

# Homework 7

## Solutions

note, if  $a_n = \alpha a_{n-1} + \beta a_{n-2}$

then either  $a_n = c_1 a^n + c_2 b^n$

where  $a, b$  are roots of  $r^2 - \alpha r - \beta$

or, if  $r^2 - \alpha r - \beta = (r-a)(r-a)$  i.e.  $a=b$ , then

$$a_n = c_1 a^n + c_2 n a^n$$

4. a)  $a_n + 7a_{n-1} = 0$

roots of char poly:  $r^2 + 7r = 0 \Rightarrow r(r+7) = 0 \Rightarrow r=0$  or  $-7$ ,

so general sol<sup>n</sup> is  $a_n = c(-7)^n$   $c$  a constant

b)  $a_n - 5a_{n-1} = 0$  almost same as (a) - get  $a_n = c 5^n$

c)  $a_n = 2a_{n-1} + 8a_{n-2}$

roots of char poly:  $r^2 - 2r - 8 = (r-4)(r+2) \Rightarrow a_n = c_1 4^n + c_2 (-2)^n$   $c_1, c_2$  constants

d)  $a_n = 3a_{n-1} - 2a_{n-2}$

roots of char poly:  $r^2 - 3r + 2 = 0 \Rightarrow (r-2)(r-1) = 0 \Rightarrow a_n = c 2^n + d$   $c, d$  constants

6. a)  $a_n = 9a_{n-1} - 20a_{n-2} \quad n \geq 2 \quad a_0 = 0, a_1 = 5$

char poly:  $r^2 - 9r + 20 = (r-5)(r-4)$

general sol<sup>n</sup>  $a_n = c 5^n + d 4^n$

$a_0 = c 5^0 + d 4^0 = c + d = 0 \Rightarrow c = -d$

$a_1 = c 5^1 + d 4^1 = 5c + 4d = 5$   $\left\{ \begin{array}{l} \text{substitute} \\ 5c - 4c = 5 \\ c = 5 \end{array} \right.$

$5c - 4c = 5$   
 $c = 5$

so  $a_n = 5 \times 5^n - 5 \times 4^n$

$= 5(5^n - 4^n)$

b)  $a_n = 7a_{n-1} - 12a_{n-2} \quad n \geq 2$

$a_0 = 2, a_1 = 5$

char poly:  $r^2 - 7r + 12 = (r-4)(r-3) = 0 \Rightarrow a_n = c 4^n + d 3^n$

$a_0 = c + d = 2 \Rightarrow c = 2 - d$

$a_1 = 4c + 3d = 5$  sub  $c = 2 - d$ :  $4(2 - d) + 3d = 5$   
 $= 8 - 4d + 3d = 5$   
 $8 - d = 5$   
 $d = 3 \Rightarrow c = 2 - 3 = -1$

$d = 3 \Rightarrow c = 2 - d \Rightarrow c = -1$

$-d = -3 \Rightarrow d = 3$

so  $a_n = -4^n + 3 \cdot 3^n = 3^{n+1} - 4^n$

c)  $a_n = 2a_{n-1} + 2a_{n-2} \quad n \geq 2 \quad a_0 = 2, a_1 = 1$

char poly:  $r^2 - 2r - 2 = 0 \Rightarrow r = \frac{2 \pm \sqrt{4+8}}{2} = \frac{1 \pm \sqrt{3}}{2}$

$\Rightarrow a_n = c \left(\frac{1+\sqrt{3}}{2}\right)^n + d \left(\frac{1-\sqrt{3}}{2}\right)^n$   
so  $a_n = \left(\frac{1+\sqrt{3}}{2}\right)^n + \left(\frac{1-\sqrt{3}}{2}\right)^n$

$a_0 = c + d = 2$

$a_1 = c \left(\frac{1+\sqrt{3}}{2}\right) + d \left(\frac{1-\sqrt{3}}{2}\right) = 1$   $c = d = 1$  solved by

d)  $a_n = -a_{n-1} + a_{n-2} \quad n \geq 2, a_0 = 4, a_1 = 2$

char poly:  $r^2 + r - 1 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2}$

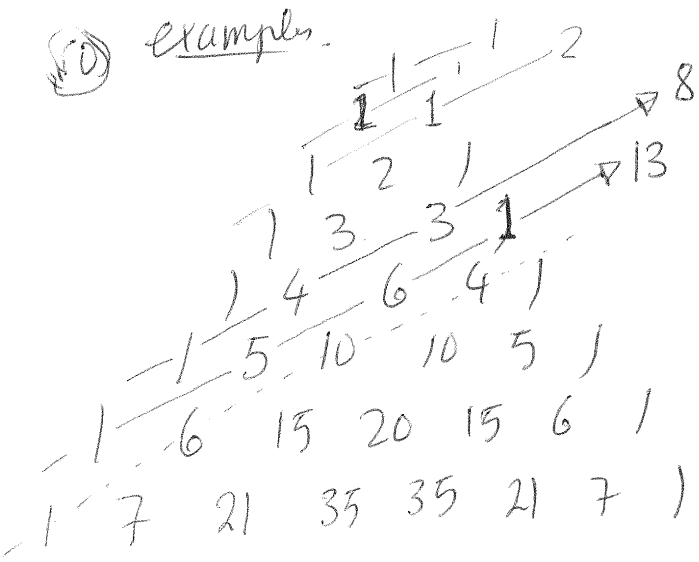
$\Rightarrow a_n = c \left(\frac{-1+\sqrt{5}}{2}\right)^n + d \left(\frac{-1-\sqrt{5}}{2}\right)^n$

