Kernel estimates and L^p spectral independence of differential and integral operators

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Abstract

We prove L^p spectral independence for bounded integral operators whose kernels satisfy certain estimates. As a corollary, we prove L^p spectral independence for unbounded operators under the assumption that one of their resolvents satisfies these estimates. In particular, we reobtain L^p spectral independence for generators of C_0-semigroups satisfying upper heat kernel estimates.

Other applications include first order differential operators and integral operators on the half line.

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1 Introduction and main results

Let $\Omega \subset \mathbb{R}^d$ be open. Suppose that Q is a linear operator that extends to bounded operators Q_p on each $L^p(\Omega)$, $p \in [1, \infty[$. It is natural to ask if the spectrum $\sigma(Q_p)$ is independent of p. In general, the answer is negative (see [6], [13] and the discussion in section 4). Partial positive results have been obtained under additional assumptions.

If $\Omega = \mathbb{R}^d$ and the operator is the convolution with an $L^1(\mathbb{R}^d)$-function, or if, on $\Omega :=]0, \infty[$, it has a kernel $1_{[0,\infty[}(y)k(x-y)$ where $k \in L^1(\mathbb{R})$ then L^p spectral independence is known to hold (see [13], the latter result is due to M.G. Krein). In both cases the special form of the kernel allows to use Fourier transform techniques.

Another way to prove L^p spectral independence is to use properties of the spectrum. This has been done by Halberg and Taylor ([10]) in the case that all the operators Q_p are compact, and by Schaefer ([15]) under the assumption that the spectra $\sigma(Q_p)$ are 0-dimensional. The latter result has been extended to the Fredholm spectrum in [5].

The question of L^p spectral independence is also of considerable interest for unbounded operators. If the operators Q_p are resolvents $R(\mu, A_p)$ of a family of unbounded operators A_p (here μ is fixed), the p-invariance of $\sigma(Q_p)$ is equivalent to the p-invariance of the spectra $\sigma(A_p)$ (see e.g. [7], Lemma 2.11).

L^p spectral independence has been studied for certain elliptic differential operators on Ω in [11], [9], [3], [12], [14]. The proofs given there use the fact that the operators A_p under consideration generate consistent C_0-semigroups T_p on all L^p, $1 \le p < \infty$, which are given by kernels that satisfy an upper heat kernel estimate (see section 4.1). W. Arendt ([3]) introduced perturbations of the semigroup by exponential weights to obtain L^p spectral independence in case of a connected resolvent set $\rho(A_2)$. By a refinement of Arendt's method the author could remove this restriction ([14]).

In this paper we exploit the method of perturbation by weights further and derive L^p spectral independence for an unbounded operator A from kernel estimates for the resolvents of A which need not be a semigroup generator. We show that a suitable estimate for the kernel of a *single* resolvent operator of A is sufficient. Moreover, the resolvent estimate we need is weaker than the exponential decay which is implied by a heat kernel estimate on the semigroup (see section 4.1).

Since it is much more convenient to deal with bounded operators we state our main result on L^p spectral independence for bounded integral operators whose kernels satisfy a certain estimate. This has the advantage that we also cover certain integral operators which are not resolvents of unbounded operators. We then derive L^p spectral independence for unbounded operators from L^p spectral independence for one of their resolvents.

Precisely, we shall prove the following (for the definition of Ω-admissible functions which force g to have a certain decay see Definition 2.1).

Theorem 1.1 *Let Q be an integral operator on $L^2(\Omega)$ which is given by a measurable kernel $k : \Omega \times \Omega \to \mathbb{C}$ that satisfies*

$$|k(x,y)| \le g(x-y), \quad x, y \in \Omega, \tag{1}$$

where $g : \mathbb{R}^d \to \mathbb{R}$ is a measurable function satisfying

$$\exp(\rho(|\cdot|))g \in L^1(\mathbb{R}^d) \cap L^{p_0}(\mathbb{R}^d) \tag{2}$$

for some Ω-admissible function ρ and some $p_0 \in]1,\infty]$. Then Q extends to consistent operators Q_p on all $L^p(\Omega)$, $p \in [1,\infty]$, and the spectrum $\sigma(Q_p)$ does not depend on p.

Applying this theorem to the resolvents of unbounded operators we get the following.

Theorem 1.2 *Let $(A_p)_{p\in[1,\infty[}$ be a family of unbounded operators in $L^p(\Omega)$. Assume that there is $\mu \in \bigcap_p \rho(A_p)$ such that $(R(\mu, A_p))$ are consistent operators with kernel k satisfying* (1) *and* (2)*. Then $\sigma(A_p)$ does not depend on p.*

Observe that g can be assumed to be zero outside $\{x-y : x, y \in \Omega\}$. If the closed linear operator A is the generator of a C_0-semigroup in $L^2(\Omega)$ that satisfies an upper heat kernel estimate of order m, then, for μ sufficiently large, the kernel of $R(\mu, A)$ satisfies (1) and (2) with $\rho(t) := \alpha t$ for some $\alpha > 0$ and some $p_0 > 1$ depending on the order m and the dimension d (see section 4.1).

The paper is organized as follows. In section 2 we introduce Ω-admissible functions, discuss their properties and collect a few simple facts that we shall need in the proof of

Theorem 1.1 which is presented in section 3. Finally, in section 4 we apply our result to generators of semigroups satisfying heat kernel estimates, to first order differential operators and to certain integral operators on the real line. Moreover, we discuss the assumptions of Theorem 1.1 and propose further directions of research.

We use the following notation. If X is a Banach space and A is a closed linear operator in X which may be bounded or not, then $\mathcal{L}(X)$ denotes the space of all bounded linear operators on X, $\sigma(A)$ denotes the spectrum of A, i.e. the set of all $\lambda \in \mathbb{C}$ for which $\lambda - A$ is not invertible in $\mathcal{L}(X)$, $\rho(A) := \mathbb{C} \setminus \sigma(A)$ denotes the resolvent set, and $R(\lambda, A) := (\lambda - A)^{-1}$ denotes the resolvent for $\lambda \in \rho(A)$.

Throughout this work we will tacitly assume that functions on Ω are extended by zero outside Ω when considered as functions on $\mathbb{R}^d$.

