Spectral inclusions for semigroups of closed operators

Peer Christian Kunstmann

Abstract

We show that several spectral inclusions known for C_0-semigroups fail for semigroups of closed operators, even if they can be regularized. We introduce the notion of spectral completeness for the regularizing operator C which implies equality of the spectrum and the C-spectrum of the generator. We prove spectral inclusions under this additional assumption. We give a series of examples in which the regularizing operator is spectrally complete including generators of integrated semigroups, of distribution semigroups, and of some semigroups that are strongly continuous for $t > 0$.

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1 Introduction

Suppose that (T_t) is a C_0-semigroup in a Banach space X with generator A. Then it is well known that the spectral inclusion

$$\exp(t\sigma(A)) \subset \sigma(T_t), \quad t > 0 \tag{1}$$

holds (see e.g. [3], Theorem 2.16). Here $\sigma(A) := \mathbb{C} \setminus \rho(A)$ is the spectrum of A and $\rho(A)$ denotes the resolvent set of A, i.e. the set of all complex λ such that $\lambda - A$ has an inverse in $L(X)$, the space of all linear bounded operators on X.

If (T_t) is just a semigroup of closed operators in X, i.e. $T_0 = Id_X$ and $T_sT_t \subset T_{s+t}$ for all $s, t \geq 0$, but is regularizable, the generator A of (T_t) is a closed linear operator in X (see section 2) and it is natural to ask whether (1) still holds.

In Theorem 1 of [5], the spectral inclusion (1) was claimed in the case that there is a regularizing operator C with dense range such that (T_tC) is exponentially bounded in $L(X)$.

However, since there are well known examples of semigroups of bounded operators that are strongly continuous for $t > 0$ whose generators A have empty resolvent, this claim is false in general. We give an example in 5.1 which also shows that the assertion of Theorem 4 in [5] is false in general.

Hence one needs additional assumptions for (1). In this paper we present conditions on the regularizing operator which are sufficient for (1) to hold. These conditions are mild enough to be satisfied in a number of examples such as for the generators of integrated semigroups, distribution semigroups and the generators of some semigroups that are strongly continuous for $t > 0$. The key ingredient is the notion of spectral completeness for A which we introduce for regularizing operators C. If C is spectrally complete for A then $\sigma(A)$ coincides with the C-spectrum $\sigma_C(A)$ of A. This allows to use arguments similar to the case of C_0-semigroups and then to switch from the C-spectrum to the spectrum. We conjecture that this can be done in a number of results known for C_0-semigroups.

As an example, we present an analogue of a result of G. Greiner (see [10], p.94) that characterized the resolvent set of the operators T_t in the case of a C_0-semigroup. From this result, which gives also conditions for the inclusion reverse to (1), we derive the spectral inclusion (1). Our results make no assumptions on exponential boundedness and we discuss the assumption of $\operatorname{im} C$ being dense.

The paper is organized as follows. Section 2 contains basic facts for semigroups of closed operators. In section 3 we introduce the notion of spectral completeness and investigate the relations between the spectrum and the C-spectrum of the generator A and the operators T_t. In Section 4 we prove the analogue of Greiner's result from which we derive a characterization of the resolvent set $\rho(T_t)$ in case of a semigroup that is strongly continuous for $t > 0$. This characterization seems to be new even for C_0-semigroups. Finally, section 5 contains our results on spectral inclusions and a discussion on the fine structure of the spectrum.

Throughout this paper X will denote a complex Banach space.

2 Semigroups of closed linear operators

Let $(T_t)_{t\geq 0}$ be a semigroup of closed linear operators in X, i.e. a family of closed linear operators satisfying $T_0 = Id_X$ and $T_sT_t \subset T_{s+t}$. Then (T_t) is a *semigroup of unbounded operators* in the sense of [6] if

$$D := \{x \in \bigcap_{s,t>0} D(T_sT_t) : t \mapsto T_tx \in C([0,\infty), X)\} \neq \{0\}.$$

Our main assumption on the semigroup is that (T_t) can be *regularized*, i.e. that there is an injective operator $C \in L(X)$ such that

$$\operatorname{im} C \subset D \text{ and } C^{-1}T_tC = T_t \text{ for all } t \geq 0. \tag{2}$$

