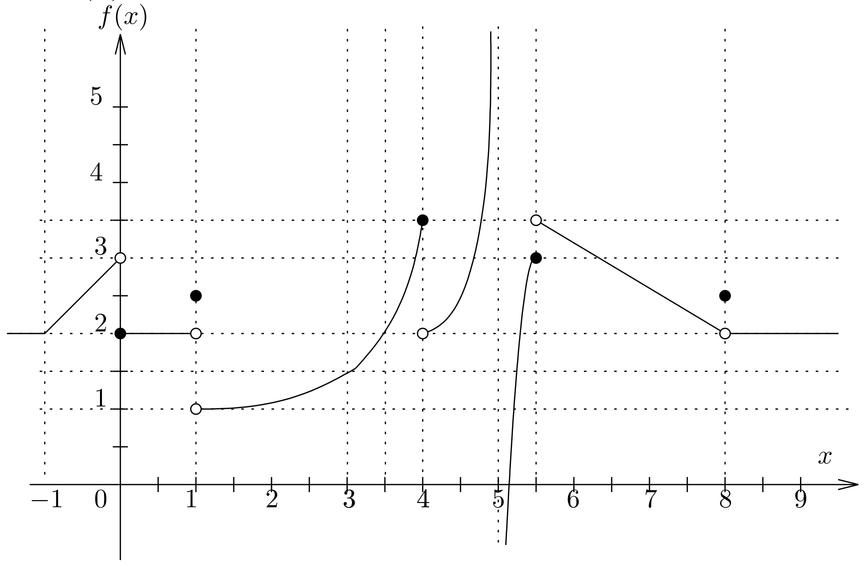
SAMPLE FIRST TEST ANSWERS :

Calculus 1550, Section 5, Fall 2004.
[In an actual test, more space would be given for writing answers.]

This is a list of possible questions; a test would not be this long.

Q1. A function $f(x)$ has domain $\mathbb{R}$, and is given by the following graph for $x \in [-1, 9]$, and by $f(x) = 2$ otherwise.



i. What are all the discontinuities of $f(x)$?

$f(x)$ is discontinuous at $x = 0, 1, 4, 5, 5.5$ and 8.

ii. Complete the following table:

a	$f(a)$	$\lim_{x\to a^-} f(a)$	$\lim_{x\to a^+} f(a)$	$\lim_{x\to a} f(a)$	right continuous at a?	left continuous at a?
-1	**2**	**2**	**2**	**2**	**Yes**	**Yes**
4	**3.5**	**3.5**	**2**	**Does not exist**	**No**	**Yes**
8	**2.5**	**2**	**2**	**2**	**No**	**No**

Q2. Evaluate the following limits, using limit laws and techniques for computing limits exactly. Show your working, or explain your reasoning, but it is not necessary to write down which laws you are using. If a limit does not exist, explain why not.

i. $\lim_{x\to 2} \sqrt{x^2 + x + 3}$

Since polynomials are continuous, and roots of continuous functions are continuous where they are defined, the function $\sqrt{x^2 + x + 3}$ is continuous as long as x is in the domain of this function. So we can just plug in the value, and we obtain $\lim_{x\to 2} \sqrt{x^2 + x + 3} = \sqrt{2^2 + 2 + 3} = \sqrt{9} = 3$

However, this uses continuity, rather than limit laws (though giving any sensible justification would be accepted). From the limit laws, we have

$$\begin{array}{rrcl}
(i) & \lim_{x\to 2}\sqrt{x^2+x+3} & = & \sqrt{\lim_{x\to 2}(x^2+x+3)} \text{ (the limit of a root is a root of the limits)} \\
(ii) & \lim_{x\to 2}(x^2+x+3) & = & \lim_{x\to 2}(x^2)+\lim_{x\to 2}(x)+\lim_{x\to 2}(3) \text{ (the limit of a sum is a sum of the limits)} \\
(iii) & \lim_{x\to 2}(x^2) & = & \left(\lim_{x\to 2}(x)\right)^2 \text{ (the limit of a product is a product of the limits)} \\
(iv) & \lim_{x\to 2}(x) & = & 2 \text{ (the limit of } f(x)=x \text{ as } x \text{ approaches } a \text{ is } a) \\
(v) & \lim_{x\to 2}(3) & = & 3 \text{ (the limit of a constant as } x \text{ approaches anything is the constant)}
\end{array}$$