2 Preliminaries

Definition 2.1 Let $\rho : [0, \infty[\to [0, \infty[$ be a non-decreasing function. We call ρ an Ω-*admissible function*, if ρ is subadditive, ρ is continuous at 0 with $\rho(0) = 0$, and $e^{-\rho(\varepsilon|\cdot|)} \in L^1(\tilde{\Omega})$ for all $\varepsilon \in (0, 1]$, where $\tilde{\Omega} := \{x - y : x, y \in \Omega\}$.

Remark 2.2 (a) If $\rho : [0, \infty[\to [0, \infty[$ is concave and $\rho(0) = 0$ then ρ is subadditive.

(b) The function $\rho_\alpha(t) := \alpha \log(1+t)$ where $\alpha \geq 0$ is non-decreasing and concave, hence subadditive. It is Ω-admissible for all Ω if $\alpha > d$. More general, if l is an integer between 1 and d then ρ_α is Ω-admissible for $\alpha > l$ if Ω is bounded in $d - l$ directions. If Ω is bounded then $\rho_0 = 0$ is Ω-admissible.

(c) A concave function $\rho : [0, \infty[\to [0, \infty[$ satisfying $\rho(0) = 0$ and $\lim_{t\to\infty} \log(t)/\rho(t) = 0$ is Ω-admissible for all dimensions d and all $\Omega \subset \mathbb{R}^d$. In particular, $\rho(t) := t$ defines such a function.

(d) If $\rho : [0, \infty[\to [0, \infty[$ is monotone and subadditive, we have for all $x, y, z \in \mathbb{R}^d$ and all $\varepsilon > 0$:

$$\exp(-\rho(\varepsilon|x - y|) \leq \exp(\rho(\varepsilon|x - z|) - \rho(\varepsilon|y - z|)) \leq \exp(\rho(\varepsilon|x - y|)), \tag{3}$$

which is implied by the triangle inequality.

Let E and F be Banach spaces such that there is a Hausdorff topological vector space G with $E \hookrightarrow G$ and $F \hookrightarrow G$ where "$\hookrightarrow$" denotes continuous embedding. Assume that $E \cap F$ is dense in both E and F and that $Q_E \in \mathcal{L}(E)$ and $Q_F \in \mathcal{L}(F)$ are consistent operators, i.e. that $Q_E x = Q_F x$ for all $x \in E \cap F$. It is well-known that, under these assumptions, consistency of the resolvents of Q_E and Q_F fails in general (see, e.g., the examples in [3]). In order to assure consistency of *all* resolvents we shall use the following lemma.

Lemma 2.3 *If $E \subset F$ then $R(\lambda, Q_E)$ and $R(\lambda, Q_F)$ are consistent for all $\lambda \in \rho(Q_E) \cap \rho(Q_F)$.*

Proof. By $E \subset F$ and consistency we have $Q_E \subset Q_F$, hence $\lambda - Q_E \subset \lambda - Q_F$. Hence $(\lambda - Q_E)^{-1} \subset (\lambda - Q_F)^{-1}$ which implies the assertion. □

Remark 2.4 The conclusion of the lemma holds under the weaker assumption $Q_E^k(E) \subset F$ for some integer $k \geq 0$ which can be proved just as Proposition 2.5.(b) in [3].

We will also use the following converse statement.

Lemma 2.5 *Let $\lambda \in \rho(Q_E)$ and assume that there is $R \in \mathcal{L}(F)$ which is consistent with $R(\lambda, Q_E)$. Then $\lambda \in \rho(Q_F)$ and $R(\lambda, Q_F) = R$.*

Proof. For $x \in E \cap F$ we have $(\lambda - Q_F)Rx = (\lambda - Q_E)R(\lambda, Q_E)x = x$ and $R(\lambda - Q_F)x = R(\lambda, Q_E)(\lambda - Q_E)x = x$. Since $E \cap F$ is dense in F the assertion follows. □

We recall Young's inequality and derive an estimate on iterated convolutions.

Proposition 2.6 *If $p, q, r \in [1, \infty]$ with $p^{-1} + q^{-1} = 1 + r^{-1}$ then*

$$\|f * g\|_r \leq \|f\|_p \|g\|_q, \quad f \in L^p(I\!R^d), g \in L^q(I\!R^d).$$

Corollary 2.7 *Suppose $f \in L^1(I\!R^d) \cap L^{p_0}(I\!R^d)$ for some $p_0 > 1$. Then*

$$\|f^{*(n+1)}\|_\infty \leq (\max(\|f\|_1, \|f\|_{p_0}))^{n+1} \tag{4}$$

for all $n \in I\!N_0$ with $n \geq (p_0 - 1)^{-1}$, where $f^{*1} := f$ and $f^{*(m+1)} := f^{*m} * f$ for all $m \in I\!N$.

Proof. We first show by induction that, for all $n \in I\!N_0$ with $p_0 \leq (n+1)/n$, we have

$$\|f^{*(n+1)}\|_r \leq \|f\|_{p_0}^{n+1}, \quad \text{where} \quad r = \frac{p_0}{n + 1 - np_0}. \tag{5}$$

This is clear for $n = 0$ (which means $(n+1)/n = \infty$). So let $n \in I\!N$ and suppose (5) for n. By Young's inequality we get

$$\|f^{*(n+2)}\|_r \leq \|f^{*(n+1)}\|_q \|f\|_{p_0}$$

for $q^{-1} + p_0^{-1} = 1 + r^{-1}$. Applying the assumption for $q^{-1} = (n + 1 - np_0)/p_0$ we get (5) for $n + 2$ instead of $n + 1$ and $r^{-1} = q^{-1} + p_0^{-1} - 1 = (n + 2 - (n+1)p_0)/p_0$ which proves the claim.

If $\frac{n+2}{n+1} < p_0 < \frac{n+1}{n}$ then (5) holds, and by Young we get $\|f^{*(n+2)}\|_\infty \leq \|f^{*(n+1)}\|_r \|f\|_q$ where $r^{-1} = (n+1-np_0)/p_0$ and $q^{-1} = 1 - r^{-1} = 1 - (n+1-np_0)/p_0 = (n+1)(p_0 - 1)/p_0$. We observe $q \in (1, p_0)$, hence $\|f\|_q \leq \max(\|f\|_1, \|f\|_{p_0})$, which together with $\|f^{*(n+k)}\|_\infty \leq \|f^{*(n+k-1)}\|_\infty \|f\|_1$ for $k > 2$ and (5) completes the proof of the lemma. □

We close this section by citing the well-known characterization of operators $L^1 \to L^\infty$.