$c + d = 4$   
 $c \left(\frac{-1+\sqrt{5}}{2}\right) + d \left(\frac{-1-\sqrt{5}}{2}\right) = 2$   
 $c = 2 - d$   
 $2 \left(\frac{-1+\sqrt{5}}{2}\right) + d \left(\frac{-1-\sqrt{5}}{2}\right) = 2$   
 $-1 + \sqrt{5} + d \left(\frac{-1-\sqrt{5}}{2}\right) = 2$   
 $-d\sqrt{5} = 5 - \sqrt{5} \Rightarrow d = \frac{5-\sqrt{5}}{\sqrt{5}}$

so  $a_n = \left(2 + \frac{4}{\sqrt{5}}\right) \left(\frac{-1+\sqrt{5}}{2}\right)^n + \left(2 - \frac{4}{\sqrt{5}}\right) \left(\frac{-1-\sqrt{5}}{2}\right)^n$

get  $c = 2 + \frac{4}{\sqrt{5}}$   
 $d = 2 - \frac{4}{\sqrt{5}}$

examples.



why does sum  $F(n) + F(n+1)$  give  $F(n+2)$ ?

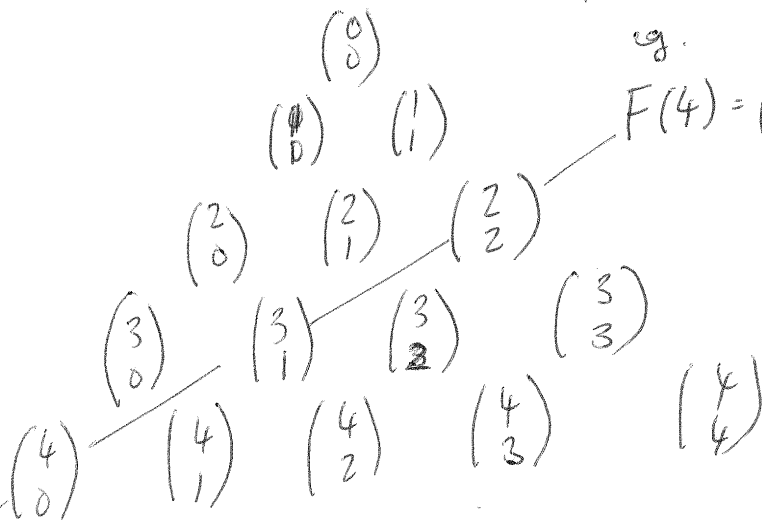
$$\begin{aligned}
 & F(5) + F(6) \\
 &= (1+4+3) + (1+5+6+1) \\
 &= 1 + 1+5 + 4+6 + 3+1 \\
 &= 1 + 6 + 10 + 4 \\
 &= F(7)
 \end{aligned}$$

$$F(n) = \binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \dots + \binom{n}{\lfloor n/2 \rfloor}$$

eg.

$$F(4) = \binom{4}{0} + \binom{4}{1} + \binom{4}{2}$$

for  $\binom{n}{r}$  would be zero, non zero,  $n-r \geq r \Rightarrow n \geq 2r \Rightarrow r \leq n/2$  so max  $r$  is  $\lfloor n/2 \rfloor$  = integer part of  $n/2$ .



use the rule

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

Proof that the  $F(n)$  satisfy same relation as Fibonacci #s  $F(n-1) + F(n)$

$$\begin{aligned}
 & F(n-1) + F(n) \\
 &= \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-4}{3} + \dots \\
 &+ \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \binom{n-4}{4} + \dots \\
 &= \binom{n}{0} + \left( \binom{n-1}{0} + \binom{n-1}{1} \right) + \left( \binom{n-2}{1} + \binom{n-2}{2} \right) + \left( \binom{n-3}{2} + \binom{n-3}{3} \right) + \dots \\
 &= \binom{n}{0} + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \dots \\
 &= 1 + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \dots \\
 &= \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \dots \\
 &= F(n+1)
 \end{aligned}$$

using  $\binom{p}{p} = \binom{p+1}{p+1}$   
use  $\binom{k}{0} = 1$  for all  $k$

$F(0) = \binom{0}{0} = 1$ ,  $F(1) = \binom{1}{0} + \binom{1}{1} = 1 + 1 = 2$   
so since  $F(0) = f_1$  &  $F(1) = f_2$  &  $F(n) = F(n-1) + F(n-2)$   
 $F(n)$  are Fibonacci numbers (shifted by 1)