STUDENT NAME: 3rd Test.

Calculus 1550, section 6.
Thursday, March 25, 2004.

This test paper has 8 pages. Points per question are given in square brackets.

Please write your initials on each sheet.

1. Use implicit differentiation to find $\frac{dy}{dx}$.

i. [5 points]

$$x^2 + \cos(y) + 1 = \ln(x)$$

$$2x + y'(-\sin(y)) = \frac{1}{x}$$

$$-\sin(y)\, y' = \frac{1}{x} - 2x$$

$$y' = \left(\frac{-1}{x} + 2x\right) / \sin(y)$$

ii. [5 points]

$$(x^4 + y)x = 1$$

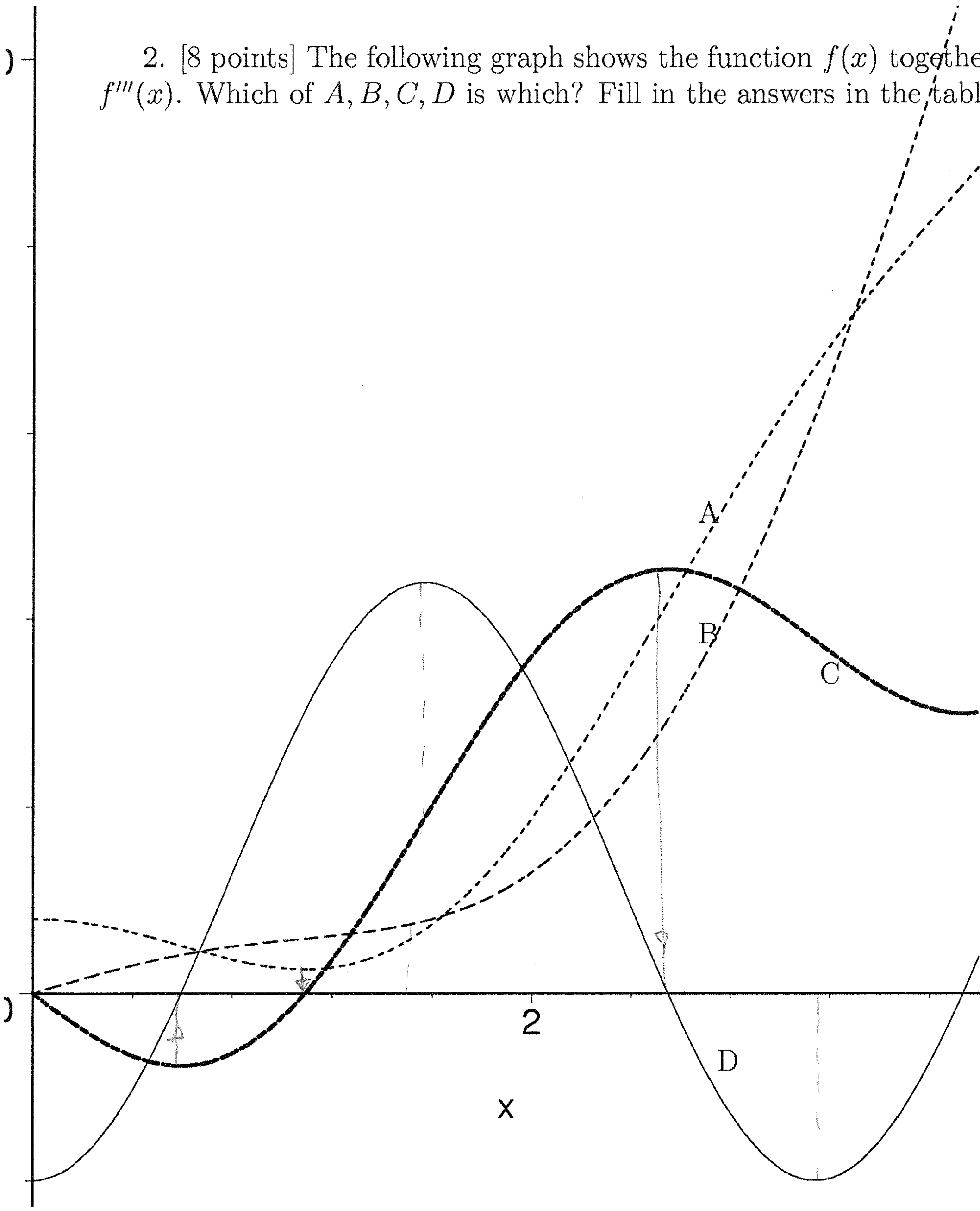
$$(4x^3 + y')x + x^4 + y = 0$$

$$(4x^3 + y')x = -(x^4 + y)$$

$$4x^3 + y' = \frac{-(x^4 + y)}{x}$$

$$y' = \frac{-(x^4 + y)}{x} - 4x^3$$