Proposition 2.8 *The map $k \mapsto T_k$, where $T_k f(x) := \int k(x,y) f(y)\, dy$, defines an isometric isomorphism $L^\infty(\Omega \times \Omega) \to \mathcal{L}(L^1(\Omega), L^\infty(\Omega))$.*

3 Proof of the main result

Let us first show how to treat the case $p = \infty$. Suppose that we have shown the assertion for $p \in [1, \infty[$. Let Q be an operator satisfying the assumptions of Theorem 1.1. Then Q^* satisfies the assumptions of Theorem 1.1 and $\sigma((Q^*)_p)$ does not depend on $p \in [1, \infty[$. Since clearly $((Q^*)_1)^* = Q_\infty$, we have

$$\sigma(Q) = \sigma(Q^*) = \sigma((Q^*)_1) = \sigma(((Q^*)_1)^*) = \sigma(Q_\infty),$$

and the assertion holds for all $p \in [1, \infty]$. Hence we exclude the case $p = \infty$ in the following.

Suppose that the assumptions of Theorem 1.1 hold. First observe that by (1), (2), and Young, we have for each $p \in [1, \infty[$

$$Q_p(L^p(\Omega)) \subset L^q(\Omega), \tag{6}$$

where $q^{-1} := p_0^{-1} + p^{-1} - 1$ if the latter term is ≥ 0 and $q := \infty$ otherwise. In any case, we have $q > p$ and hence $L^q(\Omega) \neq L^p(\Omega)$ which implies $0 \in \sigma(Q_p)$.

We now fix $p \in [1, \infty[$ and $\lambda \in \rho(Q_p)$. By the above $\lambda \neq 0$.

For $\varepsilon \in [0, 1]$ and $z \in \Omega$ we define the weight function $w_{\varepsilon,z}$ by

$$w_{\varepsilon,z}(x) := \exp(\rho(\varepsilon|x - z|)), \quad x \in \Omega, \tag{7}$$

and consider the operator $Q_{p,\varepsilon,z}$ given by the kernel

$$k_{\varepsilon,z}(x, y) := w_{\varepsilon,z}(x)k(x, y)w_{\varepsilon,z}(y)^{-1}, \quad x, y \in \Omega. \tag{8}$$

From (1), (2) and (3) we see by Young's inequality that $Q_{p,\varepsilon,z}$ is bounded on $L^p(\Omega)$. We collect the properties on the operators $Q_{p,\varepsilon,z}$ we shall need in the following lemma.

Lemma 3.1 *(a) The operators $Q_{p,\varepsilon,z}$ converge in norm to Q_p for $\varepsilon \to 0$, uniformly in $z \in \Omega$.*

(b) There is an $\varepsilon_0 \in (0, 1)$ such that $\lambda \in \rho(Q_{p,\varepsilon,z})$ for all $\varepsilon \in [0, \varepsilon_0]$, $z \in \Omega$, and

$$\lim_{\varepsilon \to 0} \|R(\lambda, Q_{p,\varepsilon,z}) - R(\lambda, Q_p)\| = 0, \tag{9}$$

where the convergence is uniformly in $z \in \Omega$.

Proof. We observe that (3) gives

$$|w_{\varepsilon,z}(x)w_{\varepsilon,z}(y)^{-1} - 1| \leq \max(e^{\rho(\varepsilon|x-y|)} - 1, 1 - e^{-\rho(\varepsilon|x-y|)} = e^{\rho(\varepsilon|x-y|)} - 1 \tag{10}$$

since $\rho \geq 0$. For $f \in L^p(\Omega)$ we have by (8) and (10)

$$\begin{aligned} |Q_{p,\varepsilon,z}f - Q_pf| &\leq \int_\Omega |(w_{\varepsilon,z}(x)w_{\varepsilon,z}(y)^{-1} - 1)k(x, y)||f|(y)\, dy \\ &\leq \int_\Omega g(x - y)(e^{\rho(\varepsilon|x-y|)} - 1)|f|(y)\, dy \\ &= ((\exp(\rho(\varepsilon|\cdot|)) - 1)g) * |f|(x) =: h_\varepsilon * |f|(x) \end{aligned}$$

which implies $\|Q_{p,\varepsilon,z} - Q_p\| \leq \|h_\varepsilon\|_1$. Thus (a) follows by dominated convergence.

Now (b) follows from (a) since inversion is continuous in the open subset of invertible elements in $\mathcal{L}(L^p(\Omega))$. $\Box$

From the resolvent equation we get for all $n \in I\!N$

$$R(\lambda, Q_p) = \lambda^{-n} Q_p^n R(\lambda, Q_p) + \sum_{j=1}^{n} \lambda^{-j} Q_p^{j-1}. \tag{11}$$

We will use Lemma 2.5, and thus have to show that $R(\lambda, Q_p)$ is a bounded operator on $L^q(\Omega)$. (Thinking a bit on the assertion of Remark 2.4 and (6) or rather its iterated version, Lemma 3.2 below, one sees that the sufficient condition of Lemma 2.5 is also necessary for $\lambda \in \rho(Q_p)$ in our case).

By (1), (2) and Young's inequality, the second term in (11) defines a bounded operator on $L^q(\Omega)$. Thus, it rests to show L^q-boundedness of the first term. In a first step, we use Corollary 2.7 to prove the following lemma.

Lemma 3.2 *If $n_0 \geq (p_0 - 1)^{-1} + 1$ then, for $\varepsilon \in [0, 1]$ and $z \in \Omega$, $Q_{p,\varepsilon,z}^{n_0}$ extends to a bounded operator $L^1(\Omega) \to L^p(\Omega)$ and to a bounded operator $L^p(\Omega) \to L^\infty(\Omega)$, and the respective norms are uniformly bounded in ε and z.*

Proof. By (8), (3), and (1), $Q_{p,\varepsilon,z}^{n_0}$ is an integral operator with kernel m satisfying $|m(x, y)| \leq h^{*n_0}(x - y)$ where $h := \exp(\rho(|\cdot|))g$. Now $h \in L^1(I\!R^d)$ by (2), hence $h^{*n_0} \in L^1(I\!R^d)$. Since we have also $h \in L^{p_0}(I\!R^d)$, we get $h^{*n_0} \in L^\infty(I\!R^d)$ by (4) and the choice of n_0, hence $h^{*n_0} \in L^q(I\!R^d)$ for all $q \in [1, \infty]$. By Young's inequality this implies the assertion. $\Box$

