Long time asymptotics of solutions to the anharmonic oscillator model from nonlinear optics

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Abstract:
The anharmonic oscillator model desribing the propagation of electromagnetic waves in an exterior domain containing a nonlinear dielectric medium is investigated. Local decay of the electromagnetic field for $t \to \infty$ in the charge free case is shown. If the potential of the oscillating particles is in addition convex it is shown that for arbitrary initial states the electromagnetic field converges for $t \to \infty$ to an electrostatic field determined by the initial data and the prescribed external current.
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1 Introduction

The subject of this paper is the anharmonic oscillator model from nonliear optics consisting of Maxwell's equations

$$\partial_t \mathbf{E} = \text{ curl } \mathbf{H} - \partial_t \tilde{\mathbf{P}} - \mathbf{j}, \quad \partial_t \mathbf{H} = - \text{ curl } \mathbf{E}, \tag{1.1}$$

on $I\!R^+ \times \Omega$ coupled with the equation

$$\alpha \partial_t^2 \mathbf{P} + \partial_t \mathbf{P} + \nabla_P V(x, \mathbf{P}) = \gamma \mathbf{E} \tag{1.2}$$

on $I\!R^+ \times G$. The initial boundary conditions

$$\vec{n} \wedge \mathbf{E} = 0 \text{ on } (0, \infty) \times \Gamma_1 \text{ and } \vec{n} \wedge \mathbf{H} = 0 \text{ on } (0, \infty) \times \Gamma_2 \tag{1.3}$$

$$\mathbf{E}(0, x) = \mathbf{E}_0(x), \mathbf{H}(0, x) = \mathbf{H}_0(x). \tag{1.4}$$

and

$$\mathbf{P}(0, x) = \mathbf{P}_0(x), \quad \partial_t \mathbf{P}(0, x) = \mathbf{P}_1(x) \text{ on } G. \tag{1.5}$$

are imposed. This system describes the propagation of electromagnetic waves in a dielectric medium occupying the set G, see [3], [14]. Here $\Omega \subset I\!R^3$ is an exterior domain, $G \subset \Omega$ a certain subset and $\Gamma_1 \subset \partial\Omega$, $\Gamma_2 \stackrel{\text{def}}{=} \partial\Omega \setminus \Gamma_1$. The unknown functions are the electric and magnetic field $\mathbf{E}, \mathbf{H}$, which depend on the time $t \geq 0$ and the space-variable $x \in \Omega$, and the dielectic polarization $\mathbf{P}$ defined on $I\!R^+ \times G$. In 1.1 the function $\tilde{\mathbf{P}}$ is the extension of $\mathbf{P}$ on $I\!R \times \Omega$ defined by zero on the set $I\!R^+ \times (\Omega \setminus G)$. The physical meaning of the boundary condition 1.3 is that Γ_1 is perfecty conducting, such that the tangential component of the electric field must vanish.

$\alpha, \gamma \in L^\infty(G)$ are coefficients depending on the space variables which take into account the possibly variable mass , electrical charge and density of the oscillating charged particles. An external current $\mathbf{j} \in L^1((0,\infty), L^2(\Omega))$ is included also. The potential energy function $V : G \times I\!R^3 \to [0,\infty)$ causes a spring force $\nabla_P(x, \mathbf{P})$, which may depend nonlinearily on $\mathbf{P}$. It is assumed that

$$0 \leq V(x,y) \leq Ky\nabla_P V(x,y) \text{ for all } x \in G, y \in I\!R^3 \tag{1.6}$$

with some constant $K > 0$.
In particular it is allowed that $|(\nabla_P V)(x,y)|$ tends to zero for $|y| \to \infty$ as in [14]. The linear case $(\nabla_P V)(x,y) = ay$ with some $a > 0$ is included also.

In [14], where $G = \Omega = I\!R^3$ and the coefficients and the potential do not depend on x, it is shown that 1.1, 1.2 admits a unique strong strong solution in $C([0,\infty), H^s(I\!R^3))$ for $s \geq 2$. Note that in our case system 1.1 does not admit classical solutions on all of $(0,\infty) \times \Omega$ due to the discontinuity of $\tilde{\mathbf{P}}$ on $\Sigma \stackrel{\text{def}}{=} (\partial G) \cap \Omega$, the inteface between the polarizable medium and the vacuum-region $\Omega \setminus G$. But if the solution is smooth on $(0,\infty) \times G$ and on $(0,\infty) \times (\Omega \setminus G)$ then 1.1 involves a transmission condition, which requires the continuity of the tangential components of $\mathbf{E}$ and $\mathbf{H}$, as well as a linking condition for the normal components of $\mathbf{D} = \mathbf{E} + \tilde{\mathbf{P}}$ and $\mathbf{H}$ on Σ. Therefore a suitable weak formulation of 1.1, 1.2 will be given in section 2, which admits discontinuous solutions similar as in the theory of shock waves. In [4] the Landau-Lifschitz equation for the magetic moment coupled with Maxwell's equations is handled analoguously. The magnetic moment is located in a bounded domain, whereas Maxwell's equations are posed on whole space in [4].

The main topic of this paper in the investigation of the long time asymptotic behaviour of the solutions. For this purpose it is assumed that

$$\gamma \in L^{3/2}(G) \text{ and } (1+|x|)\gamma \in L^{r_0}(G) \text{ with some } r_0 \in (3/2,\infty).$$

Since $\gamma \in L^\infty(G)$, this assumption is fulfilled for examlpe if $\int_G (1+|x|)^{r_0} dx < \infty$, in particular if G is bounded. Another possibilty is that G has finite measure and $\|\gamma\|_{L^\infty(\{|x|>R\})} \leq C_1 R^{-a}$ for all $R > 0$ with some $a > 1$ and $C_1 > 0$.

Let X_0 denote the set of all $(\mathbf{F}, \mathbf{G}) \in X \stackrel{\text{def}}{=} L^2(\Omega, I\!R^6)$, which satisfy

$$\text{curl } \mathbf{f} = \text{ curl } \mathbf{g} = 0 \text{ on } \Omega, \quad \vec{n} \wedge \mathbf{f} = 0 \text{ on } \Gamma_1, \vec{n} \wedge \mathbf{g} = 0 \text{ on } \Gamma_2.$$

The basic goal is to prove the decay-property

$$\int_{\{x\in\Omega:|x|\le\alpha t\}} |\mathbf{E}|^2+|\mathbf{H}|^2 dx \stackrel{t\to\infty}{\longrightarrow} 0 \text{ for all } \alpha<1, \tag{1.7}$$

in particular local energy decay, provided that the initial data satisfy

$$\int_\Omega (\mathbf{D}_1\mathbf{f}+\mathbf{H}_0\mathbf{g})dx = 0 \text{ for all } (\mathbf{f},\mathbf{g})\in X_0. \tag{1.8}$$

Here $\mathbf{D}_1 \stackrel{\text{def}}{=} \mathbf{E}_0+\tilde{\mathbf{P}}_0-\int_0^\infty \mathbf{j}(s)ds$, where $\tilde{\mathbf{P}}_0$ denotes the extension of $\mathbf{P}_0$ by zero on $\Omega\setminus G$. (Note that the propagation speed of electromagnetic waves in vacuum is normalized to 1 in 1.1.) The physical meaning of 1.7 is that the wave-packet $(\mathbf{E}(t),\mathbf{H}(t))$ is concentated near the sphere $|x|=t$ for large times

Condition 1.8 includes

$$\text{div } \mathbf{D}_1 = 0 \text{ and } \text{div } \mathbf{H}_0 = 0 \text{ on } \Omega$$

and the bondary-conditions

$$\vec{n}\mathbf{D}_1 = 0 \text{ on } \Gamma_2 \text{ and } \vec{n}\mathbf{H}_0 = 0 \text{ on } \Gamma_1.$$

By 1.1 the function $\mathbf{D} \stackrel{\text{def}}{=} \mathbf{E}+\tilde{\mathbf{P}}$ and $\mathbf{H}$ obey $\text{div } \mathbf{H}(t) = \text{div } \mathbf{H}_0 = 0$ and

$$\text{div } \mathbf{D}(t) = \text{div } \left[\mathbf{E}_0+\tilde{\mathbf{P}}_0-\int_0^t \mathbf{j}(s)ds\right] \stackrel{t\to\infty}{\longrightarrow} \text{div } \mathbf{D}_1 = 0 \text{ in } \mathcal{D}'(\Omega)$$

if condition 1.8 is fulfilled. Physically this means that the space charge $\rho \stackrel{\text{def}}{=} \text{div } \mathbf{D}$ determined by the initial-state $(\mathbf{E}_0,\mathbf{H}_0)$ and the prescribed current $\mathbf{j}$ vanishes as $t\to\infty$.

Invoking a result in [12] it can be shown that the solution $(\mathbf{E},\mathbf{H})$ of 1.1-1.5 is asymptotically free in the sense that there exists a unique
$(\mathbf{F}_0,\mathbf{G}_0)\in L^2(\mathbb{R}^3)$ with $\text{div } \mathbf{F}_0 = \text{div } \mathbf{G}_0 = 0$, such that

$$||(\mathbf{E}(t),\mathbf{H}(t))-(\mathbf{F}(t),\mathbf{G}(t))||_{L^2(\Omega)} \stackrel{t\to\infty}{\longrightarrow} 0 \tag{1.9}$$

Here $(\mathbf{F},\mathbf{G})\in C(\mathbb{R},L^2(\mathbb{R}^3,\mathbb{C}^6))$ denotes the solution to Maxwell's equations in the whole space, that is

$$\partial_t\mathbf{F} = \text{curl } \mathbf{G}, \quad \partial_t\mathbf{G} = -\text{ curl } \mathbf{F}, \tag{1.10}$$

supplemented by the initial-condition

$$\mathbf{F}(0,x)=\mathbf{F}_0(x), \mathbf{G}(0,x)=\mathbf{G}_0(x). \tag{1.11}$$

This means that the solution to 1.1-1.4 behaves asymptotically like a free space solution to equations 1.10, 1.11 as $t\to\infty$.

The proof of the decay-property 1.8 uses a result in [11]. In particular it is shown that the weak ω-limit set of $(\mathbf{E},\mathbf{H})$ is contained in X_0.

In [11] weak decay of solutions of certain semiliear hyperbolic systems including Maxwell's equations is shown using a suitable modification of the method for wave-equations in bounded domains in [5], where the ω-limit set of the trajectories is investigated, see also [2], [7].

The asymptotic behaviour of solutions to the Landau-Lifschitz equation for the magetic moment coupled with Maxwell's equations has been investigated in [4], where it is shown that all points of the ω-limit set are solutions of the corresponding stationary equations.

In the last section the asymptotic behavior for arbitrary initial-states is investigated under the additional assumption that V is convex and that 1.6 holds for $q = 2$. First it is shown that there exist some $\mathbf{P}_\infty \in L^2(G)$ and uniquely determined $(\mathbf{E}_\infty, \mathbf{H}_\infty) \in X_0$ with

$$\nabla_{\mathbf{P}}(x, \mathbf{P}_\infty) = \gamma \mathbf{E}_\infty \text{ on } G, \tag{1.12}$$

$$\int_\Omega \Big([\mathbf{D}_1 - \tilde{\mathbf{P}}_\infty - \mathbf{E}_\infty]\mathbf{f} + [\mathbf{H}_0 - \mathbf{H}_\infty]\mathbf{g} \Big)\, dx = 0 \text{ for all } (\mathbf{f}, \mathbf{g}) \in X_0. \tag{1.13}$$

Then it is shown that $(\mathbf{E}, \mathbf{H})$ converges to $(\mathbf{E}_\infty, \mathbf{H}_\infty)$ as $t \to \infty$ in the sense that

$$\int_{\{x \in \Omega : |x| \le \alpha t\}} |\mathbf{E}(t) - \mathbf{E}_\infty|^2 + |\mathbf{H}(t) - \mathbf{H}_\infty|^2 dx \stackrel{t\to\infty}{\longrightarrow} 0 \text{ for all } \alpha < 1, \tag{1.14}$$

in analogy to 1.7. A consequence of this result is that local energy decay of the electromagnetic field holds if and only if condition 1.8 is satisfied.

At least in the case where Ω is simply connected one has $X_0 = \{(\nabla\varphi, \nabla\psi) : \varphi \in Z_1, \psi \in Z_2\}$, compare Lemma 4 in [9]. Here Z_1 denotes the space of all $\varphi \in H^1_{loc}(\overline{\Omega}) \cap L^6(\Omega)$ with $\nabla\varphi \in L^2(\Omega)$, which are constant one each connected component $\Gamma_{1,1}, .., \Gamma_{1,M}$ of Γ_1. The space Z_2 is defined analogouosly. With $\mathbf{E}_\infty = -\nabla\varphi$ and $\mathbf{H}_\infty = -\nabla\psi$ equation 1.13 is the weak formulation of the mixed elliptic boundary value problem

$$\Delta\varphi = \text{ div } \tilde{\mathbf{P}}_\infty - \text{ div } \mathbf{D}_1, \quad \Delta\psi = - \text{ div } \mathbf{H}_0 \text{ on } \Omega$$

supplemented by the bondary-conditions

$$\varphi \text{ constant on } \Gamma_{1,k} \text{ , } \psi \text{ constant on } \Gamma_{2,k},$$

$$\vec{n}[\nabla\varphi - \tilde{\mathbf{P}}_\infty + \mathbf{D}_1] = 0 \text{ on } \Gamma_{2,k}, \quad \vec{n}[\nabla\psi + \mathbf{H}_0] = 0 \text{ on } \Gamma_{1,k},$$

$$\int_{\Gamma_{1,k}} \vec{n}[\nabla\varphi - \tilde{\mathbf{P}}_\infty + \mathbf{D}_1] dS = 0 \text{ and } \int_{\Gamma_{2,k}} \vec{n}[\nabla\psi + \mathbf{H}_0] dS = 0$$

for all $k \in \{1, .., N\}$.

2 Notation, assumptions and preliminaries

For an arbitrary open set $K \subset \mathbb{R}^3$ the space of all infinitely differentiable functions with compact support contained in K is denoted by $C_0^\infty(K)$. For $p \in [1, \infty)$ the dual exponent p^* is given by $p^{-1} + (p^*)^{-1} = 1$. $H^1(K)$ is the usual first order Sobolev space consisting of all functions in $L^2(K)$ whose distributional gradient belongs to $L^2(K)$.
$H_{curl}(K)$ is defined as the space of all $\mathbf{E} \in L^2(K, \mathbb{C}^3)$ with $\operatorname{curl} \mathbf{E} \in L^2(K)$ endowed with the norm

$$||\mathbf{E}||^2_{H_{curl}(K)} \stackrel{\text{def}}{=} ||\mathbf{E}||^2_{L^2(K)} + || \operatorname{curl} \mathbf{E}||^2_{L^2(K)}.$$

Let $\Omega \subset \mathbb{R}^3$ be a (connected) domain with bounded complement, such that $\mathbb{R}^3 \setminus \overline{\Omega}$ is a Lipschitz-domain and $G \subset \Omega$ a measurable set with nonempty interior. Througout this paper the following assumptions are imposed on $V : G \times \mathbb{R}^3 \to [0, \infty)$.

$$V(x, \cdot) \in C^2(\mathbb{R}^3, \mathbb{R}), \quad V(x, 0) = 0 \text{ and } (\nabla_P V)(x, 0) = 0 \tag{2.15}$$

for all fixed $x \in G$. It is assumed that $(\nabla_P V)$ Lipschitz-continuous with respect to y, i.e. there exits some $L_0 \in (0, \infty)$, such that

$$|(\nabla_P V)(x, y) - (\nabla_P V)(x, z)| \leq L_0 |y - z| \text{ for all } x \in G, y, z \in \mathbb{R}^3. \tag{2.16}$$

This condition is also required in [14], since the second- and third order derivatives of V are assumed to be globally bounded there.

