Math 2020, Discrete problems, Spring 2005

Homework 5, Solutions.

Solutions will appear in boxes, in blue

1. Questions on equivalence relations: See your lecture notes, or definitions 2.32, 2.51 in the text book for definitions of reflexive, symmetric, transitive, and equivalence relations.

Q1.A. Define a relation $\square$ on $\mathbf{Q} \setminus \{0\}$ by saying

$$a \square b \iff a/b \text{ is a square in } \mathbf{Q}$$

Note 0: $\mathbf{Q}$ is the set of rational numbers.
Note i: $\iff$ means "if and only if", and so $a\square b$ is true if $\frac{a}{b}$ is a square, and false if $\frac{a}{b}$ is not a square.)
Note ii: In class I gave this example, but I forgot to write that this is defined on $\mathbf{Q} \setminus \{0\}$. We have to exclude $\{0\}$ because we can't divide by 0.
Note iii: "a/b is the square in $\mathbf{Q}$" means that there is some rational number c/d such that $a/b = (c/d)^2$.
Now define another relation $\blacksquare$ on $\mathbf{Q} \setminus \{0\}$ by setting

$$a \blacksquare b \iff ab \text{ is a square in } \mathbf{Q}$$

Q1.A.1. Prove that $\square$ and $\blacksquare$ define exactly the same relation on $\mathbf{Q} \setminus \{0\}$, that is, show that for $\frac{a}{b}$ and $\frac{c}{d}$ in $\mathbf{Q}$, we have

$$\tfrac{a}{b} \square \tfrac{c}{d} \iff \tfrac{a}{b} \blacksquare \tfrac{c}{d}$$

Proof: Assume a, b are in $\mathbf{Q}$ and are both not zero. An example is: $8\square 2$ since $8/2 = 4 = 2^2$, and also $8\blacksquare 2$, since $8 \times 2 = 16 = 4^2$. Now we see what happens in general:
First we will show that $a\square b$ implies $a\blacksquare b$.

(first use the definition)

$$a\square b \iff a/b \text{ is a square in } \mathbf{Q}$$

(write what this means explicitly)

$$\iff a/b = m^2 \quad \text{for some } m \in \mathbf{Q}$$

(bearing in mind the definition of $\blacksquare$, try to write this in terms of ab. this can be achieved by multiplying both sides by b^2.)

$$\iff ab = m^2 b^2 = (mb)^2$$

$$\Rightarrow ab \text{ is a square in } \mathbf{Q}$$

(now use the definition of $\blacksquare$)

$$\iff a\blacksquare b.$$

Now we will show that $a\blacksquare b$ implies $a\square b$.

(first use the definition)

$$a\blacksquare b \iff ab \text{ is a square in } \mathbf{Q}$$

(write what this means explicitly)

$$\iff ab = n^2 \quad \text{for some } n \in \mathbf{Q}$$

(bearing in mind the definition of $\square$, try to write this in terms of a/b. this can be achieved by dividing both sides by b^2.)

$$\iff a/b = n^2/b^2 = (n/b)^2$$

$$\Rightarrow a/b \text{ is a square in } \mathbf{Q}$$

(now use the definition of $\square$)

$$\iff a\blacksquare b.$$

Q1.A.2.

Although $\square$ did not make sense on all of $\mathbf{Q}$, so we had to exclude 0, the relation $\blacksquare$ actually makes sense on all of $\mathbf{Q}$. Define a relation $\widetilde{\blacksquare}$ on $\mathbf{Q}$ by

$$a \mathrel{\widetilde{\blacksquare}} b \iff ab \text{ is a square of an element of } \mathbf{Q}$$

(Remark: we say that $\widetilde{\blacksquare}$ extends $\blacksquare$ from $\mathbf{Q} \setminus \{0\}$ to $\mathbf{Q}$.)

Although $\blacksquare$ is an equivalence relation on $\mathbf{Q} \setminus \{0\}$, (we discussed this in class), $\widetilde{\blacksquare}$ is not an equivalence relation.

Q1.A.2.i. Which of the properties reflexive, transitive and symmetric does $\widetilde{\blacksquare}$ not satisfy?

$\widetilde{\blacksquare}$ is reflexive, since aa is always a square; it is symmetric, since if ab is a square, then so is ba. But $\widetilde{\blacksquare}$ is not transitive.

Q1.A.2.ii. For whichever of these properties you claim $\widetilde{\blacksquare}$ does not satisfy, give an example to justify your claim.

