Math 2020, Discrete problems, Spring 2005

Schedule for March 1 2005

(page numbers are for first edition of course text book.)

1. Permutations and permutation groups (Section 4.2, pages 141–147)
2. Symmetry groups of polyhedra and polygons (E.g., Example 9.86, page 389.)
3. Definition of a group (definition 9.40, page 381, and definition page 9.11, page 369)
4. How properties of groups are seen in the multiplication table of the group
   (Note, the multiplication table tells you everything about the group. This is not discussed in enough detail in the book, but examples are given on page 390 and 132.)
5. More examples of groups (section 9.4.)

Homework 5, due Thursday 11 March.

1. Questions on equivalence relations: See your lecture notes, or definitions 2.32, 2.51 in the text book for definitions of reflexive, symmetric, transitive, and equivalence relations.

Q1.A. Define a relation $\square$ on $\mathbf{Q} \setminus \{0\}$ by saying

$$a \,\square\, b \iff a/b \text{ is a square in } \mathbf{Q}$$

Note 0: $\mathbf{Q}$ is the set of rational numbers.
Note i: $\iff$ means "if and only if", and so $a \square b$ is true if $\frac{a}{b}$ is a square, and false if $\frac{a}{b}$ is not a square.)
Note ii: In class I gave this example, but I forgot to write that this is defined on $\mathbf{Q} \setminus \{0\}$. We have to exclude $\{0\}$ because we can't divide by 0.
Note iii: "a/b is the square in $\mathbf{Q}$" means that there is some rational number c/d such that $a/b = (c/d)^2$.
Now define another relation $\blacksquare$ on $\mathbf{Q} \setminus \{0\}$ by setting

$$a \blacksquare b \iff ab \text{ is a square in } \mathbf{Q}$$

Q1.A.1. Prove that $\square$ and $\blacksquare$ define exactly the same relation on $\mathbf{Q} \setminus \{0\}$, that is, show that for $\frac{a}{b}$ and $\frac{c}{d}$ in $\mathbf{Q}$, we have

$$\tfrac{a}{b} \,\square\, \tfrac{c}{d} \iff \tfrac{a}{b} \blacksquare \tfrac{c}{d}$$

Q1.A.2.
Although $\square$ did not make sense on all of $\mathbf{Q}$, so we had to exclude 0, the relation $\blacksquare$ actually makes sense on all of $\mathbf{Q}$. Define a relation $\widetilde{\blacksquare}$ on $\mathbf{Q}$ by

$$a \,\widetilde{\blacksquare}\, b \iff ab \text{ is a square of an element of } \mathbf{Q}$$

(Remark: we say that $\widetilde{\blacksquare}$ extends $\blacksquare$ from $\mathbf{Q} \setminus \{0\}$ to $\mathbf{Q}$.)
Although $\blacksquare$ is an equivalence relation on $\mathbf{Q} \setminus \{0\}$, (we discussed this in class), $\widetilde{\blacksquare}$ is not an equivalence relation.

Q1.A.2.i. Which of the properties reflexive, transitive and symmetric does $\widetilde{\blacksquare}$ not satisfy?

Q1.A.2.ii. For whichever of these properties you claim $\widetilde{\blacksquare}$ does not satisfy, give an example to justify your claim.

Remark: Try the above questions for an equivalence relation of your choice, e.g., given an equivalence relation (you could try those listed in the next question), can you find a different way to express the relation? Can you extend the relation, or restrict it, in a natural way, to another set, so it is no longer an equivalence relation?

Q1.B.

Q1.B.1. The table on the next page gives a list of 14 different relations.
Which of the relations are equivalence relations? Mark an X in the column under "is equivalence relation?" for those which are equivalence relations.

Q1.B.2. For those which are equivalence relations, give an example of an equivalence class (i.e., choose an element x of the set, and write a list of all the elements equivalent to x, or a formula for all the elements if this is an infinite set).

Q1.B.3. For those which are equivalence relations, in the last column of the table, write down the number of equivalence classes (this could be infinite.)

2. Questions on wallpaper patterns

A glide reflection is defined to be a transformation of the plane given by first reflecting about some axis, and then moving parallel to the axis of reflection.
In fact, these transformations are the same as the set of transformations defined by reflecting, and then translating in any direction. This exercise is to illustrate this fact.

