Solving discrete problems — Math 2020

Schedule for March 10th

1 More examples of groups.
2. Definition of a cyclic group. (page 383, definition 9.58)
   This is a group where all the elements are powers of some particular element.
   This element is called a *generator* of the group.
   Not every element of a cyclic group can generate the group, just those with the same order as the order of the group.
3. Definition of a subgroup. (page 383, definition 9.51)
   This is a subset of a group that is also a group, with the same group operation.
4. Definition of a group homomorphism and isomorphism (section 9.5, page 386).
5. More multiplication tables for groups.
6. Quiz on Symmetries of a tetrahedron.
7. Proof that all elements of a finite group have finite order.
8. Proof that the order of elements of a group divides the order of the group.

Homework, due Tuesday March 29

A. Make multiplication tables for each of the following groups of order 6.

Each table should have seven rows and columns, which includes the top row, which lists all elements of the group, and the left most column, listing all elements of the group, in the same order as the top row.
The first element in the list should be the identity of the group.

For an example, see the tables of multiplication tables for groups of order 4 done in class (but since these groups have order 6 your tables will be bigger than the ones in class).

1. The symmetries of a triangle, including reflectional symmetries.
   The group operation is composition of symmetries, i.e., do one, then the next.
   (List the elements of your group as reflections, rotations, or the identity.
   Draw a triangle, and indicate your notation, e.g., for each reflection you list, show the axes of reflection, etc.)
2. The symmetries of a hexagonal based pyramid, not including any reflections.
   The group operation is composition of symmetries, i.e., do one, then the next.
   (draw a pyramid with a hexagonal base, and indicate your notation.)
3. The group of all permutations on 3 things, S_3.
   The group operation is composition of permutations, i.e., do one, then the next.
4. The group $\mathbf{Z}/6\mathbf{Z}, +$.
   the group operation is addition.
   Note, you should write the elements of this group as $0, 1, 2, 3, 4, 5$.
5. The subgroup of all elements of $\mathbf{Z}/13\mathbf{Z}^\times$ which are squares in $\mathbf{Z}/13\mathbf{Z}^\times, \times$.
   the group operation is multiplication.
   (To find the elements of this group, just find the values of all the squares of the numbers from 1 to 12, modulo 13; the first few are: $1^2 \equiv 1, 2^2 \equiv 4, 3^2 \equiv 9, 4^2 \equiv 3 \mod 13$... etc. You will only get 6 different numbers.)
   No number in your table should be less than 1 or greater than 12.
6. The group given by the set of the following six functions:
   $$f_1(x) = x, \quad f_2(x) = 1/x, \quad f_3(x) = 1 - x, \quad f_4(x) = 1/(1-x), \quad f_5(x) = 1 - \frac{1}{x}, \quad f_6(x) = \frac{x}{x-1},$$
   with the operation on this group given by composition of functions,
   e.g., $f_2 \circ f_3(x) = f_2(1-x) = \dfrac{1}{1-x} = f_4(x)$, so $f_2 \circ f_3 = f_4$.

Note, to check that you have made no mistakes in your multiplication tables, you should check the following:

- that every row and column contains the identity of the group.
- each row and each column contains every element of the group.

B. For each element in each of the the above groups, find its order, and make a table of orders. This can be included as an extra row added to the tables you have already made; write "order" at the beginning of the row.

C. Which of the groups in the first part of this question are cyclic? (This means they are generated by one element.) For the cyclic groups, say what a possible generator is (there may be more than one).

D. Which of the above groups is abelian (this means $a \circ b = b \circ a$ for all $a, b \in G$, and as explained in class, can also be seen from the symmetry of the multiplication table).

E. Write down which groups are isomorphic to each other.
(To be isomorphic, the elements must match up some way — the matching is a map ϕ called an *isomorphism*, which satisfies $\phi(a)\phi(b) = \phi(ab)$.) The multiplication tables of isomorphic groups have the same "pattern", provided you fill in the elements in the right order.