Next, let $\alpha \in L^\infty(G)$ be a uniformly positive and $\gamma \in L^\infty(G)$ be a positive, but not necessarily uniformly positve function on G. Now $\mathcal{G} \subset L^2(G)$ is the weighted $L^2(G)$-space consisting of all measurable functions $\mathbf{f} : G \to \mathbb{R}^3$ with $\int_G \gamma^{-1}(x)|\mathbf{f}(x)|^2 dx < \infty$ endowed with the norm

$$||\mathbf{f}||^2_{\mathcal{G}} \stackrel{\text{def}}{=} \int_G \gamma^{-1}(x)|\mathbf{f}(x)|^2 dx.$$

In the sequel $\mathbf{w} \in \mathbb{C}^6$ we denote by $\underline{\mathbf{w}}_1 \in \mathbb{C}^3$ the first three and by $\underline{\mathbf{w}}_2 \in \mathbb{C}^3$ the last three components of a vector $\mathbf{w} \in \mathbb{C}^6$ and $S\mathbf{w} \stackrel{\text{def}}{=} (-x \wedge \underline{\mathbf{w}}_2, x \wedge \underline{\mathbf{w}}_1)$.

Next, some function-spaces related to Maxwell's equations with mixed boundary-conditions are introduced.

W_H denotes the closure of $C_0^\infty(\mathbb{R}^3 \setminus \overline{\Gamma_2}, \mathbb{C}^3)$ in $H_{curl}(\Omega)$, where $H_{curl}(\Omega)$, is the space of all $\mathbf{E} \in L^2(\Omega, \mathbb{C}^3)$ with $\operatorname{curl} \mathbf{E} \in L^2(\Omega)$.
W_E denotes the set of all $\mathbf{E} \in H_{curl}(\Omega)$, such that

$$\int_\Omega \mathbf{E} \operatorname{curl} \mathbf{F} - \mathbf{F} \operatorname{curl} \mathbf{E} dx = 0 \text{ for all } \mathbf{F} \in W_H,$$

which includes a weak formulation of the boundary-condition $\vec{n} \wedge \mathbf{E} = 0$ on Γ_1, see [8].

Now, the following operators are defined.
Let $D(B) \stackrel{\text{def}}{=} W_E \times W_H$ and

$$B(\mathbf{E}, \mathbf{H}) \stackrel{\text{def}}{=} (\operatorname{curl} \mathbf{H}, -\operatorname{curl} \mathbf{E}) \text{ for } (\mathbf{E}, \mathbf{H}) \in D(B).$$

It turns out that B is a densely defined skew self-adjoint operator in the Hilbert-space $X \stackrel{\text{def}}{=} L^2(\Omega, \mathbb{C}^6)$ endowed with the usual scalar-product .

Let P be the orthogonal projector on $(\ker B)^\perp = \overline{\text{ran } B}$.

For $\mathbf{w} \in X = L^2(\Omega, \mathbb{R}^3)$ and $\mathbf{f} \in L^1_{loc}([0,\infty), X)$ a function $\mathbf{u} \in C([0,\infty), X)$ is called a weak soution to the initial boundary value problem

$$\partial_t \underline{\mathbf{u}}_1 = \text{ curl } \underline{\mathbf{u}}_2 + \underline{\mathbf{f}}_1, \quad \partial_t \underline{\mathbf{u}}_2 = - \text{ curl } \underline{\mathbf{u}}_1 + \underline{\mathbf{f}}_2, \tag{2.17}$$

supplemented by the initial-boundary-conditions

$$\vec{n} \wedge \underline{\mathbf{u}}_1 = 0 \text{ on } (0,\infty) \times \Gamma_1, \quad \vec{n} \wedge \underline{\mathbf{u}}_2 = 0 \text{ on } (0,\infty) \times \Gamma_2 \tag{2.18}$$

and

$$\mathbf{u}(0,x) = \mathbf{w}(x). \tag{2.19}$$

if

$$\frac{d}{dt}\langle \mathbf{u}(t), \mathbf{a}\rangle_X = -\langle \mathbf{u}(t), B\mathbf{a}\rangle_X + \langle \mathbf{f}(t), \mathbf{a}\rangle_X \text{ for all } \mathbf{a} \in D(B) \tag{2.20}$$

and 2.19 is fulfilled. This means that 2.17 is fulfilled in the sense of distributions, wheras the boundary conditions 2.18 are satisfied in the sense that $\int_0^t \mathbf{u}(s)ds \in D(B) = W_E \times W_H$ for all $t \geq 0$. It is well known that 2.20 is equivalent to the variation of constant formula

$$\mathbf{u}(t) = \exp(tB)\mathbf{w} + \int_0^t \exp((t-s)B)\mathbf{f}(s)ds \tag{2.21}$$

where $(\exp(tB))_{t \in \mathbb{R}}$ is the unitary group generated by B, see [16]. 2.21 yields the energy estimate

$$\frac{1}{2}\frac{d}{dt}||\mathbf{u}(t)||_X^2 = \langle \mathbf{f}(t), \mathbf{u}(t)\rangle_X \tag{2.22}$$

Next $\mathcal{R} : L^2(G) \to X$ is defined by

$$(\mathcal{R}\mathbf{p})(x) \stackrel{\text{def}}{=} (\mathbf{p}(x), 0) \text{ if } x \in G \text{ and } (\mathcal{R}\mathbf{p})(x) \stackrel{\text{def}}{=} 0 \text{ if } x \in \Omega \setminus G.$$

Let

$$\mathbf{j} \in L^1((0,\infty), L^2(\Omega, \mathbb{R}^3)), \quad \mathbf{P}_0 \in \mathcal{G} \text{ and } \mathbf{P}_1 \in \mathcal{G}. \tag{2.23}$$

By 2.15 and 2.16 the nonlinear composition operator $\mathbf{p} \in \mathcal{G} \to (\nabla_y V)(\cdot, \mathbf{p}(\cdot))$ is globally Lipschitz continuous as a map from $\mathcal{G}$ to $\mathcal{G}$. Therefore the initial-value-problem

$$\alpha \partial_t^2 \mathbf{P}(t) + \partial_t \mathbf{P}(t) + (\nabla_y V)(x, \mathbf{P}(t)) = \gamma \mathbf{E} \text{ on } (0,\infty) \times G \tag{2.24}$$

supplemented by the initial-conditions

$$\mathbf{P}(0) = \mathbf{P}_0, \quad \partial_t \mathbf{P}(0) = \mathbf{P}_1 \tag{2.25}$$

admits for all $\mathbf{E} \in C([0,\infty), L^2(\Omega, I\!R^3))$ a unique weak solution $\mathbf{P} \in C^2([0,\infty),\mathcal{G})$. If $\mathbf{E} \in C([0,\infty), L^2(\Omega, I\!R^3))$ and $\mathbf{F} \in C([0,\infty), L^2(\Omega, I\!R^3))$ then the Lipschitz continuity of $\nabla_y V$ yields the estimate

$$||\partial_t(\mathbf{P}(t) - \mathbf{Q}(t))||_{\mathcal{G}} \leq \int_0^t \exp\left(L(t-s)\right)\|\gamma(\mathbf{E}(t) - \mathbf{F}(s))\|_{\mathcal{G}} ds \tag{2.26}$$

$$\leq C\int_0^t \exp\left(L(t-s)\right)\|\mathbf{E}(t) - \mathbf{F}(s)\|_X ds$$

with some $L, C > 0$ independent of $\mathbf{E}$, $\mathbf{F}$ and t. (Here $\mathbf{Q} \in C^2([0,\infty),\mathcal{G})$ is the solution of 2.24 and 2.25 with $\mathbf{E}$ replaced by $\mathbf{F}$.)

Now $(\mathbf{E},\mathbf{H}) \in C([0,\infty), X)$ and $\mathbf{P} \in C^2([0,\infty),\mathcal{G})$ solve 1.1-1.5 (in the sense of 2.20), if .

$$(\mathbf{E}(t),\mathbf{H}(t)) = \exp\left(tB\right)(\mathbf{E}_0,\mathbf{H}_0) - \int_0^t \exp\left((t-s)B\right)\left[\mathcal{R}\partial_t\mathbf{P}(t) + (\mathbf{j}(t),0)\right] ds \tag{2.27}$$

and $\mathbf{P}$ solves 2.24 and 2.25. It follows from the estimates 2.22 and 2.26 using the contraction mapping principle in the space $C([0,T], X)$ that problem 2.24, 2.25 and 2.27 has a unique weak solution $(\mathbf{E},\mathbf{H}) \in C([0,\infty), X)$ and $\mathbf{P} \in C^2([0,\infty),\mathcal{G})$. If the additional regularity assumption

$$(\mathbf{E}_0,\mathbf{H}_0) \in D(B) \text{ and } \mathbf{j} \in W^{1,1}((0,\infty), L^2(\Omega)) \tag{2.28}$$

is imposed it follows that $\mathcal{R}\partial_t\mathbf{P}(\cdot) + (\mathbf{j},0) \in W^{1,1}_{loc}([0,\infty), X)$ and $(\mathbf{E}_0,\mathbf{H}_0) \in D(B)$. By the result in [16, Corollary 2.5, sect. 4.2] it follows that $(\mathbf{E},\mathbf{H}) \in C^1([0,\infty), X) \cap C([0,\infty), D(B))$ is a strong solution of
$\partial_t(\mathbf{E}(t),\mathbf{H}(t)) = B(\mathbf{E}(t),\mathbf{H}(t)) - \mathcal{R}\partial_t\mathbf{P}(t) - (\mathbf{j}(t),0)$. 2.27 and the energy estimate 2.22 yield

$$\frac{1}{2}\frac{d}{dt}||(\mathbf{E}(t),\mathbf{h}(t))||_X^2 = -\langle \mathcal{R}\partial_t\mathbf{P}(t) + (\mathbf{j}(t),0)), (\mathbf{E}(t),\mathbf{h}(t))\rangle_X \tag{2.29}$$

$$= -\int_G \mathbf{E}\partial_t\mathbf{P}dx - \int_\Omega \mathbf{E}\mathbf{j}dx$$

whereas 2.24 yields

$$\frac{d}{dt}\left(1/2||\alpha^{1/2}\partial_t P(t)||_{\mathcal{G}}^2 + \int_G \gamma^{-1}V(x,\mathbf{P})dx\right) = -\int_G \gamma^{-1}|\partial_t\mathbf{P}|^2 dx + \int_G \mathbf{E}\partial_t\mathbf{P}dx \tag{2.30}$$

2.29 and 2.30 yield the energy-estimate

$$\frac{d}{dt}\left(||(\mathbf{E}(t),\mathbf{h}(t))||_X^2 + 1/2||\alpha^{1/2}\partial_t P(t)||_{\mathcal{G}}^2 + \int_G \gamma^{-1}V(x,\mathbf{P})dx\right) \tag{2.31}$$

$$= -\int_G \gamma^{-1}|\partial_t\mathbf{P}|^2 dx - \int_\Omega \mathbf{E}\mathbf{j}dx \leq ||(\mathbf{E}(t),\mathbf{h}(t))||_X||\mathbf{j}(t)||_X - ||\partial_t\mathbf{P}(t)||_{\mathcal{G}}^2$$

In the next lemma elementary properties of the solution are shown.

Lemma 1 *i)*

$$(\mathbf{E},\mathbf{H}) \in L^\infty((0,\infty),X),$$

$$\text{and } ||\partial_t \mathbf{P}(\cdot)||^2_{\mathcal{G}} \in L^1(0,\infty) \text{and } \gamma^{-1}V(x,\mathbf{P}(\cdot)) \in L^\infty((0,\infty),L^1(G)). \tag{2.32}$$

ii) $||\partial_t \mathbf{P}(t)||_{\mathcal{G}} \overset{t\to\infty}{\longrightarrow} 0.$

iii) If 2.28 holds one has

$$(\mathbf{E},\mathbf{h}) \in W^{1,\infty}((0,\infty),X) \cap L^\infty((0,\infty),D(B))$$

Proof:
Let

$$\mathcal{E}_t \overset{\text{def}}{=} \left(||(\mathbf{E}(t),\mathbf{h}(t))||^2_X + 1/2||\alpha^{1/2}\partial_t P(t)||^2_{\mathcal{G}} + \int_G \gamma^{-1}V(x,\mathbf{P})dx\right). \tag{2.33}$$

By 2.31 one has

$$\frac{d}{dt}\mathcal{E}_t \leq \mathcal{E}_t(\mathbf{w})^{1/2}||\mathbf{j}(t)||_X - ||\partial_t \mathbf{P}(t)||^2_{\mathcal{G}}$$

Since $||\mathbf{j}(\cdot)||_X \in L^1(0,\infty)$, 2.29 yields i).

If 2.28 holds we hasve

$$\partial_t^2(\mathbf{E}(t),\mathbf{h}(t)) = B\partial_t(\mathbf{E}(t),\mathbf{h}(t)) - \mathcal{R}\partial_t^2\mathbf{P}(t) - \partial_t(\mathbf{j}(t),0) \tag{2.34}$$

weakly in the sense of 2.20. With a similar estimate as before one obtains using the global boundedness of $(D_P^2 V)$ by 2.16

$$1/2\frac{d}{dt}\left(||\partial_t(\mathbf{E}(t),\mathbf{h}(t))||^2_X + ||\alpha^{-1/2}\partial_t^2 P(t)||^2_{\mathcal{G}}\right) \tag{2.35}$$

$$= -\int_\Omega \partial_t \mathbf{E}\partial_t \mathbf{j}dx - \int_G \gamma^{-1}|\partial_t^2\mathbf{P}|^2 dx - \int_G \gamma^{-1}\partial_t^2\mathbf{P}\cdot(D_P^2V)(x,\mathbf{P})\cdot\partial_t\mathbf{P}dx$$

$$\leq ||\partial_t(\mathbf{E}(t),\mathbf{h}(t))||_X||\partial_t\mathbf{j}(t)||_X + C_1||\partial_t\mathbf{P}||^2_{\mathcal{G}} - b_0/2||\partial_t^2\mathbf{P}||^2_{\mathcal{G}}$$

With 2.32 and $||\partial_t \mathbf{j}(\cdot)||_X \in L^1(0,\infty)$ it follows that

$$\partial_t(\mathbf{E},\mathbf{h}) \in L^\infty((0,\infty),X) \tag{2.36}$$

$$\text{and } ||\partial_t^2\mathbf{P}(\cdot)||^2_{\mathcal{G}} \in L^1(0,\infty) \tag{2.37}$$

By 2.32 and 2.37 one has $\partial_t\mathbf{P} \in W^{1,2}((0,\infty),\mathcal{G})$, which implies $||\partial_t\mathbf{P}(t)||_{\mathcal{G}} \overset{t\to\infty}{\longrightarrow} 0$. This completes the proof of part ii).