2. [8 points] The following graph shows the function $f(x)$ together with $f'(x)$, $f''(x)$ and $f'''(x)$. Which of A, B, C, D is which? Fill in the answers in the table.



f	B
f'	A
f''	C
f'''	D

3.i. [9 points] Find the <u>first</u>, <u>second</u> and <u>third</u> derivatives of $f(x) = \dfrac{x+1}{x-1}$.

$$f'(x) = \frac{(x-1)\times 1 - 1\times(x+1)}{(x-1)^2} = \frac{x-1-x-1}{(x-1)^2} = \frac{-2}{(x-1)^2}$$

$$= -2(x-1)^{-2}$$

$$f''(x) = -2\times -2\times (x-1)^{-3} = +4(x-1)^{-3}$$

$$f'''(x) = -2 \cdot -2 \cdot -3 \cdot (x-1)^{-4} = -12(x-1)^{-4}$$

$$f^{(4)}(x) = 12\times 4\,(x-1)^{-5} = 4!\,(x-1)^{-5}$$

$$f^{(5)}(x) = -12\times 4\times 5\,(x-1)^{-6} = -5!\,(x-1)^{-6}$$

$$f^{(6)} = -6\times(-5!)\,(x-1)^{-7} = 6!\,(x-1)^{-7}$$

3.ii. [2 point] What is $\dfrac{d^6\left(\frac{x+1}{x-1}\right)}{dx^6}$?