Table for **Q.1.B:** Each relation in this table is a relation on a given set, listed under "set".

	set	relation	is equivalence relation?	number of equiv classes
1.	$\mathbf{R}$	$aRb \iff a-b$ is negative		
2.	$\mathbf{R}$	$aRb \iff a-b$ is even		
3.	$\mathbf{R}$	$aRb \iff \sin(a)=\sin(b)$		
4.	$\mathbf{Z}$	$aRb \iff \sin(a)=\sin(b)$		
5.	$\mathbf{R}$	$aRb \iff \sin(\pi a)=\sin(\pi b)$		
6.	$\mathbf{Z}$	$aRb \iff \sin(\pi a)=\sin(\pi b)$		
7.	$\mathbf{R}$	$aRb \iff \lfloor a \rfloor = \lfloor b \rfloor$		
8.	$\mathbf{R}$	$aRb \iff \|a\| = b$		
9.	$\mathbf{R}$	$aRb \iff \|a\| = \|b\|$		
10.	$\mathbf{R}$	$aRb \iff \|a\|b = a\|b\|$		
11.	$\mathbf{Z}$	$aRb \iff \|a\| - \|b\|$ is even		
12.	$\mathbf{Z}$	$aRb \iff a \mid b$		
13.	$\mathbf{Z}\setminus\{0\}$	$aRb \iff ab$ is a square in $\mathbf{Z}$		
14.	$\mathbf{R}\setminus\{0\}$	$aRb \iff ab$ is a square in $\mathbf{R}$		

Q2.A.

1. Any translation of the plane is given by a map $T_{a,b} : (x,y) \mapsto (x+a, y+b)$ for some fixed values of $a, b \in \mathbf{R}$.
2. A reflection about an axis given by $x = a$ is given by $R_a : (x,y) \mapsto (2a - x, y)$

If you are not familiar with these statements from a course on linear algebra, convince yourself of their truth, e.g., by drawing pictures and considering how these maps apply. E.g. For a square with vertices at $(0,0),(0,1),(1,1),(1,0)$, Where do these points map to under $T_{4,3}$? Where to they map to under R_3? Draw a picture on graph paper to show what's happening.

Q2.A.i Prove (by direct computation, i.e., compute both of $T_{0,b} \circ T_{a,0}$ and $T_{a,b}$, as applied to an arbitrary point (x,y), and compare to see if the two expressions you get are equal) that $T_{0,b} \circ T_{a,0} = T_{a,b}$ for any $a, b \in R$.
Here $\circ$ means composition of functions, i.e., first apply $T_{a,0}$ to a point, and then apply $T_{0,b}$.
Geometrically, what this means is that any translation can be achieved by first making a horizontal translation, and then making a vertical translation.

Q2.A.ii. Prove (by direct computation) that $T_{a,0} \circ R_b = R_{b+a/2}$ for any a, b in $\mathbf{R}$.
Geometrically this means that any reflection followed by a translation perpendicular to the axis of reflection is the same as a single reflection, about a different axis. (Though we're only doing this for vertical axis of reflection; if you have taken a course in linear algebra, you might like to prove this more generally.)

Q2.A.iii.
Parts i and ii of this question mean that a glide reflection can be defined either as a reflection followed by any translation (not perpendicular to axis of translation), or a reflection followed by a translation parallel to the axis of reflection. These definitions are equivalent.
This is because

$$T_{a,b}R_c = (T_{0,b} \circ T_{a,0}) \circ R_c = T_{0,b} \circ (T_{a,0} \circ R_c) = T_{0,b} \circ R_{c+a/2}$$

There are three equalities in the above statement, these follow from the results in Q2.A.i, Q2.A.ii, and from the associativity of transformations of the plane. Which equality follows from which of these three facts?

Q2.A.vi.
Choose your own values of a, b, c, and sketch a triangle B on graph paper, or on a clearly scketched grid. Write down what the values of a, b, c are, and what the vertices of the triangle are.
Sketch $R_{c+a/2}(B)$, $T_{0,b}(R_{c+a/2}(B))$ and $R_c(B)$, $T_{a,b}(R_c(B))$. (You should see how the transformations are related.)

Q2.B.

For this question, either draw a neat grid so translational symmetries can be easily seen, or use graph paper. (there is a web site at
http://www.mathematicshelpcentral.com/graph_paper.htm
where you can print out sheets of graph paper if you don't have any.)

Q2.B.i Draw (part of) a wallpaper pattern which has a symmetry group which includes rotations through 180°, but does not include rotations through 90°.
Look at the handout for lecture 11 on Escher's wallpaper patterns to see an example where there are rotations of 90°.
You can just draw a simple shape in your fundamental unit, e.g., the letter L.

Q2.B.ii For the pattern you drew in part Q2.B.i, make a list of all the kinds of symmetries it has.
(For an example, see how the symmetries were listed under the three examples in the handout for lecture 11 on Escher's wallpaper patterns, or see how the kinds of symmetries are shown in the pictures in the solution to the sample quiz on wallpaper patterns — see under "lecture 11" on the course web page.)