**A new proof for a Barbaşin's theorem in the periodic case**

by
Constantin BUŞE and Marcela GIURGIULESCU

**Abstract**

It is proved that a $q$-periodic evolutionary process $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ of bounded linear operators on a Banach space $X$ is uniformly exponentially stable if and only if the function $t \mapsto ||U(0,-t)||$ belongs to $L^1(\mathbf{R}_+)$. We would use this result for a new proof of a Barbaşin's theorem in this particular case.

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## 1. Introduction

Let $X$ be a Banach space (real or complex) and $L(X)$ the Banach algebra of all bounded linear operators acting on $X$. We denote by $||\cdot||$ the norms of vectors and operators. A strongly continuous evolutionary process on $X$ is a family of bounded linear operators $\mathcal{U} = \{U(t,s)\}_{t\geq s}$ on $X$ which satisfy the following three conditions:

**(i)** $U(t,s) = U(t,r)U(r,s)$ for all $t \geq r \geq s$;

**(ii)** $U(t,t) = Id, Id$ is the identity operator in $L(X)$, for all $t \in \mathbf{R}$;

**(iii)** for each $x \in X$ the maps $(t,s) \mapsto U(t,s)x$ are continous on the set $\{(t,s) : t \geq s\}$.

If, in addition, the family $\mathcal{U}$ satisfies the condition

**(iv)** there are $M \geq 1$ and $\omega \in \mathbf{R}$ such that

$$||U(t,s)|| \leq Me^{\omega(t-s)}, \quad \forall t \geq s,$$

then it is called strongly continuous evolutionary process with exponential growth on $X$.

Let $\mathcal{U}$ be a strongly continuous evolutionary process on $X$. If $U(t,s) = U(t-s,0)$ for all $t \geq s$ then the family $\mathcal{G} = \{U(t,0)\}_{t\in\mathbf{R}}$ is a $C_0$-group of bounded linear operators on $X$.

Let $q > 0$, fixed. A strongly continuous evolutionary process $\mathcal{U}$ on $X$ is called $q$-periodic evolutionary process if

**(v)** $U(t+q, s+q) = U(t, s)$ for all $t \geq s$.

It is easy to see that a $C_0$-group or a $q$-periodic evolutionary process are strongly continuous evolutionary process with exponential growth, see e.g. [5, Lemma 1].

**Examples. 1.** Let $A \in L(X)$ and $G(t) := e^{tA}$ for all $t \in \mathbf{R}$. The family $\mathcal{G} := \{G(t)\}_{t \in \mathbf{R}}$ is a $C_0$-group of bounded linear operators on $X$. Moreover, the function $t \mapsto e^{tA} : \mathbf{R} \to L(X)$ is continuous in the uniform topology of $L(X)$, so $\mathcal{G}$ is an uniformly continuous group of operators on $X$.

**2.** Let $A(t) \in L(X)$ for $t \in \mathbf{R}$ such that the function $t \mapsto A(t)$ is continuos and bounded. Let $P(\cdot)$ the solution of the operatorial Cauchy problem $\dot{X}(t) = A(t)X(t), X(0) = Id$. The family $\mathcal{U} = \{U(t, s) : t \geq s\}$, where $U(t, s) := P(t)P^{-1}(s)$ for $t \geq s$, is a strongly continuous evolutionary process with exponential growth.

**3** Let $A(\cdot)$ and $\mathcal{U}$ as in Example 2. If $A(t+q) = A(t)$ for all $t \in \mathbf{R}$ and some $q > 0$, then $\mathcal{U}$ is a $q$-periodic evolutionary process.

Let $\mathcal{U} = \{U(t, s) : t \geq s\}$ be a strongly continuous evolutionary process. We say that $\mathcal{U}$ is uniformly stable if there is $K > 0$ such that $||U(t, s)|| \leq K$ for all $t \geq s$. We also say that $\mathcal{U}$ is uniformly exponentially stable if there exist $N > 0$ and $\nu > 0$ such that

$$||U(t, s)|| \leq Ne^{-\nu(t-s)}, \quad \forall t \geq s.$$

It is not hard to see that the maps $\xi \mapsto ||U(t, \xi)||$ are measurable on $(-\infty, t]$ for every $t > 0$. In fact this follows by the strongly continuous property of $\mathcal{U}$ and definition of the operatorial norm.

**Theorem 1.** *A strongly continuous evolutionary process with exponential growth $\{U(t, s) : t \geq s\}$ on $X$ is uniformly exponentially stable if and only if*

$$\sup_{t>0} \int_{-\infty}^{t} ||U(t, \xi)|| d\xi = M < \infty. \tag{1}$$

For reasons of self-containedness we give the proof of Theorem 1 in some details. In fact this proof is an easy modification of the proof from [6, Theorem 3.1.2]. The following lemma would be used latter.