We call such an operator C a *regularizing operator*. Then $(S_t) := (T_tC)$ defines a C-regularized semigroup, i.e. $S_0 = C$ is injective, $S_sS_t = CS_{s+t}$ for all $s, t \geq 0$, and $t \mapsto S_tx \in C([0,\infty), X)$ for all $x \in X$. This notion has been introduced in [2], for a survey see [4]. The existence of a regularizing operator implies $D \neq \{0\}$ and $D(T_sT_t) = D(T_{s+t}) \cap D(T_t)$ for $s, t \geq 0$. The construction of the *infinitesimal generator* in [6] is based on exponentially bounded trajectories which might not exist (see [6], 3.13). We proceed in a different way and define the operator A_0 via

$$(x, y) \in A_0 \iff x, y \in D \text{ and } \lim_{t\to 0}(T_tx - x)/t = y.$$

Observe that C commutes with A_0. In general the operator A_0 is not closed, and even its closure might not be the "right" operator to work with (see [8]). We define the *generator of* (T_t) to be the operator $A := C^{-1}A_0C$, i.e.

$$(x, y) \in A \iff (Cx, Cy) \in A_0.$$

Then A is also the generator of (S_t), i.e. $(x, y) \in A \Leftrightarrow \lim_{t\to 0}(S_tx - Cx)/t = Cy$. By [4] $A = C^{-1}AC$. The regularized semigroup (S_t) is the main technical ingredient in our proofs.

For any family M of closed operators in X we denote by $M^c := \{S \in L(X) : \forall B \in M : SB \subset BS\}$ the *commutant* of M and by $M^{cc} := (M^c)^c$ the *bicommutant* of M. We require in addition that $C \in \mathcal{T} := \{T_t : t \geq 0\}^{cc}$. Then the generator A is independent of the specific regularizing operator.

Lemma 2.1 *Suppose that $\tilde{C} \in L(X)$ is injective and satisfies* (2)*. Then $\tilde{C}^{-1}A_0\tilde{C} = A$.*

Proof. By $C \in \mathcal{T}$, $\tilde{C}$ and C commute. $(\tilde{S}_t) := (T_t\tilde{C})$ is a $\tilde{C}$-regularized semigroup with generator $\tilde{A}$. By [4], Proposition 3.10, $(\hat{S}_t) := (S_t\tilde{C}) = (\tilde{S}_tC)$ is a $C\tilde{C}$-regularized semigroup generated by A and by $\tilde{A}$, hence $A = \tilde{A}$. □

We define $\mathcal{S} := \{S_t : t \geq 0\}^{cc}$ and $\mathcal{A} := \{A\}^{cc}$. The set $\{S_t : t \geq 0\}$ consists of pairwise commuting elements, hence $\mathcal{S}$ is a commutative Banach algebra.

Proposition 2.2 *We have $\mathcal{A} \subset \mathcal{T} = \mathcal{S}$.*

Proof. If $BS_t = S_tB$ for all $t \geq 0$ then we get $BC = CB$ letting $t = 0$ and we get $BA \subset AB$ recalling

$$(x, y) \in A \Longleftrightarrow \forall t > 0 : S_tx - Cx = \int_0^t S_sy\, ds$$

(see [4]). If $CB = BC$ and $BA \subset AB$ then, for $x \in X$, $u : t \mapsto S_tx$ is the solution of $u(t) = A\int_0^t u(s)\, ds + Cx$, $t \geq 0$. Hence the assumptions imply that Bu is the solution of $v(t) = A\int_0^t v(s)\, ds + CBx$ which equals $t \mapsto S_tBx$ by uniqueness (see [4]). Thus we have $BS_t = S_tB$ for all $t \geq 0$. We have thus shown $\{S_t : t \geq 0\}^c = \{B \in L(X) : BA \subset AB, CB = BC\}$.

If $BT_t \subset T_tB$ for all $t \geq 0$, then $BC = CB$ by $C \in \mathcal{T}$ which together implies $BS_t = S_tB$ for all $t \geq 0$. If $BS_t = S_tB$ for all $t \geq 0$, then $BT_t = BC^{-1}T_tC \subset C^{-1}BS_t = C^{-1}T_tCB = T_tB$. Hence $\mathcal{T} = \mathcal{S}$. $\square$

Proposition 2.2 implies that $\mathcal{S}$ is a Banach algebra that contains all resolvents of the operators T_t, $t \geq 0$. A *character* of $\mathcal{S}$ is a linear multiplicative functional $\chi : \mathcal{S} \to \mathbb{C}$ with $\chi(I) = 1$ (or, equivalently, with $\chi \not\equiv 0$). A character is automatically continuous, and we shall denote the set of all characters of $\mathcal{S}$ by $\mathcal{X}$. For all $S \in \mathcal{S}$ the spectrum $\sigma(S)$ of S in $L(X)$ coincides with the spectrum of S in $\mathcal{S}$ and by Gelfand theory with the set $\{\chi(S) : \chi \in \mathcal{X}\}$.

Remark 2.3 1. If $\rho(A) \neq \emptyset$ then $\mathcal{A} = \{R(\lambda, A) : \lambda \in \rho(A)\}^{cc}$ and the spectra of A in $\mathcal{A}$ and $\mathcal{S}$ coincide.