□

By 2.31 and 2.33, 2.32 and $||\mathbf{j}(\cdot)||_X \in L^1(0,\infty)$ one has
$\frac{d}{dt}\mathcal{E}(t) \in L^1(0,\infty)$, which implies the existence of the limit

$$\mathcal{E}_\infty \stackrel{\text{def}}{=} \lim_{t\to\infty} \mathcal{E}(t) = \lim_{t\to\infty} \left(||(\mathbf{E}(t), \mathbf{h}(t))||_X^2 + \int_G \gamma^{-1} V(x, \mathbf{P}) dx \right). \tag{2.38}$$

The physical meaning of $\mathcal{E}(t)$ is the total energy of the system, i.e. the sum of the potential and kinetic energy of the oscillating particles and the energy of the electromagnetic field.

In the sequel let $R_0 > 0$, such that $I\!R^3 \setminus \Omega \subset B_{R_0} \stackrel{\text{def}}{=} \{x \in I\!R^3 : |x| < R_0\}$ and choose $\chi_0 \in C^\infty(I\!R^3)$ with supp $\chi_0 \subset \Omega$ and $\chi_0(x) = 1$ on $I\!R^3 \setminus B_{R_0}$.
Then for $\mathbf{w} \in X$ or $\mathbf{w} \in L^2(I\!R^3)$ define $\mathcal{C}_0\mathbf{w} \stackrel{\text{def}}{=} ((\nabla\chi_0) \wedge \underline{\mathbf{w}}_2, -(\nabla\chi_0) \wedge \underline{\mathbf{w}}_1)$,
For convenience $\chi_0\mathbf{w}$ and $\mathcal{C}_0\mathbf{w}$ will be regarded as elements of $L^2(I\!R^3)$ by extending them by zero outside supp χ_0 if $\mathbf{w} \in X$.

Next let $D(B_0) \stackrel{\text{def}}{=} H_{curl}(I\!R^3) \times H_{curl}(I\!R^3)$ and $B_0(\mathbf{e}, \mathbf{h}) \stackrel{\text{def}}{=} (\text{ curl } \mathbf{h}, -\text{ curl } \mathbf{e})$.
Finally, let P_0 be the orthogonal projector on $(\ker B_0)^\perp$, which consits of all $\mathbf{u} \in L^2(I\!R^3)$ with div $\underline{\mathbf{u}}_j = 0$.

3 A weak convergence property of the solutions

In what follows the additional regeularity assumption 2.28 will be imposed on the data. The following 'unique continuation' principle has been shown in [11], which holds even for arbitrary spatial domains. As in [11] it will be used in the investigation of the weak ω-limit set of the solution of 1.1- 1.5.

Theorem 1 *Suppose that* $\mathbf{g} \in X$ *obeys*

$$\underline{(\exp{(tB)}\mathbf{g})}_1 = 0 \text{ on } G \text{ for all } t \in I\!R. \tag{3.39}$$

Then $\mathbf{g} \in$ *ker* B.

The above theorem is a suitable modification for not necessarily bounded domains of the following unique-continuation-principle, which is used in [5], [7] and [18]. Let $\mathcal{U} \subset I\!R^N$ be bounded domain and $u \in C([0,\infty), \overset{0}{H^1}(\mathcal{U})) \cap C^1([0,\infty), L^2(\mathcal{U}))$ be a solution of the wave-equation $\partial_t^2 u = \Delta u$ on $[0,\infty) \times \mathcal{U}$ with the property that $u(t,x) = 0$ on $[0,\infty) \times E$ for some subset $E \subset \mathcal{U}$ with positive measure. Then $u = 0$ on all of $[0,\infty) \times \mathcal{U}$.
Theorem 1 says that each solution $(\mathbf{e}, \mathbf{f}) \in C(I\!R, L^2(\Omega, I\!R^{M+N})$ of the evolution equation $\partial_t(\mathbf{e}, \mathbf{f}) = B(\mathbf{e}, \mathbf{f})$ with the property that $\mathbf{e}(t,x) = 0$ for all $t \in I\!R$ and $x \in G$ satisfies $(\mathbf{e}(0), \mathbf{f}(0)) \in \ker B$. In contrast to the unique continuation principle for bounded domains it is necessary to require the condition $\mathbf{e}(t,x) = 0$ on G for all $t \in I\!R$ and not only for positive times. The basic idea of the proof of Theorem 1 is to show that for each $f \in C_0^\infty(I\!R \setminus \{0\})$ the function $f(iB)\mathbf{g}$ is real-analytic and vanishes on G. This implies $f(iB)\mathbf{g} = 0$ for all $f \in C_0^\infty(I\!R \setminus \{0\})$ and hence $\mathbf{g} \in \ker B$. (Here the operator $f(iB)$ can be defined by the spectral-theorem, since iB is self-adjoint in $L^2(\Omega, \mathbb{C}^6)$.)

In the sequel let ω_0 denote the ω-limit-set of the trajectory $(\mathbf{E},\mathbf{H})$ with respect to the weak topology of X, i.e. the set of all $\mathbf{g}\in X$, such that there exists a sequence $t_n \stackrel{\text{n}\to\infty}{\longrightarrow} \infty$ with $(\mathbf{E}(t_n),\mathbf{H}(t_n)) \stackrel{\text{n}\to\infty}{\longrightarrow} \mathbf{g}$ in X weakly.

Theorem 2 *$P(\mathbf{E}(t),\mathbf{H}(t)) \stackrel{\text{t}\to\infty}{\longrightarrow} 0$ in X weakly.*

Proof:
Suppose $\mathbf{g}\in\omega_0$ and $t_n \stackrel{\text{n}\to\infty}{\longrightarrow} \infty$ with

$$(\mathbf{E}(t_n),\mathbf{H}(t_n)) \stackrel{\text{n}\to\infty}{\longrightarrow} \mathbf{g} \tag{3.40}$$

in X weakly. Let $\mathbf{u}_n(t) \stackrel{\text{def}}{=} (\mathbf{E}(t_n+t),\mathbf{H}(t_n+t))\in X$ and $\mathbf{f}_n(t) \stackrel{\text{def}}{=} (\nabla_P V)(x,\mathbf{P}(t_n+t))\in\mathcal{G}$ for $n\in I\!N$. Lemma 1yields

$$||\mathbf{f}_n(t)-\mathbf{f}_n(0)||_{\mathcal{G}} \le C_1\int_{[t_n,t_n+t]}||\partial_t\mathbf{P}(s)||_{\mathcal{G}}ds \stackrel{\text{n}\to\infty}{\longrightarrow} 0 \tag{3.41}$$

for all $t\in I\!R$. Next, we have by 2.21

$$\mathbf{u}_n(t)=\exp(tB)\mathbf{u}_n(0)-\int_{t_n}^{t_n+t}\exp((t_n+t-s)B)\left(\partial_s\mathcal{R}\mathbf{P}(s)+(\mathbf{j}(s),0)\right)ds,$$

from which one obtains by lemma 1 iii) that

$$||\mathbf{u}_n(t)-\exp(tB)\mathbf{u}_n(0)||_X \le \int_{t_n}^{t_n+t}||\mathcal{R}\partial_s\mathbf{P}(s)||_X+||\mathbf{j}(s)||_{L^2(\Omega)}ds \stackrel{\text{n}\to\infty}{\longrightarrow} 0$$

for all $t\in I\!R$ and hence by 3.40 with $\mathbf{u}_n(0)=(\mathbf{E}(t_n),\mathbf{H}(t_n))$

$$\mathbf{u}_n(t) \stackrel{\text{n}\to\infty}{\longrightarrow} \mathbf{u}_\infty(t) \stackrel{\text{def}}{=} \exp(tB)\mathbf{g} \text{ in } X \text{ weakly for all } t\in I\!R. \tag{3.42}$$

Suppose $T>0$. Then 3.41 implies for all $\varphi\in C_0^\infty((-T,T),\mathcal{G})$

$$\lim_{n\to\infty}\int_{-T}^{T}\langle\mathbf{f}_n(t),\partial_t\varphi(t)\rangle_{\mathcal{G}}dt=\lim_{n\to\infty}\int_{-T}^{T}\langle\mathbf{f}_n(0),\partial_t\varphi(t)\rangle_{\mathcal{G}}dt=0 \tag{3.43}$$

Using $\partial_t\mathbf{P}\in L^2((0,\infty),\mathcal{G})$ again one obtains from 2.24, 3.42 and 3.43 that

$$\int_{-T}^{T}\int_G\underline{(\mathbf{u}_\infty)}_1\partial_t\varphi dxdt=\lim_{n\to\infty}\int_{-T}^{T}\int_G\underline{(\mathbf{u}_n)}_1\partial_t\varphi dxdt \tag{3.44}$$

$$=\lim_{n\to\infty}\int_{-T}^{T}\langle\gamma\mathbf{E}(t_n+t),\partial_t\varphi(t)\rangle_{\mathcal{G}}dt$$

$$=\lim_{n\to\infty}\int_{-T}^{T}\langle\alpha\partial_t^2\mathbf{P}(t_n+t)+\partial_t\mathbf{P}(t_n+t)+\mathbf{f}_n(t),\partial_t\varphi(t)\rangle_{\mathcal{G}}dt=0$$

in particular

$$\partial_t\underline{(\mathbf{u}_\infty)}_1(t,x)=0 \text{ for all } x\in G, t\in(-T,T) \tag{3.45}$$

Since $T > 0$ is chosen arbitrarily, 3.45 holds for all $t \in I\!R$.
With $(\mathbf{E}(\cdot), \mathbf{H}(\cdot)) \in L^\infty((0,\infty), D(B))$ by lemma 1 i), it follows $\mathbf{g} \in D(B)$ and

$$\underline{(\exp(tB)B\mathbf{g})}_1(x) = \partial_t\underline{(\mathbf{u}_\infty)}_1(t,x) = 0 \text{ for all } t \in I\!R \text{ and } x \in G$$

by 3.45. Invoking theorem 1 one obtains $B\mathbf{g} \in \ker B$, and hence $||B\mathbf{g}||_X^2 = -\langle \mathbf{g}, B^2\mathbf{g}\rangle_X = 0$, whence $\mathbf{g} \in \ker B$. Hence

$$\omega_0 \subset \ker B. \tag{3.46}$$

Since $(\mathbf{E}(t), \mathbf{H}(t))$ is bounded in X as $t \to \infty$ by lemma 1 i) and zero is the only possible accumulation point of $P(\mathbf{E}(\cdot), \mathbf{H}(\cdot))$ by 3.46, the assertion follows.

□

4 Decay properties for in the asymptotically charge-free case

For all $\mathbf{a} \in \ker B$ one has

$$\begin{aligned}
&\langle(\mathbf{E}(t), \mathbf{H}(t)), \mathbf{a}\rangle_X = \langle \exp(tB)(\mathbf{E}_0, \mathbf{H}_0) \\
&\quad - \int_0^t \exp((t-s)B)\left(\mathcal{R}\partial_s\mathbf{P}(s) + (\mathbf{j}(t), 0)\right) ds, \mathbf{a}\Big\rangle_X \\
&= \left\langle (\mathbf{E}_0, \mathbf{H}_0) + \mathcal{R}\mathbf{P}(0) - \mathcal{R}\mathbf{P}(t) - \int_0^t (\mathbf{j}(s), 0) ds, \mathbf{a}\right\rangle_X
\end{aligned}$$

and hence

$$(1-P)\left((\mathbf{E}(t), \mathbf{H}(t)) + \mathcal{R}\mathbf{P}(t) + \int_0^t (\mathbf{j}(s), 0)ds - (\mathbf{E}_0, \mathbf{H}_0) - \mathcal{R}\mathbf{P}(0)\right) = 0. \tag{4.47}$$

Recall that $1 - P$ is the orthogonal projector on $\ker B$.

Throughout this section it is assumed that the initial-state $(\mathbf{E}_0, \mathbf{H}_0) \in X$ satisfies

$$(1-P)\left((\mathbf{E}_0, \mathbf{H}_0) + \mathcal{R}\mathbf{P}(0) - \int_0^\infty (\mathbf{j}(s), 0)ds\right) = 0. \tag{4.48}$$

This is condition 1.8 on the initial states. It follows from 4.47 and 4.48 that

$$(1-P)(\mathbf{E}(t), \mathbf{H}(t)) = (1-P)(\mathbf{J}(t) - \mathcal{R}\mathbf{P}(t)). \tag{4.49}$$

with $\mathbf{J}(t) \stackrel{\text{def}}{=} \int_t^\infty (\mathbf{j}(s), 0)ds$. The main goal of this section is to prove the decay property 1.7.

For this purpose the following additional assumptions are imposed on V, the set G and γ in what follows.

$$V(x, \mathbf{y}) \le K_2 \mathbf{y}\nabla_y V(x, \mathbf{y}) \text{ for all } x \in G, \mathbf{y} \in I\!R^3. \tag{4.50}$$

with some $K_2 \in (0,\infty)$ independent of $x, \mathbf{y}$, and

$$\gamma \in L^{3/2}(G) \text{ and } (1+|x|)\gamma \in L^{r_0}(G) \text{ with some } r_0 \in (3/2,\infty). \tag{4.51}$$

Finally it is assumed that the external current $\mathbf{j}$ is located in a fixed finite ball, i. e.

$$\mathbf{j}(t,x) = 0 \text{ for all } t \in (0,\infty), x \in I\!R^3 \setminus B_{R_1} \tag{4.52}$$

with some $R_1 > 0$.

Lemma 2 $||P(\mathbf{E}(t), \mathbf{H}(t))||_{L^2(\Omega\cap B_R)} \stackrel{t\to\infty}{\longrightarrow} 0$ *for all* $R > 0$.

Proof:
Each $\mathbf{u} \in (\ker B)^{\perp}$ satisfies

$$\text{div } (\underline{\mathbf{u}}_1) = 0, \quad \text{div } (\underline{\mathbf{u}}_2) = 0 \tag{4.53}$$

$$\text{with } \vec{n}\underline{\mathbf{u}}_1 = 0 \text{ on } \Gamma_2 \text{ and } \vec{n}\underline{\mathbf{u}}_2 = 0 \text{ on } \Gamma_1$$

in the sense that

$$\int_\Omega (\underline{\mathbf{u}}_1 \nabla\varphi + \underline{\mathbf{u}}_2 \nabla\psi)\, dx = 0$$

for all $\varphi \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_1})$ and $\psi \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_2})$.
This follows from the fact that $(\nabla\varphi, \nabla\psi) \in \ker B$
for all $\varphi \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_1})$ and $\psi \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_2})$.

Suppose $\mathbf{u} \in (\ker B)^{\perp} \cap D(B)$. Then $\underline{\mathbf{u}}_1 \in W_E$, whereas $\underline{\mathbf{u}}_2 \in W_H$. Therefore 4.53 and the compactness-theorem in [8], a generalization of the result in [19], see also [13], implie that

$$(\ker B)^{\perp} \cap D(B) \text{ is compactly imbedded in } L^2(\Omega \cap B_R) \text{ for all } R > 0. \tag{4.54}$$

Now,the result follows from lemma 1i), theorem 2 and 4.54.