Now from (iii) (iv) and (v), we can say that

$$\lim_{x\to 2}(x^2)+\lim_{x\to 2}(x)+\lim_{x\to 2}(3)=4+2+3=9,$$

and then substituting this in (i) we find that the limit is $\sqrt{(}9)=3$**. So we can conclude that**

$$\lim_{x\to 2}\sqrt{x^2+x+3}=3.$$

Note, students would not be expected to give full justification as above in a test, but you are expected to understand what the above steps mean. I will not write so many details for the following questions.

ii. $\lim_{x\to 2}\frac{x^3-8}{x^2-4}$

Factoring the numerator and denominator gives

$$\frac{x^3-8}{x^2-4}=\frac{(x-2)(x^2+2x+4)}{(x-2)(x+2)}$$

when $x\neq 2$**, we can cancel the** $(x-2)$**, and so**

$$\text{for } x\neq 2, f(x)=\frac{(x^2+2x+4)}{(x+2)}.$$

At $x=2$**, the function** $\frac{(x^2+2x+4)}{(x+2)}$ **has value** $\frac{(2^2+4+4)}{(2+2)}=12/4=3$**, so,**

$$\lim_{x\to 2}\frac{x^3-8}{x^2-4}=3$$

iii. $\lim_{x\to 0}\frac{(3+x)^2-9}{x}$ **In this case, plugging in** $x=0$ **would give** $0/0$**, which doesn't make sense. The numerator can be rewritten as**

$$(3+x)^2-9=9+6x+x^2-9=6x-x^2=x(6-x),$$

so

$$\text{for } x \neq 0, \frac{(3+x)^2 - 9}{x} = \frac{x(6-x)}{x} = 6 - x$$

Since we're taking a limit, we can now plug 0 into this expression, and obtain that the limit in question is equal to 6.

iv. $\lim_{x \to 0} \frac{|x|}{x}$

This limit does not exist, because when $x < 0$, the value of the function is -1, so $\lim_{x \to 0^-} \frac{|x|}{x} = -1$ but when $x > 0$, the value of the function is 1, so $\lim_{x \to 0^+} \frac{|x|}{x} = 1$ Since these are not equal, the limit can not exist.

If you draw a graph you'll see that there is a jump discontinuity at $x = 0$.

v. $\lim_{x \to 1} \frac{\sqrt{5-x} - 2}{x - 1}$

In order to compute this limit we must multiply top and bottom of the fraction by $(\sqrt{5-x} + 2)$. We obtain

$$\lim_{x \to 1} \frac{\sqrt{5-x} - 2}{x-1} = \lim_{x \to 1} \left(\frac{(\sqrt{5-x} - 2)(\sqrt{5-x} + 2)}{(x-1)(\sqrt{5-x} + 2)} \right) = \lim_{x \to 1} \left(\frac{(5-x) - 4}{(x-1)(\sqrt{5-x} + 2)} \right)$$

$$= \lim_{x \to 1} \left(\frac{(1-x)}{(x-1)(\sqrt{5-x} + 2)} \right) = \lim_{x \to 1} \left(\frac{-1}{(\sqrt{5-x} + 2)} \right) = \frac{-1}{\sqrt{4} + 2} = -1/4$$

Here we have used that we can cancel when $x \neq 1$, and then we've just substituted, since in the last limit, neither denominator nor numerator are zero.

Q3. Given some numbers a and b, define a function

$$f_{ab}(x) = \frac{x^3 + ax + b}{x^2 + 4x}.$$

Fill in the blanks in the table, to give values of a and b to make the last column correct. Write one number in each space.

a	b	$\lim_{x\to 0} f_{ab}(x)$
	1	undefined
		2
		0

For the answer to Q3, please see Q4 on Test 1, Fall 2003:
http://www.math.lsu.edu/ verrill/teaching/calculus1550/Fall2003/test1.pdf

Q4. In this question, you do not need to actually find the limits, but you must write down exactly which rules you would use to find the limit. You must write the shortest possible list of rules. If you write every rule, you will receive no points.