**Lemma 2.** *Let $\mathcal{U}$ be a strongly continuous evolutionary process with exponential growth on $X$. If there exists a function $g : \mathbf{R}_+ \to (0, \infty)$ with $\inf\{g(t) : t > 0\} < 1$ and $||U(t,s)|| \leq g(t-s)$, then $\mathcal{U}$ is uniformly exponentially stable.*

For the proof of Lemma 2, see e.g. [3, Lemma 4].

*Proof of Theorem 1.* Let $\omega > 0$ and $K \geq 1$ such that

$$||U(t,s)|| \leq Ke^{\omega(t-s)} \quad \forall t \geq s.$$

Firstly we prove that $\mathcal{U}$ is uniformly stable. Indeed, if it is not true, then for every $n = 1, 2, \cdots$, there are $t_n$ and $s_n$ with $||U(t_n, s_n)|| > n$. It follows that $n < Ke^{\omega(t_n - s_n)}$ so, $\lim\limits_{n\to\infty}(t_n - s_n) = \infty$. On the other hand for every $\xi \in [s_n, t_n]$, we have

$$n \leq ||U(t_n, s_n)|| \leq Ke^{\omega(\xi - s_n)}||U(t_n, \xi)||.$$

By integration on $[s_n, t_n]$ we obtain

$$n[1 - e^{-\omega(t_n - s_n)}] \leq KM\omega,$$

which is a contradiction. It results that there exists $K_1 > 0$ such that $||U(t,s)|| \leq K_1$ for all $t \geq s$. Moreover,

$$(t-s)||U(t,s)|| = \int_s^t ||U(t,s)|| \leq K_1 \int_s^t ||U(t,\xi)||d\xi \leq K_1 M,$$

so $(t - s + 1)||U(t,s)|| \leq K_1(M+1)$. Using Lemma 2 it follows that $\mathcal{U}$ is uniformly exponentially stable.

**Remark 3.** Seems to be an open question if the condition

$$\sup_{t>0} \int_{-\infty}^t ||U(t,\xi)x||d\xi < \infty, \quad \forall x \in X$$

implies the uniform exponentiall stability property of $\mathcal{U}$.

## 2. The periodic case.

Let $q > 0$, fixed, and $\mathcal{U}$ be a $q$-periodic evolutionary process on $X$. We would prove that if the function $\xi \mapsto ||U(0, -\xi)||$ belongs to $L^1(\mathbf{R}_+)$ then $\mathcal{U}$ is uniformly exponentially stable. More exactly, the following theorem holds.

**Theorem 4.** *A q-periodic evolutionary process* $\mathcal{U} = \{U(t, s) : t \geq s\}$ *on* $X$ *is uniformly exponentially stable if and only if*

$$\int_0^\infty ||U(0, -\xi)||d\xi < \infty. \tag{2}$$

*Proof:* A simple calculus give:

$$\int_0^\infty ||U(0, -\xi)||d\xi = \sum_{k=0}^\infty \int_{kq}^{(k+1)q} ||U(0, -\xi)||d\xi = \sum_{k=0}^\infty \int_0^q ||U(k + 1)q, r)||dr.$$

From Beppo-Levi's theorem if follows that there exists a null set $A_0 \subset [0, q]$ such that $\lim\limits_{k\to\infty} ||U((k+1)q, t)|| = 0$ for all $t \in [0, q]$ and $t \notin A_0$. Moreover, let $t_1 \in [0, q)$ and $t > t_1, t \in [0, q], t \notin A_0$. Then

$$||U((k + 1)q, t_1)|| \leq Ke^{\omega q}||U((k + 1)q, t)|| \to 0 \text{ as } k \to \infty$$

so, $\lim\limits_{k\to\infty} ||U((k + 1)q, s)|| = 0$ for all $s \in [0, q]$. Now, is easy to see that $\mathcal{U}$ is uniformly exponentially stable, see e.g. [5, Lemma 2].

**Remark 5.** *A q-periodic evolutionary process* $\mathcal{U} = \{U(t, s) : t \geq s\}$ *on* $X$ *is uniformly exponentially stable if and only if*

$$\int_0^\infty ||U(\xi, 0)x||d\xi < \infty, \quad \forall x \in X.$$

*Proof.* Let $\mathcal{U}_0$ the restriction of $\mathcal{U}$ to the set $\Delta := \{(t, s) : t \geq s \geq 0\}$. From [4, Theorem 2.3] it follows that $\mathcal{U}_0$ is uniformly exponentially stable. Let $N > 0$ and $\nu > 0$ such that $||U(t, s)|| \leq Ne^{-\nu(t-s)}$ for all $t \geq s \geq 0$. Then for each real numbers $\xi$ and $\eta$ with $\xi \geq \eta$, one has

$$||U(\xi, \eta)|| = ||U(\xi + nq, \eta + nq)|| \leq Ne^{-\nu(\xi-\eta)},$$

where $n \in \mathbf{N}$ is such that $\eta + nq \geq 0$, i.e. $\mathcal{U}$ is uniformly exponentially stable.