□

The aim of the following considerations is to prove the estimate

Lemma 3 *There exists a constant* $K_3 \in (0,\infty)$, *such that for all*
$\mathbf{E} \in W_E \cap W_{E,0}^{\perp}$ *the estimate*

$$||\mathbf{E}||_{L^2(\Omega\cap B_{R_0})} + ||\mathbf{E}||_{L^6(I\!R^3\setminus B_{R_0})} \leq K_3 ||\; curl\; \mathbf{E}||_{L^2(\Omega)}$$

holds.

Lemma 4 *i) The set of all* $\mathbf{F} \in W_{E,0}$ *with bounded support is dense in* $W_{E,0}$ *(with respect to the* $L^2(\Omega)$ *-norm).*

ii) Let $\mathbf{E} \in L^2(\Omega \cap B_{R_0}) \cap L^6(I\!R^3 \setminus B_{R_0})$ *with*

$$\int_\Omega \mathbf{E} \ \textit{curl} \ \mathbf{h} dx = 0 \ \textit{for all} \ \mathbf{h} \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_2}, \mathbb{C}^3) \tag{4.55}$$

and

$$\int_\Omega \mathbf{E}\mathbf{F} dx = 0 \ \textit{for all} \ \mathbf{F} \in W_{E,0} \ \textit{with bounded support.} \tag{4.56}$$

Then $\mathbf{E} = 0$ *.*

Proof:

i) Suppose $\mathbf{F} \in W_{E,0}$. Since curl $\mathbf{F} = 0$ there exists some $\varphi \in L^6(I\!R^3 \setminus B_{R_0})$ with

$$\mathbf{F} = \nabla\varphi \text{ on } I\!R^3 \setminus B_{R_0} \tag{4.57}$$

Let $\psi_1 \in C_0^\infty(B_2)$ with $\psi_1 = 1$ on B_1 and $\psi_n \stackrel{\text{def}}{=} \psi_1(x/n)$. Now define $\mathbf{F}_n(x) \stackrel{\text{def}}{=} \psi_n(x)\mathbf{F}(x)$ if $x \in \Omega \cap B_n$ and $\mathbf{F}_n(x) \stackrel{\text{def}}{=} \psi_n(x)\mathbf{F}(x) + \varphi(x)\nabla\psi_n(x)$ if $|x| \geq n$. Then $\mathbf{F}_n$ has bounded support and curl $\mathbf{F}_n = (\nabla\psi_n) \wedge \mathbf{F} + (\nabla\varphi) \wedge \nabla\psi_n = 0$ by 4.57. Since also $\mathbf{F}_n = \mathbf{F}$ near $\partial\Omega$ it follows easily that $\mathbf{F}_n \in W_{E,0}$. Next it followws from Hölder's inequality

$$||\mathbf{F}_n - \mathbf{F}||_{L^2(\Omega)} \leq ||(1-\psi_n)\mathbf{F}||_{L^2(\Omega)} + ||\varphi\nabla\psi_n||_{L^2(\Omega)}$$

$$\leq ||(1-\psi_n)\mathbf{F}||_{L^2(\Omega)} + ||\varphi||_{L^6(\{|x|>n\})}||\nabla\psi_n||_{L^3(I\!R^3)}$$

$$\leq ||(1-\psi_n)\mathbf{F}||_{L^2(\Omega)} + ||\varphi||_{L^6(\{|x|>n\})}||\nabla\psi_n||_{L^\infty(I\!R^3)}|B_{2n}|^{1/3}$$

$$\leq ||(1-\psi_n)\mathbf{F}||_{L^2(\Omega)} + C_1||\varphi||_{L^6(\{|x|>n\})} \stackrel{\text{n}\to\infty}{\longrightarrow} 0$$

with some C_1 independent of n. This completes the proof of i).

Next let $\mathbf{E} \in L^2(\Omega \cap B_{R_0}) \cap L^6(I\!R^3 \setminus B_{R_0})$ satisfy 4.55 and 4.56. Then

$$\text{curl } \mathbf{E} = 0 \text{ and } \text{div } \mathbf{E} = 0 \text{ on } \Omega. \tag{4.58}$$

Choose $\chi_0 \in C^\infty(I\!R^3)$ with
supp $\chi_0 \subset \Omega$ and $\chi_b(x) = 1$ for $|x| > R_0$. Since supp $\nabla\chi_0$ is bounded, it follows from 4.58 that curl $(\chi_0\mathbf{E}) = (\nabla\chi_0) \wedge \mathbf{E} \in L^{6/5}(I\!R^3)$ and div $(\chi_0\mathbf{E}) = (\nabla\chi_0) \cdot \mathbf{E} \in L^{6/5}(I\!R^3)$, where $\chi_0\mathbf{E}$ is extended by zero on $I\!R^3\setminus$ supp χ_0). From Lemma 1 in [10] one obtains $\chi_0\mathbf{E} \in L^2(I\!R^3)$ and hence $\mathbf{E} \in L^2(\Omega)$. By the definition of W_E and 4.55 we have $\mathbf{E} \in W_{E,0}$. Since $\mathbf{E} \in L^2(\Omega)$, equation 4.56 holds for all $\mathbf{F} \in W_{E,0}$ by assertion i). But this means $\mathbf{E} \in W_{E,0}^\perp$ and therefore $\mathbf{E} = 0$.

□

Lemma 5 *Let $\{\mathbf{E}_n\}_{n\in I\!N}$ be a sequence in $W_E \cap W_{E,0}^{\perp}$, which is bounded in $L^2(\Omega \cap B_{R_0}) \cap L^6(I\!R^3 \setminus B_{R_0})$, such that $\{$ curl $\mathbf{E}_n\}_{n\in I\!N}$ is precompact in $L^2(\Omega)$. Then $\{\mathbf{E}_n\}_{n\in I\!N}$ is precompact in $L^2(\Omega \cap B_{R_0}) \cap L^6(I\!R^3 \setminus B_{R_0})$.*

Proof:
Let $\tilde{\Omega} \stackrel{\text{def}}{=} B_{3R_0} \cap \Omega$ and choose $\chi_b \in C^\infty(I\!R^3)$ with $\chi_b(x) = 0$ on B_{R_0}, $\chi_b(x) = 1$ for $|x| > 2R_0$ and let $\chi_a \in C_0^\infty(B_{3R_0})$ with $\chi_a(x) = 1$ on B_{2R_0}, in particular

$$\chi_a(x) = 1 \text{ on supp } (\nabla \chi_b) \tag{4.59}$$

Let $S_1 \stackrel{\text{def}}{=} \Gamma_1 \cup \partial B_{3R_0}$ and $S_2 \stackrel{\text{def}}{=} \Gamma_2 = \partial\tilde{\Omega} \setminus S_1$. In analogy to the definition of W_E let $\mathcal{W}_E$ be the space of all $\mathbf{E} \in H_{curl}(\tilde{\Omega})$ with $\vec{n} \wedge \mathbf{E} = 0$ on S_1 in the sense that

$$\int_{\tilde{\Omega}} \mathbf{E} \text{ curl } \mathbf{F} - \mathbf{F} \text{ curl } \mathbf{E} dx = 0 \text{ for all } \mathbf{F} \in C_0^\infty(I\!R^3 \setminus \overline{S_2}, \mathcal{C}^3).$$

Now, it follows from the assumptions that

$$\{\chi_a \mathbf{E}_n\}_{n\in I\!N} \text{ is bounded in } \mathcal{W}_E. \tag{4.60}$$

Since $\mathbf{E}_n \in W_{E,0}^{\perp}$ one has also

$$\{ \text{ div } [\chi_a \mathbf{E}_n]\}_{n\in I\!N} = \{\mathbf{E}_n \nabla \chi_a\}_{n\in I\!N} \text{ is bounded in } L^2(\tilde{\Omega}) \tag{4.61}$$

and $\chi_a \vec{n} \mathbf{E} = 0$ on S_2, in the sense that

$$-\int_{\tilde{\Omega}} \chi_a \mathbf{E}_n \nabla \varphi dx = \int_{\tilde{\Omega}} (\text{ div } [\chi_a \mathbf{E}_n]) \varphi dx = \int_{\tilde{\Omega}} (\mathbf{E}_n \nabla \chi_a) \varphi dx$$

for all $\varphi \in C_0^\infty(I\!R^3 \setminus \overline{S_1}, \mathcal{C}^3)$.
Since $\tilde{\Omega} = B_{3R_0} \cap \Omega$ is a Lipschitz-domain, it follows from 4.60, 4.61 and the result in [5] that the sequence

$$\{\chi_a \mathbf{E}_n\}_{n\in I\!N} \text{ is precompact in } L^2(B_{R_0} \cap \Omega). \tag{4.62}$$

Let $\mathbf{f}_n(x) \stackrel{\text{def}}{=} \chi_b \mathbf{E}_n(x)$ if $x \in I\!R^3 \setminus B_{R_0} = \Omega \setminus B_{R_0}$ and $\mathbf{f}_n(x) \stackrel{\text{def}}{=} 0$ if $x \in B_{R_0}$.
Now it follows from 4.60 and 4.62 and the compactness assumption on $\{$ curl $\mathbf{E}_n\}_{n\in I\!N}$ that

$$\{ \text{ curl } \mathbf{f}_n\}_{n\in I\!N} = \{(\nabla \chi_b) \wedge \mathbf{E}_n + \chi_b \text{ curl } \mathbf{E}_n\}_{n\in I\!N} \text{ is precompact in } L^2(I\!R^3) \tag{4.63}$$

and

$$\{ \text{ div } \mathbf{f}_n\}_{n\in I\!N} = \{\mathbf{E}_n \nabla \chi_b\}_{n\in I\!N} \text{ is precompact in } L^2(I\!R^3). \tag{4.64}$$

By 4.63 and 4.64 it follows from Sobolev's inequality or directly Lemma 1 in [10] that the sequence $(\mathbf{f}_n)_{n\in I\!N}$ is precompact in $L^6(I\!R^3)$ and hence

$$(\mathbf{E}_n)_{n\in I\!N} \text{ is precompact in } L^6(\Omega \setminus B_{R_0}), \tag{4.65}$$

since $\chi_b(x) = 1$ for $|x| > R_0$.

$\square$

Proof of lemma 3
Suppose that the estimate were not correct, i.e. would exist a sequence $\mathbf{E}_n \in W_E \cap W_{E,0}^{\perp}$, $n \in I\!N$ with

$$1 = ||\mathbf{E}_n||_{L^2(\Omega \cap B_{R_0})} + ||\mathbf{E}_n||_{L^6(I\!R^3 \setminus B_{R_0})} \geq n||\text{ curl }\mathbf{E}_n||_{L^2} \text{ for all } n \in I\!N \tag{4.66}$$

By theorem 5 the sequence $\{\mathbf{E}_n\}_{n \in I\!N}$ is precompact in $L^2(\Omega \cap B_{R_0}) \cap L^6(I\!R^3 \setminus B_{R_0})$, i.e. there exist $\mathbf{E} \in L^2(\Omega \cap B_{R_0}) \cap L^6(I\!R^3 \setminus B_{R_0})$ and a subsequence $\mathbf{E}_{n_k}, k \in I\!N$ with

$$||\mathbf{E}_{n_k} - \mathbf{E}||_{L^2(\Omega \cap B_{R_0})} + ||\mathbf{E}_{n_k} - \mathbf{E}||_{L^6(I\!R^3 \setminus B_{R_0})} \overset{k\to\infty}{\longrightarrow} 0, \tag{4.67}$$

in particular

$$||\mathbf{E}||_{L^2(\Omega \cap B_{R_0})} + ||\mathbf{E}||_{L^6(I\!R^3 \setminus B_{R_0})} = 1. \tag{4.68}$$

From 4.66 and 4.67 it follows that

$$\int_\Omega \mathbf{E} \text{ curl } \mathbf{h} dx = \lim_{k\to\infty} \int_\Omega \mathbf{E}_{n_k} \text{ curl } \mathbf{h} dx \tag{4.69}$$

$$= \lim_{k\to\infty} \int_\Omega \mathbf{h} \text{ curl } \mathbf{E}_{n_k} dx = 0 \text{ for all } \mathbf{h} \in C_0^\infty(I\!R^3 \setminus \overline{\Gamma_2}, \mathbb{C}^3).$$

Furthermore

$$\int_\Omega \mathbf{E}\mathbf{F} dx = \lim_{k\to\infty} \int_\Omega \mathbf{E}_{n_k}\mathbf{F} dx = 0 \tag{4.70}$$

for all $\mathbf{F} \in W_{E,0}$ with bounded support. Now it follows from 4.69, 4.70 and lemma 4 ii) that $\mathbf{E} = 0$. This contradicts 4.68.

$\square$

Lemma 6 $t^{-1} \int_0^t \langle P(\mathbf{E}(s), \mathbf{H}(s)), \mathcal{R}\mathbf{P}(s) \rangle_X ds \overset{t\to\infty}{\longrightarrow} 0.$

Proof:
Let $\mathbf{u}(t) \overset{\text{def}}{=} P(\mathbf{E}(t), \mathbf{H}(t))$ and $\mathbf{A}(t) \overset{\text{def}}{=} \int_0^t \underline{\mathbf{u}}_1(s) ds$. Since $\mathbf{u}(t) \in (\text{ ker } B)^\perp$ one has $\mathbf{A}(t) \in W_{E,0}^\perp$. Hence it follows from Theorem 3 using $\text{curl } \underline{\mathbf{u}}_1(s) = -\underline{(B\mathbf{u}(s))}_2 = -\underline{[B(\mathbf{E}(t), \mathbf{H}(t))]}_2 = \text{ curl } \mathbf{E}(s) = -\partial_t \mathbf{H}(s)$ that

$$||\mathbf{A}(t)||_{L^2(\Omega \cap B_{R_0})} + ||\mathbf{A}(t)||_{L^6(I\!R^3 \setminus B_{R_0})} \leq K_3 || \text{ curl } \mathbf{A}(t)||_{L^2(\Omega)}$$

$$= K_3 || \int_0^t \text{ curl } \underline{\mathbf{u}}_1(s) ds ||_{L^2(\Omega)} = K_3 ||\mathbf{H}(0) - \mathbf{H}(t)||_{L^2(\Omega)}$$

Now, it follows from lemma 1 and the previous estimate that

$$||\mathbf{A}(t)||_{L^2(\Omega\cap B_{R_0})} + ||\mathbf{A}(t)||_{L^6(I\!R^3\setminus B_{R_0})} \leq C_1 \text{ for all } t\in(0,\infty). \tag{4.71}$$

with some constant C_1 independent of t.