Now we take $n_0 \geq (p_0 - 1)^{-1} + 1$ and consider (11) for $n := 2n_0$. We observe $Q_p^n R(\lambda, Q_p) = Q_p^{n_0} R(\lambda, Q_p) Q_p^{n_0}$ and get, applying Lemma 3.1 (b) and Lemma 3.2, that there is an $\varepsilon_0 \in (0, 1)$ such that

$$c := \sup_{\varepsilon \leq \varepsilon_0, z \in \Omega} \|Q_{p,\varepsilon,z}^{n_0} R(\lambda, Q_{p,\varepsilon,z}) Q_{p,\varepsilon,z}^{n_0}\|_{\mathcal{L}(L^1, L^\infty)} < \infty. \tag{12}$$

By Proposition 2.8 the operators $Q_{p,\varepsilon,z}^n R(\lambda, Q_{p,\varepsilon,z})$ ($\varepsilon \leq \varepsilon_0$, $z \in \Omega$) are given by kernels $m_{\varepsilon,z}$ satisfying $|m_{\varepsilon,z}(x, y)| \leq c$ for a.a. $x, y \in \Omega$. Now $m_{0,z}$ is independent of z and we write simply m_0. In the following we will show

$$m_{\varepsilon,z}(x, y) = w_{\varepsilon,z}(x) w_{\varepsilon,z}(y)^{-1} m_0(x, y), \quad \text{a.a. } x, y \in \Omega. \tag{13}$$

Having this established, we conclude from (12) taking $z = y$ that

$$|m_0(x, y)| \leq c \exp(-\rho(\varepsilon_0 |x - y|)),$$

which implies by the assumption on ρ that m_0, which is the kernel of $Q_p^n R(\lambda, Q_p)$, defines a bounded operator in $L^q(\Omega)$. Hence also $R(\lambda, Q_p)$ is bounded in $L^q(\Omega)$ and by Lemma 2.5 we finished the proof of Theorem 1.1.

For the proof of (13) we consider the operators $\tilde{Q}_{p,\varepsilon,z}$ given by the kernels $m_{\varepsilon,z}$ in $L^p_{\varepsilon,z} := L^p(\Omega, w^{-p}_{\varepsilon,z}(x)\,dx)$. Those are consistent with the respective operators $Q_{p,\varepsilon,z}$ in $L^p(\Omega)$ and are clearly similar to the operator Q_p in $L^p(\Omega)$ ($f \mapsto w^{-1}_{\varepsilon,z} f$ defines an isometric isomorphism $L^p_{\varepsilon,z} \to L^p(\Omega)$). Hence $\rho(\tilde{Q}_{p,\varepsilon,z}) = \rho(Q_p)$, and $\tilde{Q}^{n_0}_{p,\varepsilon,z} R(\lambda, \tilde{Q}_{p,\varepsilon,z}) \tilde{Q}^{n_0}_{p,\varepsilon,z}$ has the kernel $w_{\varepsilon,z}(x) w_{\varepsilon,z}(y)^{-1} m_0(x,y)$. But from $L^p(\Omega) \subset L^p_{\varepsilon,z}$ and Lemma 2.3 we get that $R(\lambda, Q_{p,\varepsilon,z})$ and $R(\lambda, \tilde{Q}_{p,\varepsilon,z})$ are consistent which proves (13).

4 Applications

4.1 Generators of semigroups satisfying heat kernel estimates

Suppose T is a C_0-semigroup generated by an operator A in $L^2(\Omega)$ and T satisfies an upper heat kernel estimate, i.e. we assume that the operators $T(t)$ are integral operators with kernels $k(t,x,y)$ which for some integer $m > 1$ and some $C > 0$ satisfy

$$|k(t,x,y)| \leq C t^{-d/m} \exp(\omega t) \exp(-b \frac{|x-y|^{m/(m-1)}}{t^{1/(m-1)}}) \tag{14}$$

for all $t > 0$, and a.a. $x, y \in \Omega$ (see [12]). Then for $p \in [1, \infty[$ the family

$$T_p(t) f := T(t) f, \quad f \in L^2(\Omega) \cap L^p(\Omega),\ t \geq 0,$$

defines a C_0-semigroup in $L^p(\Omega)$ with generator A_p. The estimate (14) is known to hold e.g. for uniformly elliptic self adjoint operators ([8]), Schrödinger operators ([16]) and certain non-self adjoint elliptic operators ([4]). See also the applications mentioned in [3], [9], [12], and [14].

We let $\mu > \omega$ and get the kernel $k(x,y)$ of $Q_p := R(\mu, A_p)$ by integration $\int_0^\infty e^{-\mu t} k(t,x,y)\,dt$, and (14) implies (1) for

$$g_{\mu,b}(x) := C \int_0^\infty t^{-d/m} e^{-(\mu-\omega)t} \exp(-b \frac{|x|^{m/(m-1)}}{t^{1/(m-1)}})\,dt. \tag{15}$$

Using the continuous Minkowski inequality and substituting $x = t^{1/m} y$ we get $\|g_{\mu,b}\|_p < \infty$ for all $p \in [1, \infty]$ that satisfy $p^{-1} > (d-m)/d$. This means in particular that $g_{\mu,b}(x) < \infty$ for a.a. $x \in I\!R^d$. Observe that the range of p we obtained does not depend on b. For any $c \in (0, b)$ and $a > 0$ we have

$$\sup_{s \geq 0} as - ct^{-1/(m-1)} s^{m/(m-1)} = t \cdot \left(\frac{a}{m}\right)^m \left(\frac{m-1}{c}\right)^{m-1} =: t \cdot \gamma(a,m,c).$$

Thus, choosing $a > 0$ and $c \in (0, b)$ such that $\gamma(a,m,c) < \mu - \omega$, we have

$$|e^{a|x|} g_{\mu,b}(x)| \leq g_{\mu-\gamma(a,m,c),b-c}(x), \quad x \in I\!R^d$$

which by the argument above means that (2) holds for $\rho(r) := ar$ and $p_0 > 1$ such that $p_0^{-1} > (d-m)/d$. Hence we obtain from Theorem 1.2 the following.

