)\, dy \\ &= (1+t)^{\alpha-\epsilon} \left[\int_{-\infty}^{\frac{t}{2}} |f(y)|^p (1+t-y)^{-\alpha}\, dy + \int_{\frac{t}{2}}^{\infty} |f(y)|^p\, dy \right];\end{aligned}$$

the second term is zero for t sufficiently large, since f is of compact support, while the first term converges to zero, as $t \to \infty$, by dominated convergence.

This example is particularly surprising if we choose $p = 2$ and $\alpha > 0$ arbitrarily small. We then have, for arbitrary $\epsilon > 0$, a strongly continuous group $\{e^{tA}\}_{t \in \mathbf{R}}$ on a Hilbert space such that

$$\|e^{tA}\| \leq (1+|t|)^\epsilon \quad \text{for all } t \in \mathbf{R},$$

$\{e^{tA}\}_{t \geq 0}$ is a strongly continuous stable semigroup of contractions, while $X^*(A^*, i\mathbf{R}) = X^*$ and $X(A, i\mathbf{R}) = X$. This is surprising because, if ϵ were zero, iA would be self-adjoint and Proposition 1.3 would hold.

Example 3.13. Example 1.1, with μ chosen to be two-dimensional Lebesgue measure, is an example of an operator A satisfying (a)–(f) of Theorem 3.7 such that $i\mathbf{R} \subseteq \sigma(A)$. (Compare with Propositions 1.0 and 1.4.) Here is a very similar, but simpler, example.

Let X be the space of all continuous functions on the right half-plane

$$\mathbf{C}^+ \equiv \{z \in \mathbf{C} \,|\, Re(z) \geq 0\}$$

satisfying $\lim_{|z|\to\infty} f(z) = 0$. Then

$$\left[e^{tA} f\right](z) \equiv e^{-zt} f(z) \ \ (t \geq 0, z \in \mathbf{C}^+)$$

defines a bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ generated by $(Af)(z) \equiv -zf(z)$, with maximal domain.

Compare the following immediate corollary of Theorem 3.7 with Proposition 1.0.

Corollary 3.14. *Suppose A generates a bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$, A and A^* are decomposable and $\sigma(A) \cap i\mathbf{R}$ is empty. Then there exists a dense subspace E, of X, such that, for all $x \in E$, $\{e^{tA}x\}_{t\geq 0}$ decays exponentially.*

IV. LOCAL ANALOGUES OF COUNTABLE PERIPHERAL SPECTRUM. In this section, we will present a local analogue of Proposition 1.4, involving, as in Section III, $X^*(A^*, i\mathbf{R})$. Proposition 4.1 is a countable analogue of Proposition 3.3.

Proposition 4.1. *If*

(1) *for all $x \in X(B, i\mathbf{R})$, $\sigma(x, B)$ is countable, and*

(2) *$P\sigma(B) \cap i\mathbf{R}$ is empty,*

then

(3) *there exist no nontrivial complete bounded trajectories for B.*

Proof: Suppose (1) holds and (3) fails. We will show that (2) must then fail.

Let u be a nontrivial bounded complete trajectory for B. By translating u if necessary, we may assume $u(0) \neq \vec{0}$. Since, by Proposition 3.3, $\sigma(u(0), B) \subseteq i\mathbf{R}$, we have

$$u(0) \in X(B, i\mathbf{R});$$

thus, by (1) and Lemma 2.11(1), $\sigma(u(0), B)$ is a nonempty, closed, countable subset of $i\mathbf{R}$. This implies that $\sigma(u(0), B)$ has an isolated point $\lambda_0 \in i\mathbf{R}$, so that again by Lemma 2.11, $\lambda_0 \in P\sigma(B)$, so that (2) fails, as desired. $\square$

Theorem 4.2. *Suppose A generates a bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$, $P\sigma(A^*)\cap i\mathbf{R}$ is empty and for all $x^* \in X^*(A^*, i\mathbf{R}), \sigma(x^*, A^*)$ is countable. Then $\{e^{tA}\}_{t\geq 0}$ is stable.*

Proof: Apply Proposition 4.1 to $B \equiv A^*$, and use Proposition 2.3. □

Proposition 1.4 follows quickly.

Proof of Proposition 1.4: It is clear from the definition of local spectrum that $\sigma(x^*, A^*) \subseteq \sigma(A^*)$, for any $x^* \in X^*$; in particular, if $x^* \in X^*(A^*, i\mathbf{R})$, then

$$\sigma(x^*, A^*) \subseteq i\mathbf{R} \cap \sigma(A^*) = i\mathbf{R} \cap \sigma(A),$$

which, by hypothesis, is countable. Thus we may apply Theorem 4.2. □

Remark 4.3. In [B-N-R, Corollary 2.5], it is shown that, if $\{e^{tA}\}_{t\geq 0}$ is bounded, $P\sigma(A^*)\cap i\mathbf{R}$ is empty, and there exists a dense set Z such that $\sigma(A, x)\cap i\mathbf{R}$ is countable, for all $x \in Z$, then e^{tA} is stable. Even for A or A^* decomposable, it is not clear what the relationship between this result and Theorem 4.2 is.

V. SPECTRAL EQUIVALENCES FOR STABILITY. When A^* has no real negative eigenvalues, we obtain a characterization of generators of stable strongly continuous semigroups that is very similar to the Hille-Yosida theorem.