$$t^{-1}\int_0^t \langle P(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X ds = t^{-1}\int_0^t\int_G \underline{\mathbf{u}}_1(s)\mathbf{P}(s)dxds \tag{4.72}$$

$$= t^{-1}\int_0^t\int_G \partial_t\mathbf{A}(s)\mathbf{P}(s)dxds = t^{-1}\int_G \mathbf{A}(t)\mathbf{P}(t)dx - t^{-1}\int_0^t\int_G \mathbf{A}(s)\partial_t\mathbf{P}(s)dxds$$

$$\leq C_1 t^{-1}\left(||\mathbf{P}(t)||_{L^2(G)} + ||\mathbf{P}(t)||_{L^{6/5}(G)}\right)$$

$$+C_1 t^{-1}\int_0^t \left(||\partial_t\mathbf{P}(s)||_{L^2(G)} + ||\partial_t\mathbf{P}(s)||_{L^{6/5}(G)}\right)ds$$

$$\leq C_1 t^{-1}\left(||\gamma^{1/2}||_{L^\infty(G)} + ||\gamma^{1/2}||_{L^3(G)}\right)||\mathbf{P}(t)||_{\mathcal{G}}$$

$$+C_1 t^{-1}\int_0^t \left(||\gamma^{1/2}||_{L^\infty(G)} + ||\gamma^{1/2}||_{L^3(G)}\right)||\partial_t\mathbf{P}(s)||_{\mathcal{G}}ds$$

by assumption 4.51 and Hölder's inequality. Next Lemma 1 yields

$$||\mathbf{P}(t)||_{\mathcal{G}} \leq ||\mathbf{P}_0||_{\mathcal{G}} + \int_0^t ||\partial_s\mathbf{P}(s)||_{\mathcal{G}}ds$$

$$\leq C_2 + C_2 t^{1/2}||\partial_s\mathbf{P}||_{L^2((0,\infty),\mathcal{G})} \leq C_3 + C_3 t^{1/2}$$

With 4.72 and Lemma 1 again one obtains

$$t^{-1}\int_0^t \langle P(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X ds$$

$$\leq C_4(t^{-1}+t^{-1/2}) + C_4 t^{-1/2}||\partial_s\mathbf{P}||_{L^2((0,\infty),\mathcal{G})} \overset{t\to\infty}{\longrightarrow} 0,$$

$\square$

Lemma 7 *One obtains*

$$t^{-1}\int_0^t \left(\int_G \gamma^{-1}V(x,\mathbf{P}(s))dx + ||(1-P)(\mathbf{E}(t),\mathbf{H}(t))||_X^2\right)ds \overset{t\to\infty}{\longrightarrow} 0,$$

in particular

$$\lim_{t\to\infty} t^{-1}\int_0^t ||(\mathbf{E}(s),\mathbf{H}(s))||_X^2 ds = \lim_{t\to\infty} t^{-1}\int_0^t ||P(\mathbf{E}(s),\mathbf{H}(s))||_X^2 ds = \mathcal{E}_\infty.$$

Proof:
It follows from 4.49 that

$$||(1-P)\left(\mathbf{E}(t),\mathbf{H}(t)\right)||_X^2 = \langle(\mathbf{E}(t),\mathbf{H}(t)),(1-P)\left[(\mathbf{J}(t),0)-\mathcal{R}\mathbf{P}(t)\right]\rangle_X$$

$$= \langle P(\mathbf{E}(t),\mathbf{H}(t)),\mathcal{R}\mathbf{P}(t)\rangle_X + \langle(\mathbf{E}(t),\mathbf{H}(t)),(1-P)(\mathbf{J}(t),0)\rangle_X - \int_G \mathbf{E}(t)\mathbf{P}(t)dx$$

$$\leq \langle P(\mathbf{E}(t),\mathbf{H}(t)),\mathcal{R}\mathbf{P}(t)\rangle_X$$

$$+C_1||\mathbf{J}(t)||_{L^2(\Omega)} - \int_G \gamma^{-1}\left[\alpha\partial_t^2\mathbf{P}(t)+\partial_t\mathbf{P}(t)+(\nabla_y V)(x,\mathbf{P}(t))\right]\mathbf{P}dx$$

$$\leq \langle P(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X + C_1||\mathbf{J}(t)||_{L^2(\Omega)}$$

$$-\int_G \gamma^{-1}\left(\alpha\partial_t^2\mathbf{P}(t)+\partial_t\mathbf{P}(t)\right)\mathbf{P}(t)dx - K_2^{-1}\int_G \gamma^{-1}V(x,\mathbf{P}(t))dx$$

by assumption 4.50. Now,

$$t^{-1}\int_0^t \left(||(1-P)\left(\mathbf{E}(s),\mathbf{H}(s)\right)||_X^2 + K_2^{-1}\int_G \gamma^{-1}V(x,\mathbf{P}(s))dx\right)ds \tag{4.73}$$

$$\leq t^{-1}\int_0^t \left(\langle P(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X + C_1||\mathbf{J}(s)||_{L^2(\Omega)}\right)ds$$

$$+C_2/t - 1/2t^{-1}\int_G \gamma^{-1}|\mathbf{P}(t)|^2dx - 1/2t^{-1}\int_G \alpha\gamma^{-1}\partial_t\mathbf{P}(t)\mathbf{P}(t)dx$$

$$+t^{-1}\int_0^t\int_G \alpha\gamma^{-1}|\partial_t\mathbf{P}(s)|^2dxds$$

$$\leq t^{-1}\int_0^t \left(\langle P(\mathbf{E}(s),\mathbf{H}(s)),\mathcal{R}\mathbf{P}(s)\rangle_X + C_1||\mathbf{J}(s)||_{L^2(\Omega)}\right)ds$$

$$+C_3/t + C_3t^{-1}\int_G \gamma^{-1}|\partial_t\mathbf{P}(t)|^2dx + t^{-1}\int_0^t\int_G \alpha\gamma^{-1}|\partial_t\mathbf{P}(s)|^2dxds$$

In the previous estimates C_j are constants independent of t. Finally, the assertion follows from lemma 1 i) , lemma 6 and 4.73.

□

From lemma 2and the previous lemma one obtains now easily

Corollary 1 $t^{-1}\int_0^t ||(\mathbf{E}(s),\mathbf{H}(s))||_{L^2(\Omega\cap B_R)}ds \overset{t\to\infty}{\longrightarrow} 0$ *for all* $R>0$.

The following estimate will be used in what follows.

Lemma 8 *Let $s \in [0,1]$. Then there exists a constant $K_1 \in (0,\infty)$ such that*

$$|\langle (1+|x|)^s \mathbf{f}, P_0 \mathbf{g}\rangle_{L^2(I\!R^3)}| \leq K_1 ||\mathbf{f}||_{H^1(I\!R^3)} ||(1+|x|)^s \mathbf{g}||_{L^{q_0}(I\!R^3)}$$

for all $\mathbf{g} \in C_0^\infty(I\!R^3)$. Here $1/(2r_0) + 1/2 = q_0^{-1}$, where $r_0 > 0$ as in assumption 4.51.

Proof:
Recall that $2r_0 > 3$. Let $p_1 \stackrel{\text{def}}{=} (\frac{1-s}{3} - \frac{1-s}{2r_0})^{-1} \in (\frac{3}{1-s}, \infty]$ ($p_1 = \infty$ for $s = 1$) and $p_2 \stackrel{\text{def}}{=} (\frac{s}{3} - \frac{s}{2r_0})^{-1}$ Since $(s-1)p_1 < -3$ and $sp_2 > 3$ one has

$$(1+|x|)^{s-1} \in L^{p_1}(I\!R^3) \text{ and } (1+|x|)^{-s} \in L^{p_2}(I\!R^3). \tag{4.74}$$

Now $\mathbf{F} \stackrel{\text{def}}{=} (1+|x|)^s P_0 \mathbf{f} - P_0\left((1+|x|)^s \mathbf{f}\right)$ obeys

$$B_0 \mathbf{F} = s(1+|x|)^{s-1}|x|^{-1} S P_0 \mathbf{f} + (1+|x|)^s B_0 \mathbf{f} - B_0\left((1+|x|)^s \mathbf{f}\right)$$

$$= s(1+|x|)^{s-1}|x|^{-1} S[P_0 \mathbf{f} - \mathbf{f}]$$

and hence

$$||(1+|x|)^{1-s} B_0 \mathbf{F}||_{L^2(I\!R^3)} \leq s||(P_0 - 1)\mathbf{f}||_{L^2(I\!R^3)} \leq s||\mathbf{f}||_{L^2(I\!R^3)}$$

and a similar estimate using div $\underline{(P_0 \mathbf{f})}_j = 0$ yields

$$||(1+|x|)^{1-s} \operatorname{div} \underline{\mathbf{F}}_j||_{L^2(I\!R^3)} = s||x|x|^{-1}\underline{(P_0 \mathbf{f})}_j||_{L^2(I\!R^3)} \leq s||\mathbf{f}||_{L^2(I\!R^3)}$$

With $1/q \stackrel{\text{def}}{=} 1/p_1 + 1/2$ we obtain by 4.74 and Hölder's inequality

$$||\operatorname{curl} \underline{\mathbf{F}}_j||_{L^q(I\!R^3)} \leq C_1 ||B_0 \mathbf{F}||_{L^q(I\!R^3)}$$

$$\leq C_1 ||(1+|x|)^{s-1}||_{L^{p_1}(I\!R^3)} ||(1+|x|)^{1-s} B_0 \mathbf{F}||_{L^2(I\!R^3)} \leq C_2 ||\mathbf{f}||_{L^2(I\!R^3)}$$

and

$$||\operatorname{div} \underline{\mathbf{F}}_j||_{L^q(I\!R^3)} \leq C_2 ||\mathbf{f}||_{L^2(I\!R^3)}$$

By Sobolev's inequality or directly Lemma 1 in [10] one obtains

$$||\mathbf{F}||_{L^r(I\!R^3)} \leq C_3 ||D\mathbf{F}||_{L^q(I\!R^3)} \tag{4.75}$$

$$\leq C_4 \left(||\operatorname{curl} \underline{\mathbf{F}}_j||_{L^q(I\!R^3)} + ||\operatorname{div} \underline{\mathbf{F}}_j||_{L^q(I\!R^3)}\right) \leq C_3 ||\mathbf{f}||_{L^2(I\!R^3)}$$

Here $r \stackrel{\text{def}}{=} (\frac{1}{q} - \frac{1}{3})^{-1} = (1/6 + \frac{1-s}{3} - \frac{1-s}{2r_0})^{-1} \in (2,6)$. Now, $\frac{1}{r} + \frac{1}{p_2} + \frac{1}{q_0} = 1$.
Therefore assumption 4.51, 4.74, 4.75 and Hölder's inequality yield

$$|\langle \mathbf{F}, \mathbf{g}\rangle_{L^2(I\!R^3)}| \leq ||\mathbf{F}||_{L^r(I\!R^3)} ||(1+|x|)^{-s}||_{L^{p_2}(I\!R^3)} ||(1+|x|)^s \mathbf{g}||_{L^{q_0}(I\!R^3)} \tag{4.76}$$

$$\leq C_5||\mathbf{f}||_{L^2(I\!R^3)}||(1+|x|)^s\mathbf{g}||_{L^{q_0}(I\!R^3)}$$

Since $q_0 \geq 6/5$ one has $q_0^* \leq 6$. Therefore, it follows from Hölder's inequality, 4.76 and the embedding $H^1(I\!R^3) \hookrightarrow L^{q_0^*}(I\!R^3)$ that

$$|\langle(1+|x|)^s\mathbf{f}, P_0\mathbf{g}\rangle_{L^2(I\!R^3)}| \leq |\langle P_0\mathbf{f}, (1+|x|)^s\mathbf{g}\rangle_{L^2(I\!R^3)} + \langle \mathbf{F}, \mathbf{g}\rangle_{L^2(I\!R^3)}| \tag{4.77}$$

$$\leq C_5||P_0\mathbf{f}||_{H^1(I\!R^3)}||(1+|x|)^s\mathbf{g}||_{L^{q_0}(I\!R^3)} + |\langle \mathbf{F}, \mathbf{g}\rangle_{L^2(I\!R^3)}|$$

$$\leq C_6||\mathbf{f}||_{H^1(I\!R^3)}||(1+|x|)^s\mathbf{g}||_{L^{q_0}(I\!R^3)}$$

$\square$

The next theorem says roughly speaking that the asymptotic propagation-speed of the wave-packet $(\mathbf{E}(t), \mathbf{H}(t))$ does not exceed 1 as $t \to \infty$.

Theorem 3 *Suppose $b > 1$. Then*

$$\int_{\{|x|\geq bt\}} |(\mathbf{E}(t), \mathbf{H}(t))|^2 dx \stackrel{t\to\infty}{\longrightarrow} 0.$$

Proof:
The proof is based on an energy estimate. Let $g \in C^\infty(I\!R)$ with $g(u) = 1$ for $u \geq (1+b)/2$ and $g(u) = 0$ for $u \leq 1$. For $R > 0$ define

$$\mathcal{E}_R(t) \stackrel{\text{def}}{=} \int_\Omega g((t+R)^{-1}|x|)\left[|\mathbf{E}(t)|^2 + |\mathbf{H}(t)|^2\right] dx$$

$$+ \int_G \gamma^{-1} g((t+R)^{-1}|x|)\left(\alpha|\partial_t\mathbf{P}|^2 + 2V(x,\mathbf{P})\right).$$

Then one obtains from the basic equations using assumption 4.52 for all $t \geq R_1$

$$\mathcal{E}_R'(t) = 2\left\langle g((t+R)^{-1}|x|)(\mathbf{E}(t), \mathbf{H}(t)), B(\mathbf{E}(t), \mathbf{H}(t)) - \mathcal{R}\partial_t\mathbf{P}(t) - (\mathbf{j}(t), 0)\right\rangle_X \tag{4.78}$$

$$-(t+R)^{-2}\int_\Omega |x|g'((t+R)^{-1}|x|)\left[|\mathbf{E}(t)|^2 + |\mathbf{H}(t)|^2\right] dx$$

$$+\int_G 2g((t+R)^{-1}|x|)\left(\mathbf{E}\partial_t\mathbf{P} - \gamma^{-1}|\partial_t\mathbf{P}|^2\right)$$

$$-(t+R)^{-2}\int_G \gamma^{-1}|x|g'((t+R)^{-1}|x|)\left(\alpha|\partial_t\mathbf{P}|^2 + 2V(x,\mathbf{P})\right)$$

$$\leq 2\left\langle g((t+R)^{-1}|x|)(\mathbf{E}(t), \mathbf{H}(t)), B(\mathbf{E}(t), \mathbf{H}(t))\right\rangle_X$$

$$-(t+R)^{-2}\int_\Omega |x|g'((t+R)^{-1}|x|)\left[|\mathbf{E}(t)|^2 + |\mathbf{H}(t)|^2\right] dx$$

$$\leq 2(t+R)^{-1}\int_\Omega |x|^{-1}g'((t+R)^{-1}|x|)\mathbf{E}(t)\cdot(x\wedge\mathbf{H}(t))dx$$

$$-(t+R)^{-2}\int_\Omega |x|g'((t+R)^{-1}|x|)\left[|\mathbf{E}(t)|^2+|\mathbf{H}(t)|^2\right]dx\leq 0$$

Now, it follows from 4.78 that

$$\mathcal{E}_R(t)\leq\mathcal{E}_R(R_1) \tag{4.79}$$

Since $b>1$ one has by 4.79 for all $R>0$

$$\limsup_{t\to\infty}\int_{\{|x|\geq bt\}}|(\mathbf{E}(t),\mathbf{H}(t))|^2dx\leq\limsup_{t\to\infty}\int_{\{|x|\geq(b+1)(t+R)/2\}}|(\mathbf{E}(t),\mathbf{H}(t))|^2dx$$

$$\leq\limsup_{t\to\infty}\mathcal{E}_R(t)\leq\mathcal{E}_R(R_1).$$

Since $g(0)=0$ it follows $\mathcal{E}_R(R_1)\overset{\mathrm{R}\to\infty}{\longrightarrow}0$. Hence the assertion follows from the previous estimate letting $R\to\infty$.