Theorem 4.1 *If A is the generator of a C_0-semigroup T in $L^2(\Omega)$ that is given by kernels $k(t,x,y)$ satisfying (14), then the spectra $\sigma(A_p)$ of the generators A_p of the induced C_0-semigroups T_p in $L^p(\Omega)$ are independent of $p \in [1,\infty[$.*

This has been obtained for Schrödinger operators A in [11], for self-adjoint A in [9], for A such that $\rho(A)$ is connected in [3], under certain commutator estimates on the resolvents in [12], and in the general case with a different proof in [14].

We want to remark that the argument above applies to fractional powers. If $\mu > \omega$ and $\alpha > 0$ then the operator

$$(\mu - A)^{-\alpha} = \int_0^\infty \frac{t^{\alpha-1}}{\Gamma(\alpha)} e^{-\mu t} T_t \, dt$$

is the resolvent $R(0,(\mu - A)^\alpha)$ of the fractional power $(\mu - A)^\alpha$ and has a kernel which can be estimated against

$$g_{\mu,\alpha}(x) := C \int_0^\infty t^{\alpha-1-d/m} e^{-(\mu-\omega)t} \exp(-b\frac{|x|^{m/(m-1)}}{t^{1/(m-1)}}) \, dt.$$

By the way indicated above we get $\|g_{\mu,\alpha}\|_{p_0} < \infty$ for $1/p_0 > (d - \alpha m)/d$. Choosing a, c, ρ as above and applying Theorem 1.2, we get L^p spectral independence for the operator $(\mu - A)^\alpha$.

4.2 Generators of semigroups satisfying polynomial type bounds

We continue to suppose that T is a C_0-semigroup generated by the operator A on $L^2(\Omega)$ and that the kernels $k(t,x,y)$ of the operators $T(t)$ satisfy a certain estimate which is now given by

$$|k(t,x,y)| \leq t^{-d/m} h(\frac{|x-y|}{t^{1/m}}) \tag{16}$$

where $m > 0$ and h is a bounded, decreasing, continuous and strictly positive function satisfying

$$\lim_{r\to\infty} r^{2d+\delta} h(r) = 0 \tag{17}$$

for some $\delta > 0$. Let $\mu > 0$, then $R(\mu, A)$ has the kernel $\int_0^\infty \exp(-\mu t) k(t,x,y)$, and (16), (17) imply (1) for

$$g_\mu(x) := \int_0^\infty e^{-\mu t} t^{-d/m} h(\frac{|x|}{t^{1/m}}) \, dt.$$

By the continuous Minkowski inequality and the substitution $x = yt^{1/m}$ we get $\|g_\mu\|_p < \infty$ for $p^{-1} > (d-m)/d$ and $\|x \mapsto |x|^{d+\epsilon} g_\mu(x)\|_p < \infty$ for those p where $\epsilon := \delta/2$. Hence (2) is satisfied for $p_0 > 1$ with $p_0^{-1} > (d-m)/d$ and $\rho(r) := (d+\epsilon)\log(1+r)$.

Hence we obtain by Theorem 1.2 that $\sigma(A_p)$ does not depend on p. As in the previous subsection we can treat fractional powers $(\mu - A)^\alpha$ for $\mu > 0$, $\alpha > 0$.

4.3 First order differential operators

It is well known that L^p spectral independence fails for $S := x(d/dx)$ with maximal domain on $]0,\infty[$ (see [1] and [3]). To shed some light on this phenomenon, we consider in this subsection operators of the form $a(x)(d/dx)$ on $\Omega :=]0,\infty[$ with a degeneracy at $x = 0$. For notational reasons, we introduce $\psi := 1/a$. Precisely, we shall assume that ψ is a complex valued measurable function on $[0,\infty[$ such that

$$w(s) := \operatorname*{ess\,sup}_{x\geq s} |\psi(x)| < \infty \tag{18}$$

for all $s > 0$, and

$$w \in L^{p_0}(0,1) \quad \text{for some } p_0 > 1. \tag{19}$$

Observe that w is non-increasing. Then $\varphi(x) := \int_0^x \psi(y)\, dy$ is well defined for $x \geq 0$, bounded on finite intervals and absolutely continuous with $\varphi' = \psi$ almost everywhere. We require that there are constants $\gamma, c > 0$ such that

$$\operatorname{Re}(\varphi(x) - \varphi(y)) \geq \gamma(x - y) - c, \quad x \geq y \geq 0. \tag{20}$$

We will discuss these assumptions.

Remark 4.2 The conditions (18) and (19) are satisfied if $|\psi(x)| \leq C|x|^{-\alpha}$ for a.a. $x \in [0,1]$ for some $\alpha \in]0,1[$, $C > 0$, and ψ is essentially bounded on $[1,\infty[$. The first condition means that if $a(x) = 1/\psi(x) \to 0$ for $x \to 0$ then this convergence is not too fast. See the examples below. A sufficient condition for (20) is

$$\delta := \operatorname*{ess\,lim\,inf}_{x\to\infty} \operatorname{Re}(\psi(x)) > 0, \tag{21}$$

since then there is an $x_0 > 0$ such that $\operatorname{Re}\psi(x) \geq \gamma := \delta/2$ for a.a. $x \geq x_0$ and we get (20) for $c := \gamma x_0 + 2\operatorname{ess\,sup}_{0\leq\xi\leq x_0} |\varphi(\xi)|$. But (21) is not necessary for (20) as is shown in Remark 4.3 (a).

For $p \in [1,\infty]$ and $\lambda > 0$, we consider the operator $R_p(\lambda)$ on $L^p(0,\infty)$ which is given by the kernel

$$k_\lambda(x, y) := 1_{[x,\infty[}(y)\psi(y) \exp(-\lambda(\varphi(y) - \varphi(x))), \quad x, y \geq 0. \tag{22}$$

We then have the estimate

$$|k_\lambda(x, y)| \leq w(|x - y|) \exp(-\lambda\gamma|x - y| + \lambda c), \quad \text{a.a. } x, y \geq 0,$$

and letting $g := w(|\cdot|) \exp(-\lambda\gamma|\cdot|+\lambda c)$, $\rho(s) := \alpha s$ where $0 < \alpha < \lambda\gamma$ we clearly have that ρ is Ω-admissible and $\exp(\rho(|\cdot|)g \in L^1(\mathbb{R}) \cap L^{p_0}(\mathbb{R})$. Hence the assumptions of Theorem 1.1 hold and we conclude that the spectrum of $R_p(\lambda)$ does not depend on $p \in [1,\infty]$.