$$= 6!\,(x-1)^{-7} = \frac{6.5.4.3.2}{(x-1)^7}$$

4. Differentiate the following functions

i.[7 points] $f(x) = \sin(\ln(x))$

$$f'(x) = \frac{1}{x} \cos(\ln(x))$$

ii.[7 points] $f(x) = \ln(\sin^3(x))$

$$f'(x) = \frac{3\sin^2(x)\cos(x)}{\sin^3(x)} = \frac{3\cos(x)}{\sin(x)}$$

iii. [7 points] $f(x) = \tan(x)^x$

$$\ln f(x) = x \ln(\tan(x))$$

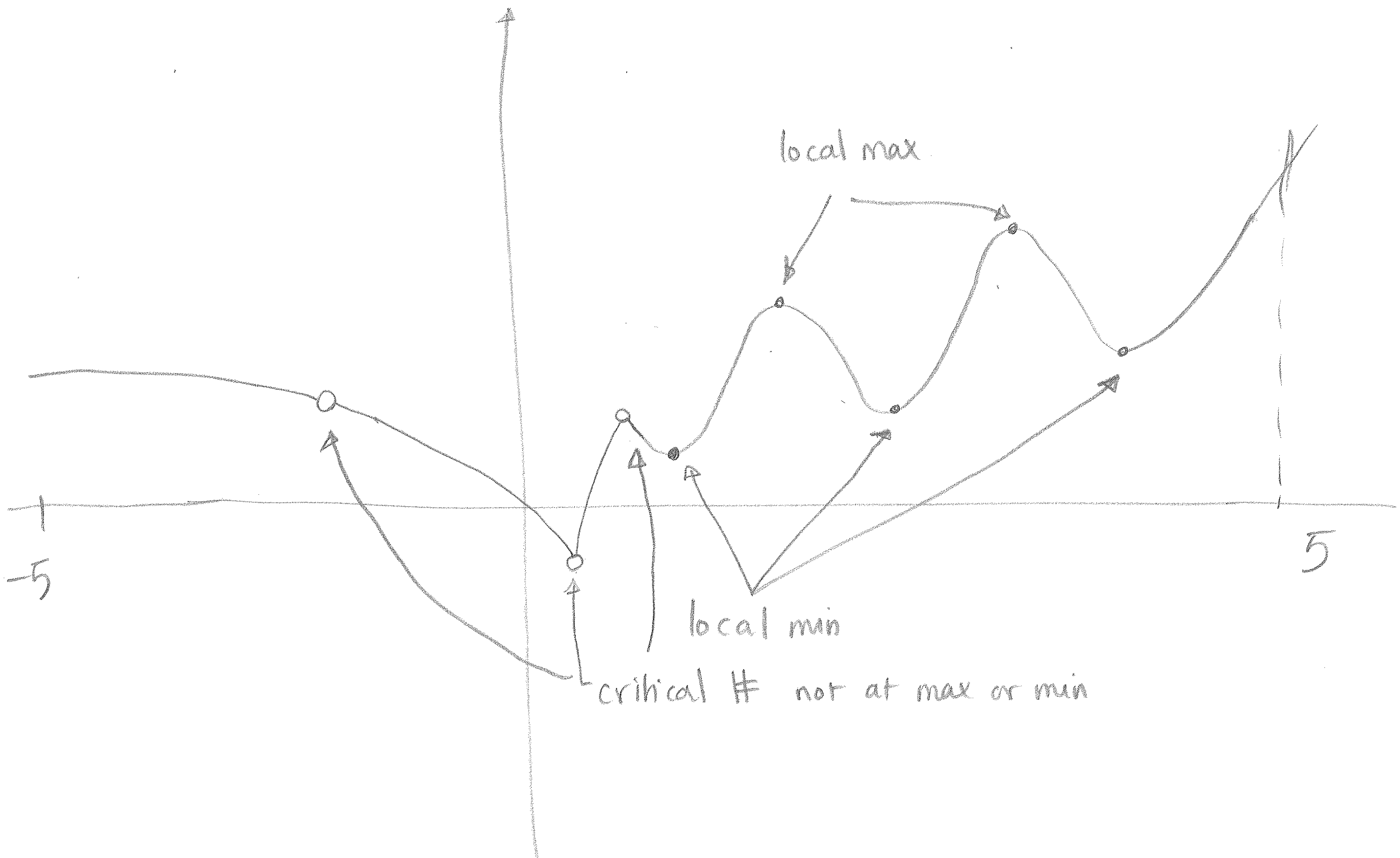
$$\frac{f'(x)}{f(x)} = x \frac{(\tan(x))'}{\tan(x)} + \ln(\tan(x))$$

$$= \frac{x \sec^2(x)}{\tan(x)} + \ln(\tan(x))$$

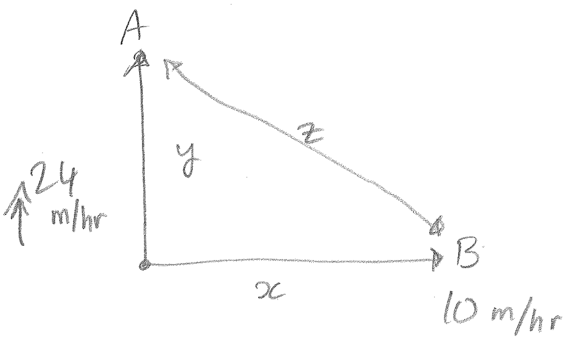
$$f'(x) = \tan(x)^x \left(x \frac{\sec^2(x)}{\tan(x)} + \ln(\tan(x)) \right)$$

5. [10 points] Sketch the graph of a function on the interval $[-5, 5]$ which has on this interval:

3 local minimum
2 local maximum
no absolute minimum
8 critical numbers



3. [10 points] Two ships start from the same point at 12 noon. One travels north at 24 miles/hour and the other travels east at 10 miles/hour How fast are they going away from each other at 2pm? Show your working, including a labeled diagram to explain your notation.



let x = distance of ship A from start

" y = ——"—— B ——"——

$x' = 10$ m/h, $y' = 24$ m/h

let z = distance between ships.