□

Recall that P_0 is the orthogonal-projector on $(\ker B_0)^\perp$ and that $\chi_0\in C^\infty(I\!R^3)$ with $\chi_0(x)=1$ on $I\!R^3\setminus B_{R_0}$ and supp $\chi_0\subset\Omega$. Since supp $\chi_0\subset\Omega$, lemma 1 yields $\chi_0(\mathbf{E}(\cdot),\mathbf{H}(\cdot))\in L^\infty(0,\infty),D(B_0))$ and hence

$$P_0\chi_0(\mathbf{E}(\cdot),\mathbf{H}(\cdot))\in L^\infty(0,\infty),D(B_0)\cap(ker B_0)^\perp)\subset L^\infty(0,\infty),H^1(I\!R^3)). \tag{4.80}$$

□

Lemma 9 *The following holds*

$$t^{-1}\int_0^t||(1-P_0)\chi_0(\mathbf{E}(s),\mathbf{H}(s))||_{L^2(I\!R^3)}ds\overset{\mathrm{t}\to\infty}{\longrightarrow}0.$$

in particular

$$t^{-1}\int_0^t||P_0\chi_0(\mathbf{E}(s),\mathbf{H}(s))||^2_{L^2(I\!R^3)}ds\overset{\mathrm{t}\to\infty}{\longrightarrow}\mathcal{E}_\infty.$$

$$\text{and } t^{-1}\int_0^t||P_0\chi_0(\mathbf{E}(s),\mathbf{H}(s))||_{L^2(\{|x|\geq as\})}ds\overset{\mathrm{t}\to\infty}{\longrightarrow}0 \text{ for all } a\in(1,\infty).$$

Proof:
First the following estimate is proved.

$$||(1-P_0)\chi_0\mathbf{u}||_{L^2(I\!R^3)} \leq K_1\left(||(1-P)\mathbf{u}||_X + ||\mathbf{u}||_{L^2(B_{R_0})}\right) \tag{4.81}$$

for all $\mathbf{u} \in X$ with some constant independent of $\mathbf{u}$. For this porpose let $\mathbf{u} \in X$ and define $\mathbf{g} \stackrel{\text{def}}{=} (1-P_0)\chi_0\mathbf{u} \in \ker B_0$, i.e. curl $\underline{\mathbf{g}}_j = 0$ on $I\!R^3$. Hence there exist $\varphi_j \in L^6(I\!R^3)$ with $\nabla\varphi_j \in L^2(I\!R^3)$ such that

$$\underline{\mathbf{g}}_j = \nabla\varphi_j \tag{4.82}$$

Hence

$$||\mathbf{g}||^2_{L^2(I\!R^3)} = \langle \chi_0\mathbf{u}, \mathbf{g}\rangle_{L^2(I\!R^3)} = \int_{I\!R^3} \chi_0 \cdot (\underline{\mathbf{u}}_1\nabla\varphi_1 + \underline{\mathbf{u}}_2\nabla\varphi_2)\, dx$$

$$= \langle \mathbf{u}, (\nabla[\chi_0\varphi_1], \nabla[\chi_0\varphi_2])\rangle_X - \int_{I\!R^3} (\underline{\mathbf{u}}_1(\nabla\chi_0)\varphi_1 - \underline{\mathbf{u}}_2(\nabla\chi_0)\varphi_2)\, dx$$

Since $(\nabla[\chi_0\varphi_1], \nabla[\chi_0\varphi_2]) \in \ker B$, it follows

$$||(1-P_0)\chi_0\mathbf{u}||^2_{L^2(I\!R^3)} = ||\mathbf{g}||^2_{L^2(I\!R^3)} \tag{4.83}$$

$$\leq C_1||(1-P)\mathbf{u}||_X \left(||\nabla\varphi_1||_{L^2(I\!R^3)} + ||\varphi_1||_{L^6(I\!R^3)} + ||\nabla\varphi_2||_{L^2(I\!R^3)} + ||\varphi_1||_{L^6(I\!R^3)}\right)$$

$$+C_1||\mathbf{u}||_{L^2(B_{R_0})}\left(||\varphi_1||_{L^2(B_{R_0})} + ||\varphi_2||_{L^2(B_{R_0})}\right)$$

$$\leq C_2\left(||(1-P)\mathbf{u}||_X + ||\mathbf{u}||_{L^2(B_{R_0})}\right)||\mathbf{g}||_{L^2(I\!R^3)}.$$

This completes the proof of 4.81. Now, the assertions follow immediately from lemma 7, corollary 1, theorem 3 and inequality 4.81.

□

Lemma 10 *Suppose $g \in C_0^\infty(I\!R)$ with $g(u) = 1$ on a neighbourhood of $[0,1]$. Then*

$$t^{-1}\langle Sg(|x|/t)P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t)), P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))\rangle_{L^2(I\!R^3)} \stackrel{t\to\infty}{\longrightarrow} \mathcal{E}_\infty.$$

Recall that $\mathcal{E}_\infty = \lim_{t\to\infty} ||(\mathbf{E}(t), \mathbf{H}(t))||^2_X$ and $S\mathbf{u} \stackrel{\text{def}}{=} (-x\wedge\underline{\mathbf{u}}_2, x\wedge\underline{\mathbf{u}}_1)$.

Proof: Define

$$F(t) \stackrel{\text{def}}{=} \langle Sg(|x|/t)P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t)), P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))\rangle_{L^2(I\!R^3)}.$$

Then

$$F'(t) = 2\,\langle Sg(|x|/t)P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t)), \tag{4.84}$$

$$P_0\chi_0\left(B(\mathbf{E}(t),\mathbf{H}(t)) - (\mathbf{j}(t),0) - \mathcal{R}\partial_t\mathbf{P}(t)\right)\rangle_{L^2(I\!R^3)}$$

$$-t^{-2}\left\langle S|x|g'(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

$$= \sum_{j=0}^{2} h_j(t) + 2\left\langle Sg(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), B_0P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

$$-t^{-2}\left\langle S|x|g'(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

Here

$$h_0(t) \stackrel{\text{def}}{=} -2\left\langle Sg(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), P_0\chi_0(\mathbf{j}(t),0)\right\rangle_{L^2(I\!R^3)}, \tag{4.85}$$

$$h_1(t) \stackrel{\text{def}}{=} -2\left\langle Sg(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), P_0\chi_0\mathcal{R}\partial_t\mathbf{P}(t)\right\rangle_{L^2(I\!R^3)}, \tag{4.86}$$

$$h_2(t) \stackrel{\text{def}}{=} 2\left\langle Sg(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), P_0\mathcal{C}_0(\mathbf{E}(t),\mathbf{H}(t))\right\rangle_{L^2(I\!R^3)} \tag{4.87}$$

For $\mathbf{u} \in (\ker B_0)^\perp \cap D(B_0) \subset H^1(I\!R^3)$ one has $\operatorname{div} \underline{\mathbf{u}}_j = 0$. Therefore, it follows from the identity $x \wedge \operatorname{curl} \mathbf{a} = \nabla(x\mathbf{a}) - \mathbf{a} - (x\nabla)\mathbf{a}$ that

$$\left\langle Sg(|x|/t)\mathbf{u}, B_0\mathbf{v}\right\rangle_{L^2(I\!R^3)} + \left\langle Sg(|x|/t)B_0\mathbf{u}, \mathbf{v}\right\rangle_{L^2(I\!R^3)}$$

$$= \int_{I\!R^3} g(|x|/t)\mathbf{u}\cdot(x\wedge \operatorname{curl} \underline{\mathbf{v}}_1, x\wedge \operatorname{curl} \underline{\mathbf{v}}_2)dx$$

$$+ \int_{I\!R^3} g(|x|/t)(x\wedge \operatorname{curl} \underline{\mathbf{u}}_1, x\wedge \operatorname{curl} \underline{\mathbf{u}}_2)\cdot\mathbf{v}dx$$

$$= \int_{I\!R^3} g(|x|/t)\mathbf{u}\cdot(\nabla[x\underline{\mathbf{v}}_1], \nabla[x\underline{\mathbf{v}}_2])\,dx + \int_{I\!R^3} g(|x|/t)\,(\nabla[x\underline{\mathbf{u}}_1], \nabla[x\underline{\mathbf{u}}_2])\cdot\mathbf{v}dx$$

$$-2\int_{I\!R^3} g(|x|/t)\mathbf{u}\cdot\mathbf{v}dx - \int_{I\!R^3} g(|x|/t)(x\nabla)[\mathbf{u}\mathbf{v}]dx$$

$$= -2t^{-1}\left\langle \tilde{S}g'(|x|/t)\mathbf{u}, \mathbf{v}\right\rangle_{L^2(I\!R^3)} + \left\langle [g(|x|/t) + t^{-1}|x|g'(|x|/t)]\mathbf{u}, \mathbf{v}\right\rangle_{L^2(I\!R^3)}$$

with $\tilde{S}\mathbf{u} \stackrel{\text{def}}{=} |x|^{-1}\left([x\underline{\mathbf{u}}_1]x, [x\underline{\mathbf{u}}_2]x\right)$.
Hence

$$2\left\langle Sg(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), B_0P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

$$= \left\langle [g(|x|/t) + t^{-1}|x|g'(|x|/t)]P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

$$-2t^{-1}\left\langle \tilde{S}g'(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

With 4.84 -4.87 it follows

$$F'(t) = ||P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))||^2_{L^2(I\!R^3)} + \sum_{j=0}^{3} h_j(t) \tag{4.88}$$

where

$$h_3(t) \stackrel{\text{def}}{=} \Big\langle [g(|x|/t) - 1 + t^{-1}|x|g'(|x|/t)]P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t)), \tag{4.89}$$

$$P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))\rangle_{L^2(I\!R^3)}$$

$$-t^{-1}\left\langle (2\tilde{S} + t^{-1}|x|S)g'(|x|/t)P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t)), P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))\right\rangle_{L^2(I\!R^3)}$$

In the following estimates C_j are constants independent of s. Lemma 8 and 4.80 yield by Hölder's inequality and 4.51

$$|h_1(s)| \leq C_1||(1+|x|)^{-1/2}Sg(|x|/s)P_0\chi_0(\mathbf{E}(s), \mathbf{H}(s))||_{H^1(I\!R^3)}$$

$$||(1+|x|)^{1/2}\mathcal{R}\partial_s\mathbf{P}(s)||_{L^{q_0}(I\!R^3)}$$

$$\leq C_2||(1+|x|)^{1/2}g(|x|/s)P_0\chi_0(\mathbf{E}(s), \mathbf{H}(s))||_{H^1(I\!R^3)}$$

$$||(1+|x|)^{1/2}\gamma^{1/2}||_{L^{2r_0}(I\!R^3)}||\gamma^{-1/2}\partial_s\mathbf{P}(s)||_{L^2(G)}$$

$$= C_2 s^{1/2}||(1+|x|)\gamma||^{1/2}_{L^{r_0}(I\!R^3)}||\partial_s\mathbf{P}(s)||_{\mathcal{G}}$$

By lemma 1 i) one obtains for all $T > 0$

$$t^{-1}\int_1^t |h_1(s)|ds \leq t^{-1}\int_1^T |h_1(s)|ds + C_3 t^{-1}\int_T^t s^{1/2}||\partial_s\mathbf{P}(s)||_{\mathcal{G}}ds$$

$$\leq t^{-1}\int_1^T |h_1(s)|ds + C_3\left(\int_T^t ||\partial_s\mathbf{P}(s)||^2_{\mathcal{G}}ds\right)^{1/2}$$

and hence by Lemma 1

$$\limsup_{t\to\infty} t^{-1}\int_1^t |h_1(s)|ds \leq C_3\left(\int_T^\infty ||\partial_s\mathbf{P}(s)||^2_{\mathcal{G}}ds\right)^{1/2}$$

for all $T > 0$, which implies that

$$t^{-1}\int_1^t |h_1(s)|ds \stackrel{t\to\infty}{\longrightarrow} 0. \tag{4.90}$$

Next

$$|h_0(t)| \leq C_3||(1+|x|)^{-1}Sg(|x|/t)P_0\chi_0(\mathbf{E}(t), \mathbf{H}(t))||_{H^1}||(1+|x|)\chi_0(\mathbf{j}(t), 0)||_{L^{q_0}}$$

$$\leq C_4||\mathbf{j}(t)||_{L^2(B_{R_1})} \leq C_4||\mathbf{j}(t)||_{L^2(\Omega)}$$

by assumption 4.52 which implies that

$$t^{-1}\int_0^t |h_0(s)|ds \overset{t\to\infty}{\longrightarrow} 0. \tag{4.91}$$

Similarely

$$|h_2(t)| \leq C_5||(1+|x|)^{-1}Sg(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))||_{H^1}||(1+|x|)\mathcal{C}_0(\mathbf{E}(t),\mathbf{H}(t))||_{L^{q_0}}$$

$$\leq C_6||(\mathbf{E}(t),\mathbf{H}(t))||_{L^2(B_{R_0})}$$

and hence by corollary 1

$$t^{-1}\int_0^t |h_2(s)|ds \overset{t\to\infty}{\longrightarrow} 0. \tag{4.92}$$

Since $g'(|x|/t) = 0$ and $g(|x|/t) = 1$ for $|x| \leq at$ with some $a > 1$, lemma 9 i) yields

$$t^{-1}\int_0^t |h_3(s)|ds \leq C_5 t^{-1}\int_0^t ||P_0\chi_0(\mathbf{E}(s),\mathbf{H}(s))||_{L^2(\{|x|\geq as\})}ds \overset{t\to\infty}{\longrightarrow} 0. \tag{4.93}$$

Now, it follows from 4.88-4.93 and lemma 9 that

$$\lim_{t\to\infty} t^{-1}F(t) = \lim_{t\to\infty} t^{-1}\int_1^t F'(s)ds$$

$$= \lim_{t\to\infty} t^{-1}\int_1^t ||P_0\chi_0(\mathbf{E}(s),\mathbf{H}(s))||^2_{L^2(I\!R^3)}ds = \mathcal{E}_\infty.$$

This completes the proof.

□

Now the main results of this section 1.7 can be proved.