Moreover, if we define the operator A_p in $L^p(0,\infty)$ by $A_p f(x) := (1/\psi(x))f'(x)$ with domain $D(A_p) := \{f \in L^p(0,\infty) : (1/\psi)f' \in L^p(0,\infty)\}$, then a direct calculation shows that k_λ is for $\lambda > 0$ the kernel of the resolvent of A_p. Hence $R_p(\lambda) = R(\lambda, A_p)$ for $\lambda > 0$ and we conclude from Theorem 1.2 that $\sigma(A_p)$ is independent of $p \in [1,\infty]$.

Remark 4.3 (a) If we take $\psi(x) := 1 + \cos x$ then $a := 1/\psi$ has singularities at $(2k+1)\pi$ and we have $w = 1$, $\varphi(x) = x + \sin x$ which clearly satisfies (20) with $\gamma := 1$ and $c := 2$, but (21) does not hold.

(b) As shown above, singularities of a are not as critical for L^p spectral independence as zeros, since they correspond to zeros of ψ and the only condition where this would matter is (20).

(c) We chose the formulation of (20) for simplicity. Of course, L^p spectral independence holds if $\mu\varphi$ satisfies (20) for some $\mu \in \mathbb{C}$, since if the spectrum of $\mu a(x)(d/dx)$ is p-invariant then also the spectrum of $a(x)(d/dx)$ is p-invariant. Observe that (20) is the only point where the behaviour of the argument of a comes into play.

(d) If $\psi \in L^1(0,\infty)$ then $\varphi \in L^\infty(0,\infty)$ and the kernel k_λ in (22) can for all $\lambda \in \mathbb{C}$ be estimated by $|\psi|(y)\exp(2|\lambda|\|\varphi\|_\infty)$ which means that the operator $(1/\psi(x))(d/dx)$ has empty spectrum in each $L^p(0,\infty)$. Thus L^p spectral independence is trivial in this case. Observe that this means $|a(x)| \to \infty$ fast enough as $x \to \infty$.

Example 4.4 To give further illustrations, let us consider the case where $a(x)$ equals x^α for $x \in [0,1]$ and x^β for $x \geq 1$. We shall discuss several choices of $\alpha, \beta \in \mathbb{R}$. For $\alpha < 1$ and $\beta \neq 1$ we have $\psi = 1/a$ and $\varphi(x) := \int_0^x \psi(y)\,dy$ is given by $(1-\alpha)^{-1}x^{1-\alpha}$ for $x \in [0,1]$ and $(1-\beta)^{-1}(x^{1-\beta} - 1) + (1-\alpha)^{-1}$ for $x > 1$.

(a) If $\alpha < 1$ and $\beta \leq 0$ then (18), (19) and (21) hold. Hence we have L^p spectral independence in this case. For $\alpha < 0$ we have a singularity of a at the origin.

(b) For the case $\alpha = 1$, $\beta = 0$ see Example 4.5.

(c) If $\alpha < 1$ and $\beta > 1$ then $\psi \in L^1(0,\infty)$ and we are in the situation of Remark 4.3 (d).

(d) If $\alpha < 1$ and $0 < \beta < 1$ then (20) does not hold, but $R_p(\lambda)$ is bounded on $L^p(0,\infty)$ for $\operatorname{Re}\lambda > 0$, and $\exp(\lambda\varphi)$ is an eigenfunction in $L^p(0,\infty)$ to the eigenvalue λ for $\operatorname{Re}\lambda < 0$. Hence $\sigma(A_p) = i\mathbb{R} +]-\infty, 0]$ is p-independent in this case. Observe that A_p is the generator of the C_0-semigroup $T_p(t)$ in $L^p(0,\infty)$ which is given by

$$(T_p(t)f)(x) = f(\varphi^{-1}(\varphi(x) + t)), \quad x, t \geq 0, \tag{23}$$

where φ^{-1} denotes the inverse function of φ.

(e) If $\alpha < 1$ and $\beta = 1$ then $\varphi(x) = \log x + (1-\alpha)^{-1}$ for $x > 1$, and $\exp(\lambda\varphi)$ is an eigenfunction to the eigenvalue λ which belongs to $L^p(0,\infty)$ if and only if $\operatorname{Re}\lambda < -1/p$. On the other hand A_p generates the C_0-semigroup given by (23) which can be shown to be of type $\leq -1/p$. Hence $\sigma(A_p) = i\mathbb{R} +]-\infty, -1/p]$ depends on p in this case.

(f) If $\alpha = \beta = 1$ then $\sigma(S_p) = i\mathbb{R} - 1/p$ (see [1], [3]), hence L^p spectral independence fails. In this case $\psi(x) = 1/x = w(x)$ does not even belong to $L^1(0,1)$ and its primitive $\varphi(x) := \log x$ does not satisfy (20). We have $R(1,S)f(x) = (1/x)\int_0^x f(y)\,dy$, and this operator has been treated in [6]. The operator $(d/dx)x^2(d/dx) = S^2 + S$ whose resolvent has been treated in [13], Exercise 12.11.b) (see also Example 4.8 in the next subsection), has served as a counterexample in [11]. We again refer to Example 4.5 for a similar operator where L^p spectral independence fails.

For other values of α and β it seems not to be clear if $\sigma(A_p)$ depends on p. Besides the examples mentioned in (d), (e), and (f), D. Alboth has found other operators for which L^p spectral independence fails, among them $|x|(d/dx)$ on $I\!R$ and $(1-x^2)(d/dx)$ on $]-1,1[$ (see Examples 10 and 12 in [1]). We use the results and techniques of [1] to show that in Example 4.4 L^p spectral independence fails for $\alpha = 1$ and $\beta = 0$. So the bound $\alpha < 1$ in (a) is sharp.

Example 4.5 Consider the operator $a(x)(d/dx)$ on $]0,\infty[$ where $a(x) := x$ for $x \in [0,1]$ and $a(x) := 1$ otherwise which we define in $L^p(0,\infty)$ on $C_c^\infty(0,\infty)$. We claim that for the closure A_p of this operator we have $\sigma(A_p) = iI\!R + [-(1/p), 0]$. Hence L^p spectral independence fails.