2. We have defined the generator A in terms of the semigroup (T_t) and the regularizing operator C. Now suppose that we are given a closed linear operator A in X. If there is an injective operator $C \in L(X)$ such that A generates a C-regularized semigroup (S_t) then $(T_t) := (C^{-1}S_t)$ defines a semigroup of closed linear operators in X which is regularizable by C. If C can be chosen in $\mathcal{A}$ then (T_t) is independent of the specific regularizing operator.

3. Examples for this are generators of m-times integrated semigroups where $C = R(\lambda, A)^m$ is possible for any $\lambda \in \rho(A)$, or (more generally) generators of distribution semigroups G where (in case that $[0, \infty) \subset \rho(A)$)

$$C_{b,\tau} := \exp(-\tau(-A)^b) := \frac{1}{2\pi i}\int_\Gamma \exp(-\tau(-\lambda)^b)R(\lambda, A)\, d\lambda. \tag{3}$$

is possible for any $b \in (0, 1)$, $\tau > 0$ (see [7], section 5). Here Γ is the boundary of a suitably chosen logarithmic region. In this case the semigroup (T_t) of closed operators is just $(G(\delta_t))$ (see [7]).

4. Further examples are the operators treated by R. Beals in [1] and other situations where the regularized semigroup is defined by a holomorphic functional calculus for the operator A (see [4]).

5. Observe that in all those situations where the regularized semigroup is constructed from the resolvents of A by a limit procedure in the strong topology the operators S_t belong to $\mathcal{A}$, hence $\mathcal{S} = \mathcal{A}$, and if $C \in \mathcal{T}$ we have equality in Proposition 2.2.

6. The proof of $\mathcal{S} = \mathcal{T}$ used the assumption $C \in \mathcal{T}$. In general we only have $\mathcal{T} \subset \mathcal{S}$. In the situation of 5. we thus have $\mathcal{T} \subset \mathcal{A}$, but the inclusion is strict in general. However, $\mathcal{T} = \mathcal{A}$ if $A = \overline{A_0}$ since $BT_t \subset T_tB$ for all $t > 0$ implies $BA_0 \subset A_0B$ for every $B \in L(X)$. In any case we see that $\mathcal{S}$ is the right algebra to work in.

3 Spectrum and C-spectrum

We recall the notions of C-spectrum and C-resolvent set (see [4]). Let $C \in L(X)$ be injective and A be a closed linear operator in X such that C commutes with A. The *C-resolvent set* $\rho_C(A)$ is defined as the set of all complex λ such that $\lambda - A$ is injective with $(\lambda - A)^{-1}C \in L(X)$ (or, equivalently, with $\operatorname{im} C \subset \operatorname{im}(\lambda - A)$), and the *$C$-spectrum* $\sigma_C(A)$ is $\mathbb{C} \setminus \rho_C(A)$. It is clear that $\sigma_C(A) \subset \sigma(A)$, and the examples 3.3 and 5.1 show that the inclusion is strict in general. If we denote the C-resolvent $(\lambda - A)^{-1}C$ by $R_C(\lambda, A)$ we have for all $\lambda, \mu \in \rho_C(A)$ the C-resolvent equation

$$C(R_C(\lambda, A) - R_C(\mu, A)) = (\mu - \lambda)R_C(\lambda, A)R_C(\mu, A) \tag{4}$$

and for $\lambda \in \rho(A)$ we have $R_C(\lambda, A) = R(\lambda, A)C = CR(\lambda, A)$.

Given a closed linear operator A, we shall denote its point spectrum by $P\sigma(A)$, its residual spectrum by $R\sigma(A)$ and its continuous spectrum by $C\sigma(A)$.

We investigate the relation of $\sigma(A)$ and $\sigma_C(A)$ for the generator A of the semigroup (T_t) and shall need the following notion.

Definition 3.1 Let B be a closed linear operator in X whose resolvents belong to $\mathcal{S}$. We call an injective $C \in \mathcal{S}$ *spectrally complete for* B if

$$\forall z \in \sigma(B)\ \exists \chi \in \mathcal{X},\ \lambda \in \rho(B) : \chi(R(\lambda, B)) = \frac{1}{\lambda - z}, \chi(C) \neq 0. \tag{5}$$

Observe that (5) implies that $\rho(B) \neq \emptyset$.