Theorem 4 *For all $a < 1$ and $b > 1$ one has*

$$||(\mathbf{E}(t),\mathbf{H}(t))||_{L^2(\Omega\cap B_{at})} \overset{t\to\infty}{\longrightarrow} 0 \tag{4.94}$$

and $||(\mathbf{E}(t),\mathbf{H}(t)) - t^{-1}S\chi_{\{at\leq|x|\leq bt\}}(\mathbf{E}(t),\mathbf{H}(t))||_X \overset{t\to\infty}{\longrightarrow} 0.$

Moreover

$$||(1-P_0)\chi_0(\mathbf{E}(t),\mathbf{H}(t))||_{L^2(I\!R^3)} \overset{t\to\infty}{\longrightarrow} 0.$$

Proof: Suppose $\delta > 0$. Choose $g \in C_0^\infty(I\!R, [0,\infty))$ with $g(y) = 1$ on $[0, 1+\delta/2]$ and $g(u) = 0$ for all $u > 1+\delta$. Then

$$||(1-P_0)\chi_0(\mathbf{E}(t),\mathbf{H}(t))||^2_{L^2(I\!R^3)}$$

$$= ||\chi_0(\mathbf{E}(t),\mathbf{H}(t))||^2_{L^2(I\!R^3)} - ||P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))||^2_{L^2(I\!R^3)}$$

$$\leq ||(\mathbf{E}(t),\mathbf{H}(t))||^2_X - ||P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))||^2_{L^2(I\!R^3)}$$

$$\leq ||(\mathbf{E}(t),\mathbf{H}(t))||^2_X - (1+\delta)^{-1}t^{-1}$$

$$\langle Sg(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\rangle^2_{L^2(I\!R^3)}$$

Lemma 10 yields

$$\limsup_{t\to\infty} ||(1-P_0)\chi_0(\mathbf{E}(t),\mathbf{H}(t))||^2_{L^2(I\!R^3)} \leq \left(1-(1+\delta)^{-1}\right)\mathcal{E}_\infty.$$

By letting $\delta \to 0$ this yields

$$\lim_{t\to\infty} ||(1-P_0)\chi_0(\mathbf{E}(t),\mathbf{H}(t))||^2_{L^2(I\!R^3)} = 0 \tag{4.95}$$

Now it follows from lemma 9 ii), theorem 3 lemma 10 and 4.95 that for all $\beta > 1$

$$\lim_{t\to\infty} t^{-1}\langle S\chi_{\{|x|\leq\beta t\}}(\mathbf{E}(t),\mathbf{H}(t)), (\mathbf{E}(t),\mathbf{H}(t))\rangle_X \tag{4.96}$$

$$= \lim_{t\to\infty} t^{-1}\langle Sg(|x|/t)(\mathbf{E}(t),\mathbf{H}(t)), (\mathbf{E}(t),\mathbf{H}(t))\rangle_X$$

$$= \lim_{t\to\infty} t^{-1}\langle Sg(|x|/t)P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)), P_0\chi_0(\mathbf{E}(t),\mathbf{H}(t))\rangle_{L^2(I\!R^3)} = \mathcal{E}_\infty.$$

Here $g \in C_0^\infty(I\!R, [0,\infty))$ with $g(y) = 1$ on $[0,\beta]$ and $g(u) = 0$ for all $u > 2\beta$ is chosen.

Let $\beta > 1$. Then one obtains from 4.96

$$\int_{\Omega\cap B_{at}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx \leq \int_{\Omega\cap B_{\beta t}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx$$

$$-\beta^{-1}t^{-1}\int_{\{at\leq|x|\leq\beta t\}} |x| \quad |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx$$

$$\leq ||(\mathbf{E}(t),\mathbf{H}(t))||^2_X - \int_{\{|x|\geq\beta t\}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx$$

$$-\beta^{-1}t^{-1}\int_{\Omega\cap B_{\beta t}} |x| \quad |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx + \beta^{-1}a\int_{\Omega\cap B_{at}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx$$

$$\leq ||(\mathbf{E}(t),\mathbf{H}(t))||^2_X - \int_{\{|x|\geq\beta t\}} |(\mathbf{E}(t),\mathbf{H}(t))|^2 dx$$

$$-\beta^{-1}t^{-1}\langle S\chi_{\{|x|\leq\beta t\}}(\mathbf{E}(t),\mathbf{H}(t)),(\mathbf{E}(t),\mathbf{H}(t))\rangle_X + a\int_{\Omega\cap B_{at}}|(\mathbf{E}(t),\mathbf{H}(t))|^2dx$$

Hence

$$(1-a)\int_{\Omega\cap B_{at}}|(\mathbf{E}(t),\mathbf{H}(t))|^2dx$$

$$\leq ||(\mathbf{E}(t),\mathbf{H}(t))||_X^2 - \int_{\{|x|\geq\beta t\}}|(\mathbf{E}(t),\mathbf{H}(t))|^2dx$$

$$-\beta^{-1}t^{-1}\langle S\chi_{\{|x|\leq\beta t\}}(\mathbf{E}(t),\mathbf{H}(t)),(\mathbf{E}(t),\mathbf{H}(t))\rangle_X$$

Next it follows from theorem 3 and 4.96 that

$$(1-a)\overline{\lim_{t\to\infty}}\int_{\Omega\cap B_{at}}|(\mathbf{E}(t),\mathbf{H}(t))|^2dx \leq (1-\beta^{-1})\mathcal{E}_\infty \text{ for all } \beta>1.$$

By letting $\beta\to 1$ this yields

$$||(\mathbf{E}(t),\mathbf{H}(t))||_{L^2(\Omega\cap B_{at})} \overset{\mathrm{t\to\infty}}{\longrightarrow} 0. \tag{4.97}$$

Together with lemma 9 this completes the proof of the first assertion 4.94.

Suppose $\beta>1$. Then it follows from theorem 3 that

$$\overline{\lim_{t\to\infty}}t^{-1}||S\chi_{\{|x|\leq bt\}}(\mathbf{E}(t),\mathbf{H}(t))||_X \leq \overline{\lim_{t\to\infty}}t^{-1}||S\chi_{\{|x|\leq\beta t\}}(\mathbf{E}(t),\mathbf{H}(t))||_X$$

$$\leq \beta\lim_{t\to\infty}||(\mathbf{E}(t),\mathbf{H}(t))||_X = \beta\mathcal{E}_\infty^{1/2}$$

Letting $\beta\to 1$ this yields

$$\overline{\lim_{t\to\infty}}t^{-1}||S\chi_{\{|x|\leq bt\}}(\mathbf{E}(t),\mathbf{H}(t))||_X \leq \mathcal{E}_\infty^{1/2}$$

By 4.96 one obtains

$$\overline{\lim_{t\to\infty}}||t^{-1}S\chi_{\{|x|\leq bt\}}(\mathbf{E}(t),\mathbf{H}(t)) - (\mathbf{E}(t),\mathbf{H}(t))||_X^2$$

$$= \overline{\lim_{t\to\infty}}\Big(t^{-2}||S\chi_{\{|x|\leq bt\}}(\mathbf{E}(t),\mathbf{H}(t))||_X^2$$

$$-2t^{-1}\Big\langle S\chi_{\{|x|\leq bt\}}(\mathbf{E}(t),\mathbf{H}(t)),(\mathbf{E}(t),\mathbf{H}(t))\Big\rangle_X + ||(\mathbf{E}(t),\mathbf{H}(t))||_X^2\Big) \leq 0.$$

Together with lemma 9 this completes the proof of the second assertion.

□

Next 1.9 is proved.

Theorem 5 *Suppose that in addition*

$$(1+|x|)^{1+\alpha_0}\gamma^{1/2} \in L^{r_1}(G) \tag{4.98}$$

for some $\alpha_0 > 0$ *and* $r_1 \in (3,\infty)$. *Then the strong limit*

$$\mathbf{U} \stackrel{\text{def}}{=} \lim_{t\to\infty} \exp(-t\mathcal{B}_0)J^*(\mathbf{E}(t),\mathbf{H}(t))$$

exists in $L^2(I\!R^3)$ *and* $\mathbf{U} \in (\mathit{ker}\ \mathcal{B}_0)^\perp$.

Here $J^* : L^2(\Omega) \to L^2(I\!R^3)$ provides the extension by zero on $I\!R^3 \setminus \Omega$. It follows from theorem 4 that for all $a < 1 < b$

$$\lim_{t\to\infty} ||t^{-1}S\chi_{\{at\le|x|\le bt\}}\mathcal{P}_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)) - J^*(\mathbf{E}(t),\mathbf{H}(t))||_{L^2} = 0. \tag{4.99}$$

Let g be defined by $g(t,u) \stackrel{\text{def}}{=} c_\alpha t^{-1-\alpha}u^\alpha$ for $u \le (1+\alpha)^{-1}\alpha t$ and $g(t,u) \stackrel{\text{def}}{=} u^{-1}$ for $u \ge (1+\alpha)^{-1}\alpha t$. Here $\alpha \stackrel{\text{def}}{=} \alpha_0/2 > 0$ with α_0 as in assumption 4.98 and $c_\alpha > 0$, such that g is continuous at $u = (1+\alpha)^{-1}\alpha t$.
Since $g(t,t) = t^{-1}$, it follows easily from 4.99, theorem 3 and 4 that

$$\lim_{t\to\infty} ||Sg(t,|x|)\mathcal{P}_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)) - J^*(\mathbf{E}(t),\mathbf{H}(t))||_{L^2} = 0. \tag{4.100}$$

Next corollary 1 and theorem 4 yield
$\lim_{t\to\infty} ||(1-\mathcal{P}_0)J^*(\mathbf{E}(t),\mathbf{H}(t))||_{L^2} = 0$, and hence with 4.100

$$\lim_{t\to\infty} ||\mathcal{P}_0Sg(t,|x|)\mathcal{P}_0\chi_0(\mathbf{E}(t),\mathbf{H}(t)) - J^*(\mathbf{E}(t),\mathbf{H}(t))||_{L^2} = 0. \tag{4.101}$$

Now, it suffices by 4.101 to prove the existence of the limit

$$\lim_{t\to\infty} \exp(-t\mathcal{B}_0)L(t)\chi_0(\mathbf{E}(t),\mathbf{H}(t)) \tag{4.102}$$

Here $L(t) \stackrel{\text{def}}{=} \mathcal{P}_0Sg(t,|x|)\mathcal{P}_0 \in B(L^2,L^2)$, where g is defined as above.
In [12] has been proved

Theorem 6 *Suppose* $\mathbf{u} \in L^\infty((0,\infty),D(B)) \cap C([0,\infty),X)$ *is a weak solution of* $\partial_t\mathbf{u} = B\mathbf{u} + \mathbf{f}$, *where* $\mathbf{f} \in L^1_{loc}([0,\infty),X)$ *obeys*
$(1+|x|))^{1+\alpha_0}\mathbf{f} \in L^1((0,\infty),L^{q_1}(\Omega)) + L^\infty((0,\infty),L^{q_1}(\Omega))$.
Then the strong limit

$$\lim_{t\to\infty} \exp(-t\mathcal{B}_0)L(t)\chi_0\mathbf{u}(t) \tag{4.103}$$

exists with respect to the $L^2(I\!R^3)$*-topology.*

Here $q_1 \in [6/5, 2)$ is defined by $1/q_1 = 1/2 + 1/r_1$, where $\alpha_0 > 0$ and $r_1 \in [3, \infty)$ are as in assumption 4.98.

Proof of theorem 5:
Let $\mathbf{u}(t) \stackrel{\text{def}}{=} (\mathbf{E}(t), \mathbf{H}(t))$ and $\mathbf{f}(t) \stackrel{\text{def}}{=} \partial_t \mathcal{R}\mathbf{P}(t) + (\mathbf{j}(t), 0)$. With 4.98, 4.52 and lemma 1 one has $(1 + |x|))^{1+\alpha_0}\mathbf{f} \in L^\infty((0,\infty), L^{q_1}(\Omega))$. Hence $\mathbf{u}$ satisfies the conditions of Theorem 4.103. By 4.101 one obtains the existence of the limit

$$\mathbf{U} \stackrel{\text{def}}{=} \lim_{t\to\infty} \exp(-t\mathcal{B}_0) J^*(\mathbf{E}(t), \mathbf{H}(t)) \text{ in } L^2(I\!R^3). \tag{4.104}$$

Since ran $\mathcal{P}_0 \subset$ ran $\mathcal{B}_0$, it follows from 4.101 and 4.104 that
$\mathbf{U} \in \overline{\text{ran } \mathcal{B}_0} = (\ker \mathcal{B}_0)^\perp$, i.e. div $(\underline{\mathbf{U}}_j) = 0$ on $I\!R^3$.

□

Now, it follows easily that $(\mathbf{F}, \mathbf{G}) \stackrel{\text{def}}{=} \exp(-t\mathcal{B}_0)\mathbf{U}$ satisfies equation 1.9 in the intrduction.

5 Convergence to stationary states

In this section it is assumed that $V(x,\cdot)$ is convex and that

$$|\mathbf{y}|^2 \le \gamma(x)g(x) + KV(x,\mathbf{y})] \text{ for all } x \in G, \mathbf{y} \in I\!R^3. \tag{5.105}$$

with some $g \in L^1(G)$ and $K \in (0,\infty)$ independent of $x, \mathbf{y}$. Lemma 1 and assumption 5.105 yield

$$\mathbf{P} \in L^\infty((0,\infty), \mathcal{G}) \subset L^\infty((0,\infty), L^2(G)). \tag{5.106}$$

In what follows let

$$\mathbf{a}_0 \stackrel{\text{def}}{=} (1-P)\left[(\mathbf{E}_0, \mathbf{H}_0) + \mathcal{R}\mathbf{P}(0) - \int_0^\infty (\mathbf{j}(s), 0)\right] ds$$

In contrast to the previous section it is not required that $\mathbf{a}_0 = 0$. In analogy to 4.49one has

$$(1-P)(\mathbf{E}(t), \mathbf{H}(t)) = \mathbf{a}_0 + (1-P)(\mathbf{J}(t) - \mathcal{R}\mathbf{P}(t)). \tag{5.107}$$

with $\mathbf{J}(t) \stackrel{\text{def}}{=} \int_t^\infty (\mathbf{j}(s), 0) ds$.