Proof. Let $\rho : I\!R \to]0,\infty[$ be given by $\rho(x) = \exp(x)$ if $x < 0$ and $\rho(x) = x + 1$ if $x \geq 0$. Then ρ is a homeomorphism and $\omega := \rho^{-1}$ is given by $\omega(y) = \log y$ if $y \in]0,1[$ and $\omega(y) = y - 1$ if $y > 1$. Observe that $\omega'(y) = 1/y$ for $y < 1$ and $\omega'(y) = 1$ for $y > 1$ which means $\omega' = 1/a$, i.e. ω' corresponds to the function ψ above. Letting $\Phi(x,y) := \rho(\omega(x) + \omega(y))$ for $x, y > 0$, $(]0,\infty[, \Phi)$ is a locally compact abelian group which we shall call G. A Haar measure is given by $d\lambda_G(x) = \omega'(x)\, dx$. It is easy to see that $Tf(x) = a(x)(d/dx)$ is a translation invariant derivation with symbol $\xi \mapsto i\xi$, and hence Theorem 4 of [1] gives $\sigma(T_p) = iI\!R$ where T_p means the operator T considered in $L^p(G, d\lambda_G)$. By Lemma 5 of [1] the operator T_p is similar to the operator $S_p f(x) := a(x)f'(x) + (1/p)a'(x)f(x)$ in $L^p(0,\infty)$ by the mapping $f \mapsto \omega^{1/p} \cdot f$. Hence $\sigma(S_p) = iI\!R$. Observe that we have $a' = 1_{[0,1]}$ and $\sigma(A_p) = \sigma(\tilde{T}_p)$ where $\tilde{T}_p := T_p - (1/p)1_{[0,1]}$ is considered as an operator in $L^p(G, d\lambda_G)$.

We define operators $U_p := V_p \tilde{T}_p V_p^{-1}$ in $L^p(I\!R)$ using the isometry $V_p : L^p(G, d\lambda_G) \to L^p(I\!R), f \mapsto f \circ \omega$. Then $U_p f(x) = f'(x) - (1/p)1_{]-\infty,0]} f(x)$ and we are done if $\sigma(U_p + 1/(2p)) = iI\!R + [-(2p)^{-1}, (2p)^{-1}]$. We argue as in [1], Example 12. Let $W_p := U_p + (2p)^{-1} = (d/dx) + (2p)^{-1}\mathrm{sgn}$. d/dx generates a group of isometries, hence by the Hille-Yosida theorem $\|(z - (d/dx)^{-1}\| \leq 1/|\mathrm{Re} z|$. Since $\|\mathrm{sgn}\|_{\mathcal{L}(L^p(I\!R))} = (2p)^{-1}$ we have by a perturbation argument that any z with $|\mathrm{Re} z| > (2p)^{-1}$ belongs to $\rho(W_p)$. If, on the other hand, $|\mathrm{Re} z| < (2p)^{-1}$ then the function f given by $f(x) := \exp((z - \mathrm{sgn}(x)/(2p))x)$ is an eigenfunction in $L^p(I\!R)$ of W_p to the eigenvalue z, hence we proved the claim. □

We discuss an example on a bounded interval.

Example 4.6 If $\Omega :=]-1,1[$ and we consider $a(x)(d/dx)$ where $a(x) := (1-x^2)^\alpha$ and $\alpha < 1$ then, as in (d) of Remark 4.3, $\psi := 1/a \in L^1(\Omega)$, and φ given by $\varphi(x) := \int_{-1}^x \psi(y)\, dy$ is bounded. But now we have to consider boundary conditions.

If we take as domain $\{f \in L^p : a \cdot f' \in L^p, f(1) = 0\}$, resolvents are given by $g \mapsto \int_x^1 \psi(y) \exp(-\lambda(\varphi(y) - \varphi(x))g(y)\, dy$ for $\lambda \in \mathbb{C}$, if we take as domain $\{f \in L^p : a \cdot f' \in L^p, f(-1) = 0\}$ we have to replace $\int_x^1$ by $\int_{-1}^x$ and λ by $-\lambda$; in any case the spectrum of the operator will be empty in all $L^p(-1,1)$. This is also clear from semigroup theory since in both cases the resulting operator or its negative generates a nilpotent semigroup.

The situation changes if we consider periodic boundary conditions, i.e. if we take $\{f \in L^p : a \cdot f' \in L^p, f(-1) = f(1)\}$ as a domain. Then for $\lambda \notin (2\pi i)\mathbb{Z}/\varphi(1)$ the resolvent is given by

$$g \mapsto \exp(\lambda\varphi(x))(c(\lambda) + \int_x^1 \psi(y)\exp(-\lambda\psi(y))g(y)\ dy,$$
$$\text{where } c(\lambda) = (\exp(\lambda\varphi(1)) - 1)^{-1} \int_{-1}^1 \psi(y)\exp(-\lambda\varphi(y))g(y)\ dy,$$

and for $\lambda \in (2\pi i)\mathbb{Z}/\varphi(1)$ an eigenfunction is given by $\exp(\lambda\varphi)$. Hence the spectrum is $(2\pi i)\mathbb{Z}/\varphi(1)$ and independent of p.

The situation changes again if we let $a(x) = (1 - x^2)$. Then we don't need boundary conditions and Example 12 in [1] shows that the operator $a(x)(d/dx)$ in $L^p(-1, 1)$ has the spectrum $i\mathbb{R} + [-(2p)^{-1}, (2p)^{-1}]$, and L^p spectral independence fails. So again, in a certain sense, the bound $\alpha < 1$ is sharp.

In Example 4.4, 4.5 and 4.6 one should keep in mind that the arguments given there to compute spectra rely to a large extent on the fact that the function a takes real values only. For complex-valued a, the situation is less clear, relations like $\mathrm{Re}\,(\lambda\varphi) = \mathrm{Re}\,\lambda \cdot \varphi$ or (23) are no longer valid. Take, e.g., $a(x) = x^{-\alpha}s(x)$ for $x \in [0, 1]$ and $a(x) = s(x)$ for $x > 1$ where $s(x) = \exp(i\sin(\mu(x)))$ and μ is any measurable real-valued function. Then (18), (19), and (21) hold with $\delta = \cos 1$, hence the spectrum of A_p does not depend on p by Theorem 1.2, but it seems not so easy to compute it.