**Corollary 6.** *Let* $\mathcal{U}$ *be a q-periodic evolutionary process on* $X$*. If*

$$\sup_{t>0} \int_0^t ||U(t, \xi||d\xi < \infty,$$

*then $\mathcal{U}$ is uniformly exponentially stable.*

*Proof.* Let $L > 0$ be a constant such that

$$\sup_{n\in\mathbf{N}} \int_0^{nq} ||U(nq,\xi)||d\xi = L < \infty.$$

Then

$$\int_0^{nq} ||U(0,-\xi)||d\xi \leq L \text{ for all } n \in \mathbf{N}, \text{hence} \int_0^{\infty} ||U(0,-t)||dt < \infty.$$

Now, we apply Theorem 4 and the assertion in Corollary 6 it follows.

## 3. An application

Let the Cauchy Problem

$$\dot{x} = -(\sqrt{2} - \sin\ln(t+1) - \cos\ln(t+1))x(t), \quad x(0) = e^{-\sqrt{2}}, t \geq 0.$$

The solution of this Cauchy Problem is $x(t) = e^{-h(t)}$, where

$$h(t) := (t+1)(\sqrt{2} - \sin\ln(t+1)), \quad t \in \mathbf{R}_+.$$

Let $\mathcal{U}_0 := \{U(t,s) : t \geq s \geq 0\}$, where $U(t,s) := x(t)x^{-1}(s)$ for all $t \geq s \geq 0$. It is easy to show that $\mathcal{U}_0$ is a strongly continuous evolutionary process with exponential growth on $\mathbf{R}$. Moreover $\mathcal{U}_0$ is uniformly stable and exponentially stable (i.e. there are a constant $\nu > 0$ and a function $N : \mathbf{R}_+ \to (0,\infty)$ such that $||U(t,s)|| \leq N(s)e^{-\nu(t-s)}$ for all $t \geq s \geq 0$). However, $\mathcal{U}_0$ is not uniformly exponentially stable, see [2, Example 2.2]. We prove here using a variant of Theorem 1, that $\mathcal{U}_0$ is not uniformly exponentially stable.

Indeed from the Jensen inequality we obtain:

$$\int_0^t e^{h(s)}ds \geq te^{\frac{1}{t}\int_0^t h(s)ds} \qquad \forall t > 0.$$

But

$$\frac{1}{t}\int_0^t h(s)ds = h(t) - \frac{t}{2}\dot{h}(t) + \frac{1}{2t}\int_0^t s^2\ddot{h}(s)ds,$$

hence

$$e^{-h(t)}\int_0^t e^{h(s)}ds \geq te^{-\frac{t}{2}\dot{h}(t)+\frac{1}{2t}\int_0^t s^2\ddot{h}(s)ds} \tag{3}$$

Let $t_n := e^{2n\pi+\frac{\pi}{4}} - 1$ and $I_n := \int\limits_0^{t_n} s^2\ddot{h}(s)ds$. Then $\dot{h}(t_n) = 0$ and, in addition:

$$I_n = \int_0^{\ln(t_n+1)} (e^{2u} - 2e^u + 1)(\sin u - \cos u)du.$$

It is clear that

$$|\int_0^{\ln(1+t_n)} (\sin u - \cos u)du| \leq 4,$$

and

$$\int_0^{\ln(t_n+1)} e^u(\sin u - \cos u)du = 1 - (t_n+1)\cos\ln(t_n+1) = 1 - \frac{\sqrt{2}}{2}(t_n+1).$$

On the other hand

$$\int_0^{\ln(t_n+1)} e^{2u}(\sin u - \cos u)du = \frac{3}{5} - \frac{\sqrt{2}}{5}(t_n+1)^2 \geq \frac{3}{5}.$$

It follows:

$$I_n \geq -6 + \frac{3}{5} + \sqrt{2}(t_n+1).$$

Now, it is easy to see that there exists $N \in \mathbf{N}$ such that:

$$\frac{1}{2t_n}I_n > -1 \quad \forall n \geq N.$$

We return in (3) and obtain:

$$e^{-h(t_n)}\int_0^{t_n} e^{h(s)}ds \geq t_n \cdot e^{-1}, \quad \forall n \in \mathbf{N}.$$

Now it is clear that (1) holds and the desire result it follows from Theorem 1.

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