Theorem 7 *There exist* $(\mathbf{E}_\infty, \mathbf{H}_\infty) \in X_0$ *and* $\mathbf{P} \in \mathcal{G}$ *with*

$$(\mathbf{E}_\infty, \mathbf{H}_\infty) + (1-P)\mathcal{R}\mathbf{P}_\infty = \mathbf{a}_0 \tag{5.108}$$

and

$$(\nabla_P V)(x, \mathbf{P}_\infty(x)) = \gamma \mathbf{E}_\infty(x) \text{ on } G. \tag{5.109}$$

The functions $\mathbf{E}_\infty$ *and* $\mathbf{H}_\infty$ *are uniquely determined by 5.108 and 5.109. Moreover*

$$(\mathbf{E}(t), \mathbf{H}(t)) \stackrel{t\to\infty}{\longrightarrow} (\mathbf{E}_\infty, \mathbf{H}_\infty) \text{ in } X \text{ weakly.} \tag{5.110}$$

If in addition $V(x,\cdot)$ *is strictly convex then also*

$$\mathbf{P}(t) \stackrel{t\to\infty}{\longrightarrow} \mathbf{P}_\infty \text{ in } \mathcal{G} \text{ weakly.} \tag{5.111}$$

Proof:
By lemma 1 and 5.106 there exists a sequence $t_n \overset{\mathrm{n}\to\infty}{\longrightarrow} \infty$ with

$$(\mathbf{E}(t_n), \mathbf{H}(t_n)) \overset{\mathrm{n}\to\infty}{\longrightarrow} (\mathbf{E}_\infty, \mathbf{H}_\infty) \text{ in } X \text{ weakly.}$$

and

$$\mathbf{P}(t_n) \overset{\mathrm{n}\to\infty}{\longrightarrow} \mathbf{P}_\infty \text{ in } \mathcal{G} \text{ weakly.} \tag{5.112}$$

Lemma 2 yields

$$(\mathbf{E}(t_n), \mathbf{H}(t_n)) \overset{\mathrm{n}\to\infty}{\longrightarrow} (\mathbf{E}_\infty, \mathbf{H}_\infty) \in \ker B. \tag{5.113}$$

Suppose $T > 0$. Let $\mathbf{u}_n(t) \overset{\mathrm{def}}{=} (\mathbf{E}(t_n + t), \mathbf{H}(t_n + t)) \in X$ and $\mathbf{P}_n(t) \overset{\mathrm{def}}{=} \mathbf{P}(t_n + t) \in \mathcal{G}$ for $n \in I\!N$. Then

$$\mathbf{u}_n(t) = \exp(tB)\mathbf{u}_n(0) - \int_{t_n}^{t_n+t} \exp((t_n + t - s)B)\,(\partial_s \mathcal{R}\mathbf{P}(s) + (\mathbf{j}(s), 0))\, ds,$$

by 2.21, from which one obtains by lemma 1 iii) that

$$||\mathbf{u}_n(t) - \exp(tB)\mathbf{u}_n(0)||_X \le \int_{t_n}^{t_n+t} ||\mathcal{R}\partial_s\mathbf{P}(s)||_X + ||\mathbf{j}(s)||_{L^2} ds \overset{\mathrm{n}\to\infty}{\longrightarrow} 0$$

for all $t \in I\!R$ and hence by 5.113

$$(\mathbf{E}(t_n + t), \mathbf{H}(t_n + t)) = \mathbf{u}_n(t) \overset{\mathrm{n}\to\infty}{\longrightarrow} \exp(tB)(\mathbf{E}_\infty, \mathbf{H}_\infty) = (\mathbf{E}_\infty, \mathbf{H}_\infty) \tag{5.114}$$

in X weakly for all $t \in I\!R$. Since $\partial_t \mathbf{P} \in L^2((0,\infty), \mathcal{G})$ one has by 5.112

$$\mathbf{P}(t_n + t) \overset{\mathrm{n}\to\infty}{\longrightarrow} \mathbf{P}_\infty \text{ in } \mathcal{G} \text{ weakly for all } t \in I\!R. \tag{5.115}$$

Now, it follows from 5.114, 5.115 and 5.107 that

$$(\mathbf{E}_\infty, \mathbf{H}_\infty) + (1 - P)\mathcal{R}\mathbf{P}_\infty = \mathbf{a}_0. \tag{5.116}$$

Next it follows from 5.114 and 5.115 using $\partial_t \mathbf{P} \in L^2((0,\infty), \mathcal{G})$ as in the proof of Theorem 2 that

$$(\nabla_P V)(x, \mathbf{P}_n) \overset{\mathrm{n}\to\infty}{\longrightarrow} \gamma \mathbf{E}_\infty \text{ in } L^\infty((-T,T), \mathcal{G}) \text{ weak-}* \tag{5.117}$$

It remains to show 5.109. This does not follow directly from 5.115 and 5.117, since $(\nabla_P V)$ is nonlinear and the convergence in 5.115 is only weak. A monotonicty argument, [6, sect. 5A], will be applied using the convexity of V, i.e. the monotonicity of $\nabla_P V$ with respect to $\mathbf{P}$. Let $\chi \in C_0^\infty(-T,T)$ be a nonnegative function and define the monotone operator $\mathcal{T} : \mathcal{X} \to \mathcal{X}$ with $\mathcal{X} \overset{\mathrm{def}}{=} L^2((-T,T), \mathcal{G})$ by

$$(\mathcal{T}\mathbf{f})(t,x) \overset{\mathrm{def}}{=} \chi(t)(\nabla_P V)(x, \mathbf{f}(t,x)) \text{ for } \mathbf{f} \in \mathcal{X}.$$

Then 5.115 and 5.117 yield

$$\mathbf{P}_n \overset{\mathrm{n}\to\infty}{\longrightarrow} \mathbf{P}_\infty \text{ and } \mathcal{T}(\mathbf{P}_n) \overset{\mathrm{n}\to\infty}{\longrightarrow} \mathbf{Q}_\infty \text{ in } \mathcal{X} \text{ weakly,} \tag{5.118}$$

$$\text{where } \mathbf{Q}_\infty(t,x) \overset{\mathrm{def}}{=} \chi(t)\gamma(x)\mathbf{E}_\infty(x).$$

The aim of the following considerations is to show

$$\limsup_{n\to\infty}\langle\mathcal{T}(\mathbf{P}_n),\mathbf{P}_n\rangle_{\mathcal{X}} \leq \langle\mathbf{Q}_\infty,\mathbf{P}_\infty\rangle_{\mathcal{X}} \tag{5.119}$$

With 3.43 one obtains

$$\limsup_{n\to\infty}\langle\mathcal{T}(\mathbf{P}_n),\mathbf{P}_n\rangle_{\mathcal{X}} = \limsup_{n\to\infty}\int_{-T}^{T}\chi(t)\langle(\nabla_P)(x,\mathbf{P}_n(t)),\mathbf{P}_n(t)\rangle_{\mathcal{G}}dt \tag{5.120}$$

$$= \limsup_{n\to\infty}\int_{-T}^{T}\chi(t)\left\langle\left[\gamma\mathbf{E}_n(t)-\alpha\partial_t^2\mathbf{P}_n(t)-\partial_t\mathbf{P}_n(t)\right],\mathbf{P}_n(t)\right\rangle_{\mathcal{G}}dt$$

$$= \limsup_{n\to\infty}\int_{-T}^{T}\Big(\chi(t)\int_G\mathbf{E}_n\mathbf{P}_n dx + \chi'(t)\langle\mathbf{P}_n(t),\alpha\partial_t\mathbf{P}_n(t)\rangle_{\mathcal{G}}dt$$

$$+\chi(t)||\alpha^{-1/2}\partial_t\mathbf{P}_n(t)||_{\mathcal{G}}^2 - \chi(t)\langle\mathbf{P}_n(t),\partial_t\mathbf{P}_n(t)\rangle_{\mathcal{G}}\Big)\,dt$$

$$= \limsup_{n\to\infty}\int_{-T}^{T}\chi(t)\langle\mathbf{u}_n(t),\mathcal{R}\mathbf{P}_n(t)\rangle_X dt.$$

By 5.106 and lemma 6 one obtains

$$\int_{-T}^{T}\chi(t)\langle\mathbf{u}_n(t),P\mathcal{R}\mathbf{P}_n(t)\rangle_X dt. \tag{5.121}$$

$$\leq \int_{-T}^{T}\chi(t)\|\mathbf{P}_n(t)\|_{L^2(G)}\|P(\mathbf{E}(t_n+t),\mathbf{H}(t_n+t))\|_{L^2(G)}dt$$

$$\leq C_1\int_{-T}^{T}||P(\mathbf{E}(t_n+t),\mathbf{E}(t_n+t))||_{L^2(G)}dt \overset{\mathrm{n}\to\infty}{\longrightarrow} 0.$$

Now it follows from 5.107, 5.120 and 5.121 that

$$\limsup_{n\to\infty}\langle\mathcal{T}(\mathbf{P}_n),\mathbf{P}_n\rangle_{\mathcal{X}} \tag{5.122}$$

$$\leq \limsup_{n\to\infty}\int_{-T}^{T}\chi(t)\left\langle\mathbf{u}_n(t),(1-P)\mathcal{R}\mathbf{P}_n(t)\right\rangle_X dt$$

$$= \limsup_{n\to\infty}\int_{-T}^{T}\chi(t)\langle\mathbf{u}_n(t),\mathbf{a}_0+(1-P)\left(\mathbf{J}(t_n+t)-\mathbf{u}_n(t)\right)\rangle_X dt$$

$$= \limsup_{n\to\infty} \int_{-T}^{T} \chi(t) \left(\langle \mathbf{u}_n(t), \mathbf{a}_0 \rangle_X - ||(1-P)\mathbf{u}_n(t)||_X^2 \right) dt$$

Invoking 5.114 and 5.116 one obtains using the lower semicontinuity of $||\cdot||_X$

$$\limsup_{n\to\infty} \langle \mathcal{T}(\mathbf{P}_n), \mathbf{P}_n \rangle_{\mathcal{X}}$$

$$\leq \int_{-T}^{T} \chi(t) \left(\langle (\mathbf{E}_\infty, \mathbf{H}_\infty), \mathbf{a}_0 \rangle_X - ||(\mathbf{E}_\infty, \mathbf{H}_\infty)||_X^2 \right) dt$$

$$= \int_{-T}^{T} \chi(t) \langle (\mathbf{E}_\infty, \mathbf{H}_\infty), (1-P)\mathcal{R}\mathbf{P}_\infty \rangle_X dt$$

$$= \int_{-T}^{T} \chi(t) \langle (\mathbf{E}_\infty, \mathbf{H}_\infty), \mathcal{R}\mathbf{P}_\infty \rangle_X dt = \langle \mathbf{Q}_\infty, \mathbf{P}_\infty \rangle_{\mathcal{X}},$$

whence 5.119. By the monotony of $\mathcal{T}$ 5.118 and 5.119 yield

$$\mathcal{T}\mathbf{P}_\infty = \mathbf{Q}_\infty, \text{ i.e. } \chi(t)(\nabla_P V)(x, \mathbf{P}_\infty(x)) = \chi(t)\gamma(x)\mathbf{E}_\infty(x)$$

for all $t \in (-T, T)$ and $x \in G$, in particular it follows that the functions $\mathbf{E}_\infty$ and $\mathbf{P}_\infty$ obey also 5.109. In order to show that $(\mathbf{E}_\infty, \mathbf{H}_\infty) \in X_0$ are uniquely determined suppose that $(\mathbf{F}_\infty, \mathbf{G}_\infty) \in X_0$ and $\mathbf{Q} \in \mathcal{G}$ are further solutions of 5.108 and 5.109. Then

$$||(\mathbf{E}_\infty - \mathbf{F}_\infty, \mathbf{H}_\infty - \mathbf{G}_\infty)||_X^2 = -\langle (\mathbf{E}_\infty - \mathbf{F}_\infty, \mathbf{H}_\infty - \mathbf{G}_\infty), (1-P)\mathcal{R}(\mathbf{P}_\infty - \mathbf{Q}_\infty) \rangle_X$$

$$= -\langle (\mathbf{E}_\infty - \mathbf{F}_\infty, \mathbf{H}_\infty - \mathbf{G}_\infty), \mathcal{R}(\mathbf{P}_\infty - \mathbf{Q}_\infty) \rangle_X = -\int_G (\mathbf{E}_\infty - \mathbf{F}_\infty)(\mathbf{P}_\infty - \mathbf{Q}_\infty) dx$$

$$= -\int_G \gamma^{-1} \left[(\nabla_P V)(x, \mathbf{P}_\infty) - (\nabla_P V)(x, \mathbf{Q}_\infty) \right] \left[\mathbf{P}_\infty - \mathbf{Q}_\infty \right] dx \leq 0.$$

This completes the proof of uniqueness. In particular it follows that $(\mathbf{E}_\infty, \mathbf{H}_\infty)$ is the only possible accumulation point of $(\mathbf{E}, \mathbf{H})$, whence 5.110. The above proof shows also that $\mathbf{P}_\infty$ is uniquely determined if $(\nabla_P)(x, \cdot)$ is strictly monotone, from which 5.110 follows.

□

Now the assertion 1.14 can be proved.

Theorem 8 *For all $a < 1$ one has*

$$||(\mathbf{E}(t) - \mathbf{E}_\infty, \mathbf{H}(t) - \mathbf{H}_\infty)||_{L^2(\Omega \cap B_{at})} \overset{\mathrm{t}\to\infty}{\longrightarrow} 0 \tag{5.123}$$

in particular

$$(\mathbf{E}(t), \mathbf{H}(t)) \overset{\mathrm{t}\to\infty}{\longrightarrow} (\mathbf{E}_\infty, \mathbf{H}_\infty) \textit{ in } L^2_{loc}(\Omega).$$

Moreover the other assertions of Theorem 4 hold for $(\mathbf{E}(t) - \mathbf{E}_\infty, \mathbf{H}(t) - \mathbf{H}_\infty)$.

Proof:
Let $W(x,y) \stackrel{\text{def}}{=} V(x,y+\mathbf{P}_\infty(x))-V(x,\mathbf{P}_\infty(x))-y(\nabla_y V)(x,\mathbf{P}_\infty(x))$ for $x \in G$ and $y \in I\!R^3$. First some properties of W are summarized.

$$W(x,0)=0, \quad (\nabla_y W)(x,0)=0 \text{ for all } x \in G,$$

$$\|(\nabla_y W)(x,y)-(\nabla_y W)(x,z)\|$$

$$\leq \|(\nabla_y V)(x,y+\mathbf{P}_\infty(x))-(\nabla_y V)(x,z+\mathbf{P}_\infty(x))\| \leq L\|y-z\|$$

for all $x \in G, y, z \in I\!R^3$. Due to the convexity of V one has further

$$W(x,y) \geq 0, \quad y(\nabla_y W)(x,y)-W(x,y)$$

$$= V(x,z(x)-y)-[V(x,z(x))-y(\nabla_y V)(x,z(x))] \geq 0$$

for all $x \in G, y \in I\!R^3$, where $z(x) \stackrel{\text{def}}{=} y+\mathbf{P}_\infty(x)$. Hence W satisfies all the assumptions 2.15, 2.16 and 4.50. Now it follows from 5.109 that $\mathbf{F} \stackrel{\text{def}}{=} \mathbf{E}-\mathbf{E}_\infty$, $\mathbf{G} \stackrel{\text{def}}{=} \mathbf{H}-\mathbf{H}_\infty$, $\mathbf{Q} \stackrel{\text{def}}{=} \mathbf{P}-\mathbf{P}_\infty$ solve

$$\partial_t(\mathbf{F}(t),\mathbf{G}(t)) = B(\mathbf{F}(t),\mathbf{G}(t))-\mathcal{R}\partial_t\mathbf{Q}(t)-(\mathbf{j}(t),0) \tag{5.124}$$

and

$$\alpha\partial_t^2\mathbf{Q}(t)+\partial_t\mathbf{Q}(t)+(\nabla_y W)(x,\mathbf{Q}(t)) = \gamma\mathbf{F} \text{ on } (0,\infty)\times G \tag{5.125}$$

supplemented by the initial-conditions

$$(\mathbf{F}(0),\mathbf{G}(0),\mathbf{Q}(0),\partial_t\mathbf{Q}(0)) = (\mathbf{F}_0,\mathbf{G}_0,\mathbf{Q}_0,\mathbf{P}_1) \tag{5.126}$$

with $\mathbf{F}_0 \stackrel{\text{def}}{=} \mathbf{E}_0-\mathbf{E}_\infty$, $\mathbf{G}_0 \stackrel{\text{def}}{=} \mathbf{H}_0-\mathbf{H}_\infty$, and $\mathbf{Q}_0 \stackrel{\text{def}}{=} \mathbf{P}_0-\mathbf{P}_\infty$. By the definition of $\mathbf{a}_0$ and 5.108 these initial data obey

$$(1-P)\left[(\mathbf{F}_0,\mathbf{G}_0)+\mathcal{R}\mathbf{Q}_0-\int_0^\infty(\mathbf{j}(s),0)\right]$$

$$= \mathbf{a}_0-(\mathbf{E}_\infty,\mathbf{H}_\infty)-(1-P)\mathcal{R}\mathbf{P}_\infty = 0,$$

i.e. they satisfy the condition of the previous section, which completes the proof by Theorem 4.

□

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