Proposition 3.2 *Assume that the regularizing operator C is spectrally complete for A. Then $\sigma_C(A) = \sigma(A)$.*

Proof. The reverse inclusion being clear, let $z \in \sigma(A)$. By assumption, we find $\lambda \in \rho(A)$ and $\chi \in \mathcal{X}$ such that $\chi(R(\lambda, A)) = (\lambda - z)^{-1}$ and $\gamma := \chi(C) \neq 0$. For all $\mu \in \rho_C(A)$ we have by (4) and the injectivity of C

$$R(\lambda, A)C - R_C(\mu, A) = (\mu - \lambda)R(\lambda, A)R_C(\mu, A). \tag{6}$$

We apply χ, divide by γ, multiply with $\lambda - z$, and get

$$1 - (\lambda - z)\gamma^{-1}\chi(R_C(\mu, A)) = (\mu - \lambda)\gamma^{-1}\chi(R_C(\mu, A)), \tag{7}$$

from which $\chi(R_C(\mu, A)) = \gamma(\mu - z)^{-1}$. In particular, $\mu \neq z$, hence $z \notin \rho_C(A)$ and the proof is finished. $\square$

The next example shows how sensitive $\sigma_C(A)$ reacts to changes of C and that the condition (5) is sharp in a certain sense. It also shows that $\sigma_C(A)$ is in general not closed.

Example 3.3 Let A be the generator of a C_0-semigroup such that $\sigma(A) \neq P\sigma(A)$. Take $\mu \in \rho(A)$ and $w \in \sigma(A) \setminus P\sigma(A)$. Let $C := (w - A)R(\mu, A)$, then $C \in L(X)$ is injective, $C^{-1}AC = A$, $C \in \mathcal{T}$ and for all $\chi \in \mathcal{X}$ and $\lambda \in \rho(A)$ with $\chi(R(\lambda, A)) = (\lambda - w)^{-1}$ we have $\chi(C) = (w - w)(\mu - w)^{-1} = 0$. Hence (5) fails, but it only fails for $z = w$, and arguing as in the proof of Proposition 3.2 yields $\sigma_C(A) = \sigma(A) \setminus \{w\}$. The example may be extended by taking a finite subset $\{w_1, \ldots, w_n\}$ of $\sigma(A) \setminus P\sigma(A)$ and letting $C := (w_1 - A)(w_2 - A)\ldots(w_n - A)R(\mu, A)^n$. Then $\sigma_C(A) = \sigma(A) \setminus \{w_1, \ldots, w_n\}$.

The condition (5) looks very technical. But if $\rho(A) \neq \emptyset$, then for every $z \in \sigma(A)$ there is a $\chi \in \mathcal{X}$ such that for each $\lambda \in \rho(A)$ we have $\chi(R(\lambda, A)) = (\lambda - z)^{-1}$ (see [3]). Hence the condition (5) is implied by the following condition

$$\rho(A) \neq \emptyset \text{ and } \forall \chi \in \mathcal{X} :\ (\exists \lambda \in \rho(A) : \chi(R(\lambda, A)) \neq 0) \Rightarrow \chi(C) \neq 0. \tag{8}$$

Of course, (8) holds if $C = R(\mu, A)^n$ for some $\mu \in \rho(A)$ and some $n \in \mathbb{N}_0$, since $\chi(R(\lambda, A)) \neq 0$ for some $\lambda \in \rho(A)$ implies by the resolvent equation that $\chi(R(\lambda, A)) \neq 0$ for all $\lambda \in \rho(A)$. This is already an important example since A generates an $R(\mu, A)^n$-regularized semigroup for some n if and only if A generates an integrated semigroup (see [4]). Hence Proposition 3.2 contains the assertion of [12], Lemma 6.1 (a), as a special case. We give another concrete example which continues Remark 2.3 3.

Example 3.4 Let A be the generator of a distribution semigroup G with $[0, \infty) \subset \rho(A)$ and consider the operators given in (3). In section 5 of [7] it is shown that A generates a $C_{b,\tau}$-regularized semigroup for all those b and τ. By Remark 2.3 5. we have $\mathcal{S} = \mathcal{A}$, by Remark 2.3 6. we have $\mathcal{S} = \mathcal{T} = \mathcal{A}$ if G is dense. Since the integral in (3) is norm-convergent, any $\chi \in \mathcal{X}$ commutes with the integral. If $z \in \sigma(A)$ there is χ such that $\chi(R(\lambda, A)) = (\mu - z)^{-1}$ for all $z \in \rho(A)$, hence by Cauchy's theorem

$$\chi(C_{b,\tau}) = \frac{1}{2\pi i} \int_\Gamma \frac{\exp(-\tau(-\lambda)^b)}{\lambda - z}\, d\lambda = \exp(-\tau(-z)^b) \neq 0.$$

Thus for any b and τ, the operator $C_{b,\tau}$ is spectrally complete for A.

The same argument may be applied to the operators treated by R. Beals in [1] and in other situations where the semigroup is defined by an unbounded holomorphic functional calculus.