Note that a stable strongly continuous semigroup is automatically bounded, so there is no loss of generality, in characterizing stable semigroups, in assuming that A generates a bounded strongly continuous semigroup.

Theorem 5.1. *Suppose A generates a bounded strongly continuous semigroup $\{e^{tA}\}_{t\geq 0}$ and A^* has no eigenvalues in $(-\infty, 0)$. Then the following are equivalent.*

(a) *$\{e^{tA}\}_{t\geq 0}$ is stable.*

(b) *For all nontrivial x^* in $\bigcap_{s>0, n\in\mathbf{N}} Im((s + A^*)^n)$,*

$$\{s^n \|(s + A^*)^{-n} x^*\| \mid s > 0, n \in \mathbf{N}\}$$

is unbounded.

Proof: By Proposition 2.3, (a) is equivalent to there being no nontrivial complete bounded trajectories for A^*. Since $A^*|_{\overline{\mathcal{D}(A^*)}}$ generates a bounded strongly continuous semigroup, this is equivalent to there being no nontrivial one-sided bounded trajectories for $-A^*$. Since $t \mapsto (1 - A)^{-1}(u(t))$ is a bounded uniformly continuous one-sided trajectory whenever u is a bounded trajectory, this is equivalent to $Z_0(-A^*)$ (see Definition 2.14) being trivial. Propositition 2.15 now implies that this is equivalent to (b). □

We cannot resist emphasizing the similarity of our characterization to the Hille-Yosida theorem, by restating Theorem 5.1 entirely in terms of spectrum. Note that, in the following, (1) by itself is equivalent to generating a bounded strongly continuous semigroup (this is the Hille-Yosida theorem), while the addition of (2) gives us stability.

Alternative Statement of Theorem 5.1. *Suppose $\mathcal{D}(A)$ is dense and A^* has no eigenvalues in $(-\infty, 0)$. Then A generates a stable strongly continuous semigroup if and only if*

(1) $(0,\infty) \subseteq \rho(A)$ *and*

$$\{s^n \|(s-A)^{-n}\| \mid s > 0, n \in \mathbf{N}\}$$

is bounded; and

(2) *for all nontrivial x^* in $\bigcap_{s>0, n\in\mathbf{N}} Im((s+A^*)^n)$,*

$$\{s^n \|(s+A^*)^{-n} x^*\| \mid s > 0, n \in \mathbf{N}\}$$

is unbounded.

REFERENCES

[A-B] W. Arendt and C. J. K. Batty, *Tauberian theorems and stability of one-parameter semigroups,* Trans. Amer. Math. Soc. 306 (1988), 837–852.

[B-N-R] C. J. K. Batty, J. van Neerven and F. Räbiger, *Local spectra and individual stability of uniformly bounded C_0-semigroups,* Trans. Amer. Math. Soc., to appear.

[C] J. A. van Casteren, "Generators of Strongly Continuous Semigroups," Research Notes in Mathematics 115, Pitman, Boston, 1985.

[d1] R. deLaubenfels, *Incomplete iterated Cauchy problems,* J. Math. Anal. and Appl. 168 (1992), 552–579.

[d2] R. deLaubenfels, "Existence Families, Functional Calculi and Evolution Equations," Springer-Verlag Lecture Notes in Mathematics, Vol. 1570 (1994).

[Er-W] I. Erdelyi and S. Wang, "A Local Spectral Theory for Closed Operators," London Math. Soc. Lecture Notes Series 105, 1985.

[Es-S-Z] J. Esterle, E. Strouse and F. Zouakia, *Stabilité asymptotique de certains semi-groupes d'opérateurs,* J. Operator Theory 28 (1992), 203–227.

[G] J. A. Goldstein, "Semigroups of Linear Operators and Applications," Oxford, New York, 1985.

[H1] Falun Huang, *Asymptotic stability theory for linear dynamical systems in Banach spaces,* Kexue Tongbao 10 (1983), 584–586 [in Chinese].

[H2] Falun Huang, *Strong asymptotic stability of linear dynamical systems in Banach spaces,* J. Diff. Equations 104 (1993), 307–324.

[L-Mat] Yu. I. Lyubich and V. I. Matsaev, *On operators with separable spectrum,* Math. USSR-Sb. 56 (1962), 433–468.

[L-Vu] Yu. I. Lyubich and Vũ Quôc Phóng, *Asymptotic stability of linear differential equations in Banach spaces,* Studia Math. 88 (1988), 37–42.

[P] A. Pazy, "Semigroups of Linear Operators and Applications to Partial Differential Equations," Springer, New York, 1983.

[Va] F. H. Vasilescu, "Analytic Functional Calculus and Spectral Decompositions," D. Reidel, Bucharest-Dordrecht, 1982.

[Vu1] Vũ Quôc Phóng, *On the spectrum, complete trajectories and asymptotic stability of linear semi-dynamical systems,* J. Diff. Eqn. 105 (1993), 30–45.

[Vu2] Vũ Quôc Phóng, *Almost periodic and strongly stable semigroups of operators,* in "Linear Operators," Banach Center Publications, Vol. 38 (1997), 401–426.

Scientia Research Institute, P. O. Box 988, Athens, Ohio 45701, USA, Mathematics Department, Ohio University, Athens, Ohio 45701, USA, and Mathematics Department, Nanjing University, Nanjing, Jiangsu 210008, P. R. CHINA

E-mail: 72260.2403@compuserve.com, qvu@oucsace.cs.ohiou.edu and wang2598@netra.nju.edu.cn