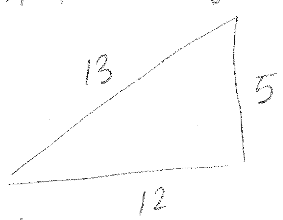
Relation:

$$z^2 = x^2 + y^2$$

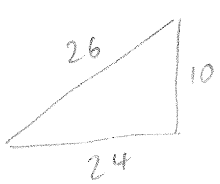
$$2zz' = 2xx' + 2yy'$$

$$z' = \frac{xx' + yy'}{z}$$

(5,12,13) triangle:



Scaled:



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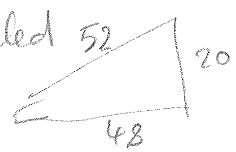

So at 2pm,

$$x = 10 \times 2, = 20$$

$$y = 24 \times 2 = 48$$

So $z = \sqrt{48^2 + 20^2} = 52$

(see picture left)

So $$z' = \frac{20 \times 10 + 48 \times 24}{\sqrt{48^2 + 20^2}}$$

$$= \frac{1}{2} \frac{20 \times 20 + 48 \times 48}{\sqrt{48^2 + 20^2}} = \frac{1}{2} \frac{(48^2 + 20^2)}{\sqrt{48^2 + 20^2}}$$

$$= \frac{1}{2}\sqrt{48^2 + 20^2} = \frac{1}{2} 52 = \boxed{26 \text{ m/hr}}$$

6. [10 points] Find the absolute maximum and absolute minimum values of the function

$$f(x) = 3x^4 - 14x^3 + 9x^2$$

on the interval $[0, 1]$.

$$3{\times}4\,x^3 - 3\times 14x^2 + 9\times 2x$$
$$= 3x(4x^2 - 14x + 6)$$
$$= 6x(2x^2 - 7x + 3)$$
$$= 6x(2x-1)(x-3)$$

roots: $\dfrac{7 \pm \sqrt{49 - 4\times 2\times 3}}{2\times 2}$

so critical values are at $0, 1/2$ & 3

only $0, 1/2 \in [0,1]$ $\boxed{f(0) = 0}$, $\boxed{f(1) = 3 - 14 + 9 = -2}$

$f(1/2) = 3\times 1/16 - 14/8 + 9/4$

$= 1/4\,(3/4 - 14/2 + 9)$

$= \frac{1}{4}(\frac{3}{4} - 7 + 9) = 1/4 \times (2 + 3/4)$

$= 1/4\,(11/4) = 11/16$

$\boxed{\begin{array}{l}\text{So max value on } [0,1] \text{ is } -2 \text{ at } 1 \\ \text{min value on } [0,1] \text{ is } 11/16 \text{ at } 1/2\end{array}}$

7. [10 points] Let

$$f(x) = x^2 - 5x + 3.$$

Find a number c in the interval $[2, 3]$ which satisfies the conclusion of the mean value theorem for the function $f(x)$ on the interval $[2, 3]$.

$f'(x) = 2x - 5$

average slope on $[2,3]$ is $\dfrac{f(3) - f(2)}{3-2} = \dfrac{9 - 15 + 3 - (4 - 10 + 3)}{1}$

$= -3 - (-3) = 0$

when does $f'(x) = 0$?

$2x - 5 = 0 \Rightarrow 2x = 5$

$\Rightarrow x = 5/2 = \boxed{2.5}$

9. [10 points] On which intervals is the function

$$f(x) = \ln|x^2 - 4x + 3|$$

increasing and decreasing?

$$f'(x) = \frac{(x^2-4x+3)'}{(x^2-4x+3)} = \frac{2x-4}{x^2-4x+3} = \frac{2(x-2)}{(x-3)(x-1)}$$

critical #s are at 1, 2, 3

	~~$(-\infty,0)$ 0 $(0,1)$~~ $(-\infty,1)$	1	$(1,2)$	2	$(2,3)$	3	$(3,\infty)$
$x-1$	−	0	+	+	+	+	+
$x-2$	−	−	−	0	+	+	+
$x-3$	−	−	−	−	−	0	+
sign of $f'(x)$	−	0	+	0	−	0	+
$f(x)$	dec		inc		dec		inc

$f(x)$ is decreasing on $(-\infty, 1)$ & $(2,3)$

& increasing on $(1,2)$ & $(3,\infty)$