We now present sufficient conditions for the equality

$$\rho_C(T_t) \setminus \{0\} = \rho(T_t) \setminus \{0\} \tag{9}$$

which does not necessarily hold if C is spectrally complete for A.

Example 3.5 Let (T_t) be a C_0-semigroup with generator A and let $\lambda \in \rho(A)$, $C := R(\lambda, A)$. Then $(S_t) := (T_t C)$ is a C-regularized semigroup with generator A and C is spectrally complete for A. Now assume that the spectral mapping theorem does not hold for (T_t), i.e. that there is $\mu \in \sigma(T_t) \setminus (\exp(t\sigma(A)) \cup \{0\}$ (take, e.g., the example in [3], 2.17). Then any character χ with $\chi(R(w, T_t)) = (w - \mu)^{-1}$ for $w \in \rho(T_t)$ vanishes on the resolvents of A which implies $\chi(C) = 0$. Hence C is not spectrally complete for T_t.

Nevertheless, there is an important example where (9) holds. This is based on the following observation.

Lemma 3.6 *(a) Let $T \in L(X)$ be injective and $n \in \mathbb{N}$. Then $\rho_{T^n}(T) = \rho(T)$ if $T^{-1} \in L(X)$, and $\rho_{T^n}(T) = \rho(T) \cup \{0\}$ otherwise.*

(b) Let $(T(t))_{t>0}$ be a semigroup of injective operators in $L(X)$ and let $T_0 := I$. Then $\rho_{T(s)}(T(t)) \setminus \{0\} = \rho(T(t)) \setminus \{0\}$ for all $s, t > 0$.

Proof. (a): It is clear that $\rho(T) \subset \rho_{T^n}(T)$ and that $0 \in \rho_{T^n}(T)$. For any non-zero $\lambda \in \sigma(T)$ we find a character $\chi \in \mathcal{X}$ such that $\chi(T) = \lambda$. We now argue just as in the proof of Lemma 3.2 to obtain $\lambda \in \sigma_{T^n}(T)$.

(b): By (a) the assertion holds for $t > 0$ and $s \in t\mathbb{N}_0$. For an arbitrary $s > 0$ we find $n \in \mathbb{N}_0$ such that $nt < s \leq (n+1)t$. Now $\operatorname{im} T(nt) \supset \operatorname{im} T(s) \supset \operatorname{im} T((n+1)t)$ implies $\rho_{T(nt)}(T(t)) \subset \rho_{T(s)}(T_t) \subset \rho_{T((n+1)t)}(T(t))$ and the assertion follows. □

Remark 3.7 We will apply this lemma to semigroups $(T(t))$ that are strongly continuous for $t > 0$. If $T(s)$ is injective and A denotes the generator of $(T(t))$ as in [9] then $T(s)$ is a regularizing operator in the sense of our definition.

4 Extending a result of G. Greiner

The following lemma (whose proof can be found in [4]) is central to all considerations of spectral mapping.

Lemma 4.1 *For all $t > 0$ and $\lambda \in \mathbb{C}$ we have*

$$(\lambda - A)B_t(\lambda) = e^{\lambda t}C - S_t = (e^{\lambda t} - T_t)C, \tag{10}$$

where $B_t(\lambda) := \int_0^t \exp(\lambda(t-s))S_s\, ds$ commutes with $\lambda - A$.

G. Greiner characterized the values $\exp(\lambda t)$ that belong to $\rho(T_t)$ ([10], p.94) if (T_t) is a C_0-semigroup generated by A. We give an extension of his result to the case of semigroups of closed operators from which we shall derive our result on spectral inclusion.

Theorem 4.2 *For any $\lambda \in \mathbb{C}$ and $t > 0$ we have $(a) \Rightarrow (b) \Rightarrow (c)$ where*

(a) $\exp(\lambda t) \in \rho(T_t)$.

(b) *$\lambda + 2\pi i\mathbb{Z}/t \subset \rho_C(A)$ and for any $x \in X$ the sequence $1/N \sum_{n=1}^{N} \sum_{k=-n}^{n} R_C(\lambda + 2\pi ik/t, A)x$ converges.*

(c) $\exp(\lambda t) \in \rho_C(T_t)$.

If $\operatorname{im} C$ is dense in X then (b) is equivalent to

(d) *$\lambda + 2\pi i\mathbb{Z}/t \subset \rho_C(A)$ and the sequence $1/N \sum_{n=1}^{N} \sum_{k=-n}^{n} R_C(\lambda + 2\pi ik/t, A)$ is bounded in $L(X)$.*

Proof. As in [10], it is sufficient to treat the case $\lambda = 0$ and $t = 2\pi$, and we proceed along the lines of the proof given there using the regularized semigroup (S_t).

