Solving discrete problems — Math 2020, Spring 2005

Schedule For Thursday, February 3, 2005.

1. Discussion on distribution of primes.
2. Some important proofs in number theory:
   - i. Proof that there are infinitely many primes.
     - ∗ Page 125 in the course book. If you don't have the book, see:
     - ∗ http://odin.mdacc.tmc.edu/∼krc/numbers/infitude.html
3. Quiz on reading and understanding proofs.

Break

4. Continuing important proofs in number theory:
   - ii. Results about the GCD (greatest common divisor).
     - ∗ (see page 121–123 or next reference below,)
   - iii. The Euclidian algorithm, and (if there is time) proof it works.
     - ∗ (see page 275 or http://www.maths.monash.edu.au/mth3122/a4lect2.pdf)
   - iv. Unique prime factorisation in the integers:
     - ∗ (see page 126 or http://www.maths.monash.edu.au/mth3122/a4lect3.pdf

Homework Due Tuesday February 15.

1. i. Find a necessary condition a and b so that $\sqrt{a+b} = \sqrt{a} + \sqrt{b}$. (Necessary means the least possible conditions).
ii. Prove that if this condition (as you found for (i)) is not satisfied, then $\sqrt{a+b} \neq \sqrt{a} + \sqrt{b}$.
iii. Write the statement of the result you proved in (ii) as a theorem in the form

$$\text{hypothesis} \rightarrow \text{conclusion}$$

iv. Give an example where the theorem (in part iii) holds (i.e., the hypothesis are satisfied), and verify by explicit computation that it holds (i.e., the conclusion is satisfied).
v. Do the same hypotheis as in (iii) imply that $\sqrt[3]{a+b} \neq \sqrt[3]{a} + \sqrt[3]{b}$?
If so, prove this. If not, give an example to show why not.

2. Find the greatest common divisor of 1940400 and 637995160 by two methods:
i. Use the prime factorisation (found on first homework).
ii. Use the Euclidean algorithm.

3. Using the fact that for an integer $m \geq 1$ there is always a prime between m and $2m$ (not including m and $2m$, unless $m = 1$.) show that $n!$ is never a square of an integer for $n > 1$. (Note, $n!$ was defined on the first homework sheet.) (Hint: First try this for $n = 2m$. What can you say about the prime factorisation of a square? What can you say about the factorisation of $n!$ for the prime between m and $2m$?)
(Note, proving that there is a prime between m and $2m$ is quite difficult, so you can assume this without proof. As an exercise (for yourself), try to find a prime between $m+1$ and $2m$ for all integers m from 1 to 10.

4. Let $\mathcal{A}$ be the set of all integers of the form $3n+1$, (as in an exercise on the first homework). Where does the proof of unique factorisation as given in class break down. I.e., what results are assumed in the proof that work for the integers, but don't work for $\mathcal{A}$.

5. What is the smallest positive integer x such that $x^2 + x + 11$ is not prime?
For a positive integer n, define $f_n(x) = x^2 + x + n$.
Find an integer m, given as a function of n, so that $f_n(m)$ is not prime.