But $\widetilde{\blacksquare}$ is not transitive. This is all the "fault" of 0. For example, since $5 \times 0 = 0 = 0^2$, and $0 \times 3 = 0 = 0^2$, we have

$$5 \mathrel{\widetilde{\blacksquare}} 0 \quad \text{and} \quad 0 \mathrel{\widetilde{\blacksquare}} 3.$$

But we do **not** have $5 \mathrel{\widetilde{\blacksquare}} 3$, since 15 is not a square in $\mathbf{Q}$.

Remark: Try the above questions for an equivalence relation of your choice, e.g., given an equivalence relation (you could try those listed in the next question), can you find a different way to express the relation? Can you extend the relation, or restrict it, in a natural way, to another set, so it is no longer an equivalence relation?

Q1.B.

Q1.B.1. The table on the next page gives a list of 14 different relations.

Which of the relations are equivalence relations? Mark an X in the column under "is equivalence relation?" for those which are equivalence relations.

Q1.B.2. For those which are equivalence relations, give an example of an equivalence class (i.e., choose an element x of the set, and write a list of all the elements equivalent to x, or a formula for all the elements if this is an infinite set).

Q1.B.3. For those which are equivalence relations, in the last column of the table, write down the number of equivalence classes (this could be infinite.)

Table for **Q.1.B:** Each relation in this table is a relation on a given set, listed under "set".

Answers to these questions are all in the table.
Note, some examples in this table given as two versions, since it may not have been clear in the original question what was meant.

	set	relation	is equivalence relation?	number of equiv classes
1.	$\mathbf{R}$	$aRb \iff a-b$ is negative	No, not symmetric	
2.	$\mathbf{Z}$	$aRb \iff a-b$ is even	X, Yes	2: even and odd numbers
2a.	$\mathbf{R}$	$aRb \iff a-b=2k$ for some integer k	X, Yes	∞ for $x \in [0,2)$ $[x]_R := \{x+2k \mid k \in \mathbf{Z}\}$,
			$e.g.$, $[0.1]_R := \{0.1, 2.1, 4.1, 6.1, \dots, -1.9, -3.9, \dots\}$	
3.	$\mathbf{R}$	$aRb \iff \sin(a) = \sin(b)$ $(a, b,$ in radians)	X, Yes	∞ for $x \in [0, \pi/2] \cup (\pi, 3\pi/2]$ $[x]_R := \{x + 2k\pi \mid k \in \mathbf{Z}\}$. $\cup\{\pi - x + 2k\pi \mid k \in \mathbf{Z}\}$.
4.	$\mathbf{Z}$	$aRb \iff \sin(a) = \sin(b)$ $(a, b,$ in radians)	X, Yes	∞ for $x \in \mathbf{Z}$, $[x]_R = \{x\}$,
			because for integers $a, b, a-b$ is never an integer multiple of π	
5.	$\mathbf{R}$	$aRb \iff \sin(\pi a) = \sin(\pi b)$ $(a, b,$ in radians)	X, Yes	∞ for $x \in [0, 1/2] \cup (\pi, 3/2]$ $[x]_R := \{x + 2k \mid k \in \mathbf{Z}\}$. $\cup\{1 - x + 2k \mid k \in \mathbf{Z}\}$.
6.	$\mathbf{Z}$	$aRb \iff \sin(\pi a) = \sin(\pi b)$ $(a, b,$ in radians)	X, Yes	1, all of $\mathbf{Z}$ because for all integers $a, \sin(\pi a) = 0$
3a.	$\mathbf{R}$	$aRb \iff \sin(a) = \sin(b)$ $(a, b,$ in degrees)	X, Yes	∞ for $x \in [0, 90] \cup (180, 270]$ $[x]_R := \{x + 360k \mid k \in \mathbf{Z}\}$. $\cup\{180 - x + 360k \mid k \in \mathbf{Z}\}$.
4a.	$\mathbf{Z}$	$aRb \iff \sin(a) = \sin(b)$ $(a, b,$ in degrees)	X, Yes	181 equivalence classes same as in 3a, but x must be an integer there are 91 integers in $[0, 90]$, and 90 in $(180, 270]$
5a.	$\mathbf{R}$	$aRb \iff \sin(\pi a) = \sin(\pi b)$ $(a, b,$ in degrees)	X, Yes	∞ (similar to 3,3a,5; just scale)
6a.	$\mathbf{Z}$	$aRb \iff \sin(\pi a) = \sin(\pi b)$ $(a, b,$ in degrees)	X, Yes	∞ (similar to 4)
7.	$\mathbf{R}$	$aRb \iff \lfloor a \rfloor = \lfloor b \rfloor$	X, Yes	∞ $[k, k+1)$ for $k \in \mathbf{Z}$
8.	$\mathbf{R}$	$aRb \iff \lvert a \rvert = b$	No, not reflexive e.g., $-2 \not{R} -2$, since $\lvert -2 \rvert \neq -2$	
9.	$\mathbf{R}$	$aRb \iff \lvert a \rvert = \lvert b \rvert$	X, Yes	∞ $\{0\}, \{1, -1\}, \{2, -2\}, \{3, -3\}, \dots$
10.	$\mathbf{R}$	$aRb \iff \lvert a \rvert b = a \lvert b \rvert$	No, not transitive e.g.,$0R1$ and $0R-1$, but $1 \not{R} -1$	
11.	$\mathbf{Z}$	$aRb \iff \lvert a \rvert - \lvert b \rvert$ is even	X, Yes	2, even numbers and odd numbers
12.	$\mathbf{Z}$	$aRb \iff a \mid b$	No, not symmetric	
13.	$\mathbf{Z} \setminus \{0\}$	$aRb \iff ab$ is a square in $\mathbf{Z}$	X, Yes	∞ $e.g.$, $[1] = \{1, 4, 9, 25, \dots\}$ $[2] = \{2, 8, 18, 50, \dots\}$ a list of all equivalence classes is given by $[n]_R = \{nk^2 \mid k \in \mathbf{Z}, k > 0\}$
		where we take n to run through all integers not divisible by any square of an integer (except 1)		
14.	$\mathbf{R} \setminus \{0\}$	$aRb \iff ab$ is a square in $\mathbf{R}$	X, Yes	2, positive numbers and negative numbers