$(a) \Rightarrow (b)$: Let $1 \in \rho(T_{2\pi})$. For any $n \in \mathbb{Z}$ we have by (10) that $in \in \rho_C(A)$ with

$$R_C(in, A) = B_{2\pi}(in)(1 - T_{2\pi})^{-1} = \int_0^{2\pi} \exp(-ins) S_s (1 - T_{2\pi})^{-1} \, ds,$$

i.e. for any $x \in X$, the vector $R_C(in, A)x$ is a multiple of the n-th Fourier coefficient of the continuous function $f : s \mapsto S_s(1 - T_{2\pi})^{-1}x$. By Fejér's Theorem the Césaro means of the series $\sum_{n=-\infty}^{\infty} \int_0^{2\pi} \exp(-ins) f(s) \, ds$ converge.

$(b) \Rightarrow (c)$: Denoting for $x \in X$ the limit of the series in (b) by Qx we have that $Q \in L(X)$. By (10) we have the equality

$$(1 - T_{2\pi}) R_C(in, A)x = \int_0^{2\pi} \exp(-ins) S_s x \, ds,$$

which, by Fejér's Theorem applied to the continuous function $s \mapsto S_s x$, leads to

$$(1 - T_{2\pi})(C + \frac{1}{\pi} Q)x = (C - S_{2\pi})x + (S_{2\pi} + C)x = 2Cx.$$

Hence $1 \in \rho_C(T_{2\pi})$ and $R_C(1, T_{2\pi}) = (1/2)(C + (1/\pi)Q)$.

By Uniform Boundedness, the implication $(b) \Rightarrow (d)$ is clear. Now let $\operatorname{im} C$ be dense in X and assume (d). It is enough to show convergence on a dense subspace of X. According to Theorem 3 of [5] (see (17) in the present work) $D := \operatorname{im}(1 - T_{2\pi})$ is dense in X. For $x = (1 - T_{2\pi})y \in D$ we have

$$R_C(in, A)x = R_C(in, A)(1 - T_{2\pi})y = B_{2\pi}(in)y = \int_0^{2\pi} \exp(-ins) S_s y \, ds,$$

so that convergence follows from Fejér's Theorem applied to the continuous function $s \mapsto S_s y$. $\square$

Combining Theorem 4.2 and Proposition 3.2 yields the following.

Corollary 4.3 *Assume that C is spectrally complete for A. Let $t > 0$ and $\lambda \in \mathbb{C}$ then $(a) \Rightarrow (b') \Rightarrow (c)$, and if in addition $\operatorname{im} C$ is dense in X then $(b') \Leftrightarrow (d')$ where (a) and (c) are as in Theorem 4.2 and*

(b') *$\lambda + 2\pi i\mathbb{Z}/t \subset \rho(A)$ and for any $x \in X$ the sequence $1/N \sum_{n=1}^{N} \sum_{k=-n}^{n} R(\lambda + 2\pi ik/t, A)Cx$ converges.*

(d') *$\lambda + 2\pi i\mathbb{Z}/t \subset \rho(A)$ and the sequence $1/N \sum_{n=1}^{N} \sum_{k=-n}^{n} R(\lambda + 2\pi ik/t, A)C$ is bounded in $L(X)$.*

As we have seen, the equality $\rho(T_t) \setminus \{0\} = \rho_C(T_t) \setminus \{0\}$, which would imply the equivalence of (a), (b) and (c) in Theorem 4.2, needs additional assumptions. From Lemma 3.6 we get the following.

Corollary 4.4 *Let $(T(t))$ be a semigroup that is strongly continuous for $t > 0$, let A be its generator. Assume that $T(s)$ is injective for some (equivalently for all) $s > 0$. Then, for any $t > 0$ and $\lambda \in \mathbb{C}$, the assertions (a) and (b) with C replaced by $T(s)$ are equivalent.*

If $\operatorname{im} T(s)$ is dense in X then (b) is equivalent to (d).

If $T(s)$ is spectrally complete for A then $(b) \Leftrightarrow (b')$ and $(d) \Leftrightarrow (d')$.