The list of rules and operations are:

Limit laws

1. The limit of a sum is the sum of the limits
2. The limit of a difference is the difference of the limits
3. The limit of a constant times a function is the constant times the limit of the function
4. The limit of a product is the product of the limits
5. The limit of a quotient is the quotient of the limits
6. The limit of a root is the root of the limit

Special cases of limits

7. The limit of a constant is the constant
8. The limit of the function $f(x) = x$ as x approaches a is a

Algebraic operations

9. factorisation
10. cancellation of common factors

Write the numbers of the rules used in the right hand column.

	limit to find	rules you would use to find the limit
i.	$\lim_{x\to 2} \frac{x^2-2}{x^3-2}$	**5, 2, 4, 7, 8**
ii.	$\lim_{x\to 2} \frac{x^3-8}{x^2-10x+16}$	**9, 10, 5, 1, 2, 3, 4, 7, 8**
iii.	$\lim_{x\to 0} \frac{(x^2-2)^2-x^4}{x^2}$	**algebraic manipulation; get 4/0, which D.N.E.**
iv.	$\lim_{x\to 0} \frac{\sqrt{x^2+3x}}{x}$	**5, 6, algebraic manipulation; get 3/0, which D.N.E.**
v.	$\lim_{x\to 1} \frac{\lvert x^2-1\rvert}{x-1}$	**piecewise defined, so must find left and right limits separately; use rules 9, 10,7,8 in both cases. Then find that left and right limits are 2 and -2, which are not equal, so limit D.N.E.**

For more details about Q4, see the working for Q3, Test 1, Spring 2004, at:
http://www.math.lsu.edu/ verrill/teaching/calculus1550/Spring2004/test1.pdf
For Q5 below, see Q1 on Test 1, Spring 2004.

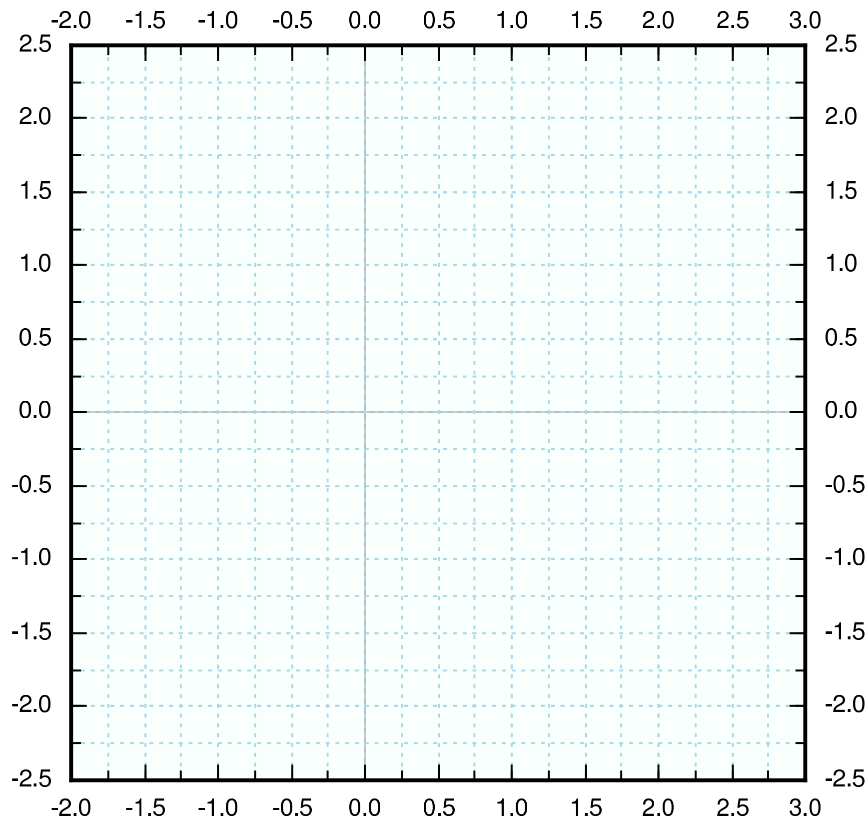
Q5. A function $f(x)$ is defined by

$$f(x) = \begin{cases} 0 & \text{if } x < -1 \\ -x & \text{if } x \in [-1, 0) \\ x^2 & \text{if } x \in (0, 1) \\ x & \text{if } x \in [1, 2] \end{cases}$$

i.
a) Where is $f(x)$ discontinuous?

b) Where is $f(x)$ not defined?

ii. Sketch the graph of $f(x)$ on the following grid:



iii. [15] Complete the following table:

a	$f(a)$	$\lim_{x\to a^-} f(a)$	$\lim_{x\to a^+} f(a)$	$\lim_{x\to a} f(a)$	right continuous at a?	left continuous at a?
-1						
0						
2						

Q6.