Note, in example 7., the $\lfloor x \rfloor$ is the largest integer smaller than x, e.g., $\lfloor 2.3 \rfloor = 2$, or $\lfloor 7.9 \rfloor = 7$, $\lfloor -2.3 \rfloor = -3$. This is also called rounding down to the integer part of x.x

2. Questions on wallpaper patterns

A glide reflection is defined to be a transformation of the plane given by first reflecting about some axis, and then moving parallel to the axis of reflection.

In fact, these transformations are the same as the set of transformations defined by reflecting, and then translating in any direction. This exercise is to illustrate this fact.

Q2.A.

1. Any translation of the plane is given by a map $T_{a,b} : (x, y) \mapsto (x + a, y + b)$ for some fixed values of $a, b \in \mathbf{R}$.
2. A reflection about an axis given by $x = a$ is given by $R_a : (x, y) \mapsto (2a - x, y)$

If you are not familiar with these statements from a course on linear algebra, convince yourself of their truth, e.g., by drawing pictures and considering how these maps apply. E.g. For a square with vertices at $(0, 0), (0, 1), (1, 1), (1, 0)$, Where do these points map to under $T_{4,3}$? Where to they map to under R_3? Draw a picture on graph paper to show what's happening.

Q2.A.i Prove (by direct computation, i.e., compute both of $T_{0,b} \circ T_{a,0}$ and $T_{a,b}$, as applied to an arbitrary point (x, y), and compare to see if the two expressions you get are equal) that $T_{0,b} \circ T_{a,0} = T_{a,b}$ for any $a, b \in R$.

Here $\circ$ means composition of functions, i.e., first apply $T_{a,0}$ to a point, and then apply $T_{0,b}$.

Geometrically, what this means is that any translation can be achieved by first making a horizontal translation, and then making a vertical translation.

Using the definitions,

$$T_{0,b} \circ T_{a,0}(x, y) = T_{0,b}(T_{a,0}(x, y)) = T_{0,b}((x + a, y)) = (x + a, y + b)$$

and

$$T_{a,b}(x, y) = (x + a, y + b),$$

so $T_{0,b} \circ T_{a,0}$ and $T_{a,b}$ are the same map, ie. $T_{0,b} \circ T_{a,0} = T_{a,b}$.

Q2.A.ii. Prove (by direct computation) that $T_{a,0} \circ R_b = R_{b+a/2}$ for any a, b in $\mathbf{R}$.