Remark 4.5 1. Suppose, under the assumptions of Corollary 4.4, that there is a regularizing operator $C \in \mathcal{T}$ which is spectrally complete for A. For any $z \in \sigma(A)$ we can choose $\mu \in \rho(A)$ and $\chi \in \mathcal{X}$ such that $\chi(R(\mu, A)) = (\mu - z)^{-1}$ and $\chi(C) \neq 0$. The function $t \mapsto T(t)R(\mu, A)C$ is norm-continuous on $[0, \infty)$ and exponentially bounded. By the choice of χ this implies that $t \mapsto \chi(T(t))$ is continuous. For large $\lambda > 0$ we have

$$\chi(R_C(\lambda, A)R(\mu, A)) = \int_0^\infty \exp(-\lambda t)\chi(T(t)R(\mu, A)C)\, dt. \tag{11}$$

On the other hand, $\chi(R_C(\lambda, A)R(\mu, A)) = \chi(C)(\lambda - z)^{-1}(\mu - z)^{-1} \neq 0$. Hence $0 \neq \chi(T(t)R(\mu, A)C) = \chi(C)\chi(T(t))(\mu - z)^{-1}$ for some $t > 0$. Since $t \mapsto \chi(T(t))$ is continuous and satisfies $\chi(T(s+t)) = \chi(T(s))\chi(T(t))$ for any $s, t > 0$, this means $\chi(T(t)) \neq 0$ for all $t > 0$. Hence, in this case, $T(s)$ is spectrally complete for A for any $s > 0$.

2. The assumption in 1. is in particular satisfied if $(T(t))$ is a C_0-semigroup. The assertions of Corollary 4.4 seem to be new even in this case. More general, the assumption of 1. is satisfied for semigroups of Class $C_{(k)}$ (see [11]) which can be regularized by an iterated resolvent of their generator.

5 Spectral inclusions

We start with the example mentioned in the introduction which shows that (1) is false in general.

Example 5.1 Let $X := l^2(\mathbb{N}) \times l^2(\mathbb{N})$, where $\mathbb{N}$ denote the positive integers, and $\beta > 2$. Define for $t > 0$ the operator T_t by multiplication with $\begin{pmatrix} \exp(-nt) & tn^\beta \exp(-nt) \\ 0 & \exp(-nt) \end{pmatrix}$. Then (T_t) is a semigroup in X which is strongly continuous for $t > 0$ (even norm-continuous), but since $\beta > 1$, (T_t) is not a C_0-semigroup. The generator A is given by multiplication with $\begin{pmatrix} -n & n^\beta \\ 0 & -n \end{pmatrix}$ with maximal domain. If we define C by multiplication with $\begin{pmatrix} n^{-\beta} & 0 \\ 0 & n^{-\beta} \end{pmatrix}$ then $S_t := T_t C$ is a C-regularized semigroup with generator A.

It is clear that $P\sigma(A) = -\mathbb{N}$ and that $(\lambda - A)^{-1}$ is for $\lambda \notin -\mathbb{N}$ given by multiplication with $\begin{pmatrix} (\lambda+n)^{-1} & n^\beta(\lambda+n)^{-2} \\ 0 & (\lambda+n)^{-1} \end{pmatrix}$. But since $\beta > 2$ this operator is unbounded and has dense range, thus $\sigma(A) = \mathbb{C}$ and $C\sigma(A) = \mathbb{C} \setminus (-\mathbb{N})$.

Arguing similarly, we see $\sigma(T_t) = P\sigma(T_t) = \{\exp(-nt) : n \in \mathbb{N}\}$, hence (1) fails. Moreover, $0 \in C\sigma(A)$, $i\mathbb{Z} \cap (P\sigma(A) \cup R\sigma(A)) = \emptyset$, but $1 \notin C\sigma(T_{2\pi})$ since $1 \in \rho(T_{2\pi})$. This shows that the assertion of Theorem 4 in [5] is false in general.

The results of the previous section imply the following.

Corollary 5.2 1. *Let (T_t) be a semigroup of closed operators with regularizing operator C and generator A. Then*

$$\exp(t\sigma_C(A)) \subset \sigma(T_t), \quad t > 0. \tag{12}$$

2. *If, in addition, C is spectrally complete for A then*

$$\exp(t\sigma(A)) \subset \sigma(T_t), \quad t > 0. \tag{13}$$

Before turning to the fine structure of the spectrum, we will prove the reverse inclusion for a strongly continuous semigroup that is norm-continuous for $t > a$ for some $a > 0$. Of course, the result is well-known for C_0-semigroups.

Theorem 5.3 *Let $(T(t))$ be a semigroup that is strongly continuous for $t > 0$ and norm-continuous for $t > a$ for some $a > 0$. Let A be its generator and suppose that there is a regularizing operator $C \in L(X)$, spectrally complete for A. Then*