4.4 Examples of integral operators

It is known that L^p spectral independence holds for convolution operators on $\mathbb{R}^d$ (convolution with an $L^1(\mathbb{R}^d)$-function) and for Wiener-Hopf operators which are of the form

$$Kf(x) := \int_0^\infty k(x-y)f(y)\ dy$$

where $k \in L^1(\mathbb{R}^d)$. The latter result is due to M.G. Krein. The proof of both results relies on Fourier transform techniques and can be found in [13].

It is clear that from Theorem 1.1 we obtain L^p spectral independence for classes of operators whose kernels can be estimated against certain convolution operators. The following is a modification of [13], Exercise 10.6.c), p. 151, which has been given there for the space of continuous functions.

Example 4.7 Let $\Omega \subset \mathbb{R}^d$ be open and $k : \Omega \times \Omega \to \mathbb{C}$ be measurable. Assume that there are $\alpha, \gamma, c > 0$ such that

$$|k(x, y)| \leq c w_\alpha(|x-y|)e^{-\gamma|x-y|} \tag{24}$$

for a.a. $x \neq y \in \Omega$, where $w_\alpha(s) = s^{\alpha-d}$ if $\alpha \in (0, d)$, $w_d(s) = \log(1 + (1/s))$, and $w_\alpha(s) = 1$ for $\alpha > d$. Then Theorem 1.1 applies with $\rho(s) := (\gamma/2)s$, $g := c w_\alpha(|\cdot|)\exp(-2\rho(|\cdot|))$ and $1 < p_0 < d/(d-\alpha)$ if $\alpha < d$, $1 < p_0 < \infty$ if $\alpha = d$ and $p_0 = \infty$ if $\alpha > d$. In fact, we already used an estimate like (24) for the resolvents in Example 4.4.

We give another concrete example, taking $\Omega =]0,\infty[$. The operator S given by the kernel

$$k(x,y) := \exp(-|x-y|) + \exp(-(x+y)), \quad x,y>0,$$

appeared in Example 9.1. in [2]. We clearly have the estimate $|k(x,y)| \le 2\exp(-|x-y|)$ and Theorem 1.1 applies, hence the spectrum of S in $L^p(\Omega)$ is independent of $p \in [1,\infty]$.

By Remark 2.2 we don't need exponential decay in $|x-y|$, a certain polynomial decay in $|x-y|$ is already sufficient.

Example 4.8 Let $\Omega \subset \mathbb{R}^d$ be bounded in $d-l$ directions where l is an integer between 1 and d. Let k be a complex-valued measurable function and assume that there are $\alpha, c > 0$ and $\beta > 2l$ such that

$$|k(x,y)| \le c w_\alpha(|x-y|)(1+|x-y|)^{-\beta}$$

for a.a. $x \ne y \in \Omega$, where w_α is as before. Then Theorem 1.1 applies with $\rho(s) := (\beta/2)\log(1+s)$, $g := c w_\alpha(|\cdot|)\exp(-2\rho(|\cdot|))$, and p_0 as above, and we obtain that the spectrum of the operator T_p associated with k in $L^p(\Omega)$ is independent of $p \in [1,\infty]$.

Let us take $\Omega =]0,\infty[$ to give two concrete examples. The operator given by the kernel

$$k(x,y) := \frac{1}{1+x^\beta+y^\beta}, \quad x,y>0,$$

appeared for $\beta = 2$ in Example 7.1. in [2]. If $\beta > 2$ then the assumptions of Theorem 1.1 are satisfied, since we have $|k(x,y)| \le (1+|x-y|^\beta)^{-1}$. Thus we obtain that the spectrum of T in $L^p(\Omega)$ does not depend on $p \in [1,\infty]$.

Again, let $\beta > 2$ and $\alpha \in (0,1)$ and define $v, w : \Omega \to \mathbb{R}$ by $v(x) := x^\alpha$ if $x \le 1$, $v(x) := x^\beta$ if $x > 1$, and $w := v^{-2}$. We consider the operator T with kernel k given by

$$k(x,y) := 1_{]0,x]}(y)v(y)w(x) + 1_{]x,\infty[}(y)v(x)w(y), \quad x,y>0.$$

Since v is increasing, we have by the definition of w the estimate $|k(x,y)| \le 1/v(|x-y|)$ for all $x,y > 0$. We can apply Theorem 1.1 with $\rho(s) := (\beta/2)s$, $g := 1/v(|\cdot|)$ and $1 < p_0 < 1/\alpha$, and obtain that the spectrum of T in $L^p(\Omega)$ does not depend on $p \in [1,\infty]$.

We compare this with [13], Exercise 12.11.b), p.195, which showed that for $\alpha = \beta = 1$, i.e. for $v(x) = x$, L^p spectral independence fails. Elementary calculations show that we have for any $f \in L^p(\Omega)$

$$\left(-\frac{v^2}{3v'}(Tf)'\right)' + \frac{2v'}{3}(Tf) = f,$$

which gives the connection to Example 4.4 (f). Observe that in this case

$$|k(x,y)| \le \frac{1}{|x-y|} \quad \text{and} \quad |\nabla k(x,y)| \le \frac{C}{|x-y|^2}, \quad x \ne y.$$

Hence L^p spectral independence fails in general for Calderon-Zygmund operators of non-convolution type, even if they are bounded on L^1 and L^∞. See also sections 13.5 and 13.6 in [13] for examples derived from the Hilbert transform where L^p spectral independence fails.

We want to close this work with some problems.

Problems 4.9 (a) Does the assertion of Theorem 1.1 hold under the assumptions (1) and $g \in L^1(\mathbb{R}^d) \cap L^{p_0}(\mathbb{R}^d)$, i.e. (2) for $\rho = 0$? This would explain Example 4.4 (d).

(b) If the answer is yes, does it hold under the assumptions (1) and $g \in L^1(\mathbb{R}^d)$?

(c) Does an analogue of Theorem 1.1 hold $g(x-y)$ in (1) is replaced by $h(x,y)$ and h satisfies

$$x \mapsto \|h(x,\cdot)\|_r \in L^\infty(\Omega), \qquad y \mapsto \|h(\cdot,y)\|_r \in L^\infty(\Omega) \tag{25}$$

for $r \in \{1, p_0\}$ where $p_0 > 1$ plus some additional decay assumption? Observe that $v(x) = x$ in the last example shows that only assuming (25) for $r = 1$ is not sufficient for L^p spectral independence .

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