Using the definitions,

$$T_{a,0} \circ R_b(x, y) = T_{a,0}(R_b(x, y)) = T_{a,0}((2b - x, y)) = (2b - x + a, y) = (2b + a - x, y) = (2(b - a/2) - x, y)$$

and

$$R_{b+a/2}(x, y) = (2(b + a/2) - x, y),$$

so $T_{a,0} \circ R_b$ and $R_{b+a/2}$ are the same map, ie. $T_{a,0} \circ R_b = R_{b+a/2}$.

Geometrically this means that any reflection followed by a translation perpendicular to the axis of reflection is the same as a single reflection, about a different axis. (Though we're only doing this for vertical axis of reflection; if you have taken a course in linear algebra, you might like to prove this more generally.)

Q2.A.iii.

Parts i and ii of this question mean that a glide reflection can be defined either as a reflection followed by any translation (not perpendicular to axis of translation), or a reflection followed by a translation parallel to the axis of reflection. These definitions are equivalent.

This is because

$$T_{a,b}R_c = (T_{0,b} \circ T_{a,0}) \circ R_c = T_{0,b} \circ (T_{a,0} \circ R_c) = T_{0,b} \circ R_{c+a/2}$$

There are three equalities in the above statement, these follow from the results in Q2.A.i, Q2.A.ii, and from the associativity of transformations of the plane. Which equality follows from which of these three facts?

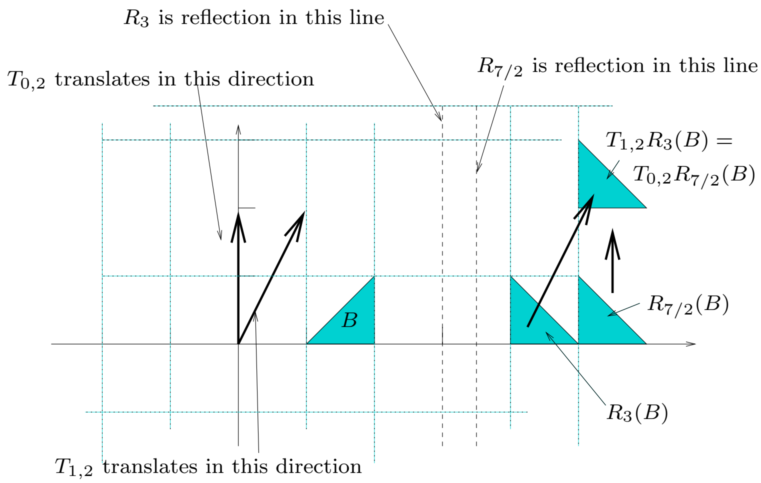
The first equality uses Q2.A.i, the second uses associativity, and the third uses Q2.Aii.

Q2.A.vi.

Choose your own values of a, b, c, and sketch a triangle B on graph paper, or on a clearly scketched grid. Write down what the values of a, b, c are, and what the vertices of the triangle are.

Sketch $R_{c+a/2}(B)$, $T_{0,b}(R_{c+a/2}(B))$ and $R_c(B)$, $T_{a,b}(R_c(B))$. (You should see how the transformations are related.)

Example with $a = 1, b = 2, c = 3$, and triangle So maps are $R_{3+1/2} = R_{7/2}$, $T_{0,2}$, R_3, $T_{1,2}$.



Q2.B.

For this question, either draw a neat grid so translational symmetries can be easily seen, or use graph paper. (there is a web site at
http://www.mathematicshelpcentral.com/graph_paper.htm
where you can print out sheets of graph paper if you don't have any.)

Q2.B.i Draw (part of) a wallpaper pattern which has a symmetry group which includes rotations through 180°, but does not include rotations through 90°.

Look at the handout for lecture 11 on Escher's wallpaper patterns to see an example where there are rotations of 90°. You can just draw a simple shape in your fundamental unit, e.g., the letter L.

Q2.B.ii For the pattern you drew in part Q2.B.i, make a list of all the kinds of symmetries it has.

(For an example, see how the symmetries were listed under the three examples in the handout for lecture 11 on Escher's wallpaper patterns, or see how the kinds of symmetries are shown in the pictures in the solution to the sample quiz on wallpaper patterns — see under "lecture 11" on the course web page.)

In the following picture, there are symmetries of translation (by any integer amount horizontally, and even integer amounts vertically), and rotation of 180 degrees, about vertices, some of which are marked in the second version, on the right. Note, there are no reflections because the diagonal line in the figure would not be preserved by any reflection in a horizontal or vertical axes.