$$\exp(t\sigma(A)) = \sigma(T(t)) \setminus \{0\}, \quad t > 0. \tag{14}$$

Proof. One inclusion being clear from Corollary 5.2 2., let $0 \neq \mu \in \sigma(T(t))$. We find $\chi \in \mathcal{X}$ such that $\chi(T(t)) = \mu$. Replacing t by tn and μ by μ^n with an integer n if necessary we can without loss of generality assume that $t > a$. Now the function $\gamma : (0,\infty) \to \mathbb{C}, s \mapsto \chi(T(s))$ satisfies $\gamma(s+r) = \gamma(s)\gamma(r)$ for all $s, r > 0$, is continuous for $s > a$, and $\gamma(t) \neq 0$. Hence $\gamma(s) = \gamma(s+t)/\gamma(t) \to 1$ for $s \to 0+$, γ is continuous, and there is $w \in \mathbb{C}$ such that $\gamma(s) = \exp(sw)$ for all $s > 0$. It is well known that $\|T(\cdot)\|$ is exponentially bounded, and hence $\lambda > 0$ large enough belongs to $\rho_C(A)$ (see [4]), i.e. to $\rho(A)$ by Proposition 3.2. For any $s > 0$ and $\lambda > 0$ sufficiently large we thus have

$$\chi(R(\lambda, A))e^{ws} = \chi(R(\lambda, A)T(s)) = \int_0^\infty e^{-\lambda r + w(r+s)}\, dr = (\lambda - w)^{-1}\, e^{ws},$$

which means $w \in \sigma(A)$. □

In Theorem 2 of [5] the spectral mapping theorem for the point spectrum $P\sigma$ is proved, namely

$$\exp(tP\sigma(A)) = P\sigma(T_t) \setminus \{0\}, \quad t > 0. \tag{15}$$

The proof given there makes use of the exponential boundedness of the regularized semigroup generated by A which is not necessary: The inclusion "$\subset$" is immediate from (10). For the reverse inclusion let $x \neq 0$ with $\exp(\lambda t)Cx = S_t x$. Since $s \mapsto \exp(-\lambda s)S_s x$ is a non-zero continuous function on $[0,t]$ it has a Fourier coefficient $a_n \neq 0$ for some integer n where

$$a_n = \frac{1}{t}\int_0^t \exp(-2\pi ins/t)\exp(-\lambda s)S_s x\, ds = \frac{\exp(-\lambda t)}{t} B_t(\lambda + 2\pi in/t)x. \tag{16}$$

From (10) we conclude $\lambda + 2\pi in/t \in P\sigma(A)$, and (15) is proved.

The residual spectrum $R\sigma(A) := \{\lambda : \overline{\operatorname{im} \lambda - A} \neq X\}$ was treated in Theorem 3 of [5] for exponentially bounded C-regularized semigroups with $\operatorname{im} C$ dense. It was proved that for $t > 0$

$$R\sigma(T_t) \setminus \{0\} \subset \exp(tR\sigma(A)) \subset R\sigma(T_t). \tag{17}$$

The proof given there applies to the non-exponentially bounded case as well, however, as the following two examples show, the assumption of $\operatorname{im} C$ being dense is essential for either inclusion in (17).

Example 5.4 Let $X := C[0,1]$ and $A := -d/dx$ with $D(A) := \{f \in C^1[0,1] : f(0) = 0\}$. Then $\rho(A) = \mathbb{C}$, and A generates the $-A^{-1}$-regularized semigroup (S_t) given by $S_t f(x) := \int_0^{x-t} f(s)\, ds$ if $1 \geq x \geq t \geq 0$ and $S_t f(x) := 0$ if $t > x$. It is easy to see that, for $t < 1$, $T_t f(x) = f(x-t)$ if $x \geq t$, $= 0$ otherwise and $D(T_t) = \{f \in X : f(0) = 0\} =: X_0$. Of course, $S_t = T_t = 0$ and $D(T_t) = X$ for $t \geq 1$. For $t < 1$ and $\mu \in \mathbb{C}$ we have $\mu - T_t : X_0 \to X_0$, $P\sigma(T_t) = \{0\}$ and $R\sigma(T_t) = \mathbb{C}$. Hence the first inclusion in (17) fails.

Example 5.5 Let $X := l^\infty \times l^\infty$ and consider the semigroup (T_t) of example 5.1 and the operators A and C in X. Then (T_t) is norm-continuous for $t > 0$, and $(T_t C)$ is a norm-continuous C-regularized semigroup generated by A. As before we have $\sigma(A) = \mathbb{C}$, $P\sigma(A) = -\mathbb{N}$ and $P\sigma(T_t) = \{\exp(-nt) : n \in \mathbb{N}\}$ for $t > 0$. Since each T_t is a bounded operator, $R\sigma(T_t)$ is bounded, but – as is easily seen – if $\lambda - A$ is injective, the domain of $(\lambda - A)^{-1}$ is not dense in X. Hence $R\sigma(A) = \mathbb{C}$, and the second inclusion in (17) fails.

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Mathematisches Institut I, Universität Karlsruhe
Englerstr. 2, D – 76128 Karlsruhe, Germany
e-mail: peer.kunstmann@math.uni-karlsruhe.de