1. What is the mathematical definition of continuity?
   A function $f(x)$ is continuous at a point $x = a$ if $f(a)$ exists, and if $\lim_{x\to a} f(x)$ exists, and if these two numbers are equal.

2. On what intervals is the function $f(x) = \frac{\sin(x)}{x(x-1)}$ continuous? Justify your answer.
   This is continuous on $(-\infty, 0), (0, 1)$ and on $(1, \infty)$. This is because $\sin(x)$ is continuous, and polynomials are continuous, and if $f(x)$ and $g(x)$ are continuous, then $f(x)/g(x)$ is continuous except when $g(x) = 0$. In this case the only place where the denominator can be zero is at $x = 0$ and 1, so these are excluded from the domain. The function is continuous elsewhere.

3. Where is $f(x) = \frac{\sin(x)}{x(x-1)}$ discontinuous? Justify your answer.
   The discontinuities are where the function is not continuous, i.e., at 0 and 1,

4. For each discontinuity of $f(x) = \frac{\sin(x)}{x(x-1)}$, state whether it is a removable discontinuity, infinite discontinuity, or jump discontinuity, or another kind of discontinuity.
   At $x = 0$, we know from class that $\sin(x)/x$ tends to 1 as x tends to 0, and at 0 the value of $x - 1$ is -1, so the limit of $\frac{\sin(x)}{x(x-1)}$ as x tends to 0 is -1. Since this limit exists, the discontinuity at 0 must be a removable discontinuity. At 1, the limit from the right is ∞, since if x is a little greater than 1, all of the numbers, $\sin(x), x$ and $x - 1$ will be positive. But if x is a little less than 1, x and $\sin(x)$ are positive, but $x - 1$ is negative. So the limit from the left is minus infinity. In any case, the limit does not exist, and this is an infinite discontinuity.

Q7.

1. Write down the mathematical definition of the limit of a function $f(x)$ as x approaches a from the right. Write your answer in complete grammatical sentences, and include a description of the mathematical symbols usually used to denote this limit.

   The value of the limit of a function as x approaches a from the left is a number L such that by making x sufficiently close to a, and keeping $x < a$, we can make $f(x)$ arbitrarily close to L. (this is sufficient, since you are not required to learn the "epsilon-delta" definition in section 2.4.) This is written as $\lim_{x\to a} f(x) = L$.

2. Give an example of a function with different left and right limits at some point.

   The function $f(x) = \frac{|x|}{x}$ has different left and right limits at $x = 0$.

   There are many other examples that would be graded correctly, this is just one possible answer.

3. Give an example of a function where the limit of $f(x)$ as x approaches a exists, but $f(a)$ does not exist.

The function $f(x) = \frac{\sin(x)}{x}$ has limit 1 at $x = 0$, but $f(0)$ is not defined.

There are many other examples that would be graded correctly, this is just one possible answer.

4. State the limit law known as the "quotient rule", and give an example of where you can use it to find a limit.

   This limit laws states that if $f(x)$ and $g(x)$ are functions and if $\lim_{x\to a} f(x)$ and $\lim_{x\to a} g(x)$ both exist, and if $\lim_{x\to a} g(x)$ is not zero, then

$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim\limits_{x\to a} f(x)}{\lim\limits_{x\to a} g(x)}.$$

   A simple case where we can use this is to say that

$$\lim_{x\to 0}\left(\frac{cos(x)}{x+1}\right) = \frac{\lim\limits_{x\to 0} \cos(x)}{\lim\limits_{x\to 0}(x+1)}.$$

   We would finish this by plugging in values to obtain that the limit is equal to $1/1 = 1$.

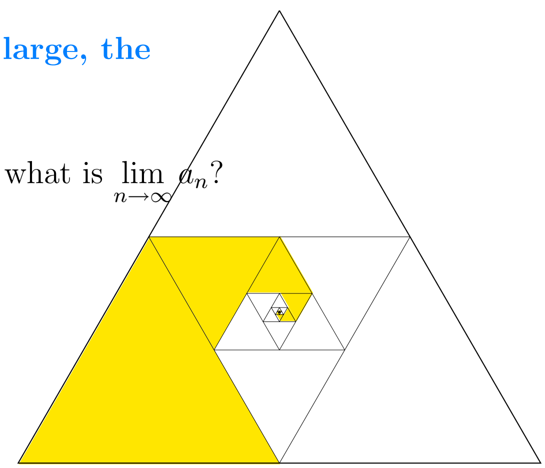
Q8.

1. If $f(x) = \dfrac{2^x - 1}{2^{x+3}}$, what is $\lim\limits_{x\to\infty} f(x)$?

   This is similar to examples in class. When x is very large, the function is close in value to $2^x/2^{x+3} = 2^{x-x-3} = 1/8$.

2. If $a_1 = 0.9$, $a_2 = 0.99$, and so on, so that $a_n = \sum\limits_{i=1}^{n} \dfrac{9}{10^i}$, then what is $\lim\limits_{n\to\infty} a_n$?

   $0.999999... = 0.\overline{9} = 1$
   This can be justified using geometric series.



3. What is the shaded spiral area in the diagram on the right, as a fraction of the area of the large triangle?
   By considering similar spirals obtained by rotating the first one, we can see that the area must be 1/3 of the area of the big triangle.

   How would you write this as a sum of infinitely many terms?

   Each triangle has 1/4 the area of the previous triangle, so as a sum of infinitely many terms, this is given as:

$$1/4 + 1/4^2 + 1/4^3 + 1/4^4 + ...$$

   In sigma notation, this is written as

$$\sum_{n=0}^{\infty} \frac{1}{4^n}$$

Q9.

In the following questions, state whether the situation is
(a) possible, but not certain, (b) definitely has happened, or (c) is impossible.

In case (a), give examples to explain why it may or may not have happened.

In case (b), justify your answer, say what results you are using, and what functions you work with. Also give an idea of how many times this has happened, e.g., just once, or many times?

In case (c), explain why it is impossible

1. Has there ever been a moment when the temperature in Sydney, Australia, is exactly the same as the temperature in Baton rouge?

   Yes, because both Sydney and Baton Rouge experience hot temperatures in summer, and cooler temperatures in winter. But when it is winter in Baton Rouge, it is summer in Sydney, and when it's summer in Sydney it's winter in Baton Rouge, so if we let $S(t)$ = temperature at time t in Sydney, and $B(t)$ = temperature at time t in Baton Rouge, on a Summer day in Baton Rouge, we will usually have $B(t) > S(t)$, so $B(t) - S(t) > 0$, but on a winter day, we will have (at some time during the day, with very high likely hood) $B(t) < S(t)$, so $B(t) - S(t) < 0$. Some time between, we must have $B(t) - S(t) = 0$, since we can apply the intermediate value theorem, since temperature is a continuous function (at least we assume it is, though it's possible there might be a very sudden plunge, e.g., snow storm in July, this is not very likely), and the difference of two continuous functions is continuous.

   So, sometime between Summer and Winter Baton Rouge and Sydney have the same temperature. It's possible that this happens only one day in 6 months, but since temperatures vary during the day, it's most likely that it occurs many times.

2. A seed, which will grow into a pine tree, sprouts its first shoot on the same day as a baby is born. 20 years later, both are living. Were they ever exactly the same height?

   Yes (except under special circumstances). By the intermediate value theorem. Let $H(t)$=height of person at time t, Let $T(t)$=height of tree at time t. Then at the birth date, $H(t) - T(t) > 0$, but 20 years later, $H(t) - T(t) > 0$, so since we assume that that growth is continuous for both the person and the tree, by the intermediate value theorem, there is some intermediate time where the height difference is zero, i.e., the heights are the same.

   It is most likely that there is only one time (or short time interval) when the two heights are equal, though it's possible this happens several times over some time interval when the tree is still fairly short. It's possible the tree is some kind of dwarf variety, and so never gets as tall as the person. It's also harder to measure the exact height of a tree, and if branches fall or are cut, this might not be continuous.

3. In 1990, Mr. X was a millionaire, whilst Mr. Y was deeply in debt. Today Mr. X has lost all his fortune and is bankrupt, while Mr. Y has just made his first million. Was there ever a moment that they had exactly the same amounts of money in their bank accounts?

   We can not conclude this. Let $X(t)$ be Mr. X's bank balance, and $Y(t)$ Mr. Y's bank balance. We want to say that at some time $X(t) = Y(t)$, but we can't conclude this (at least not from the intermediate value theorem), since these functions are not continuous; Mr X. May have suddenly lost all his money in one day.