Homework 7

Solutions

note, if $a_n = \alpha a_{n-1} + \beta a_{n-2}$

then either $a_n = c_1 a^n + c_2 b^n$

where a, b are roots of $r^2 - \alpha r - \beta$

or, if $r^2 - \alpha r - \beta = (r-a)(r-a)$ ie $a=b$, then

$a_n = c_1 a^n + c_2 n a^n$

4. a) $a_n + 7a_{n-1} = 0$

roots of char poly: $r^2 + 7r = 0 \Rightarrow r(r+7) = 0 \Rightarrow r = 0$ or -7,

so general sol^n is $a_n = c(-7)^n$ $\quad$ c a constant

b) $a_n - 5a_{n-1} = 0$ $\quad$ almost same as (a) - get $a_n = c\,5^n$

c) $a_n = 2a_{n-1} + 8a_{n-2}$

roots of char poly: $r^2 - 2r - 8 = (r-4)(r+2) \Rightarrow a_n = c_1 4^n + c_2(-2)^n$ $\quad$ c_1, c_2 constants

d) $a_n = 3a_{n-1} - 2a_{n-2}$

roots of char poly: $r^2 - 3r + 2 = 0 \Rightarrow (r-2)(r-1) = 0 \Rightarrow a_n = c2^n + d$ $\quad$ c, d constant

6. a) $a_n = 9a_{n-1} - 20a_{n-2}$ $n \geq 2$ & $a_0 = 0, a_1 = 5$:

char poly: $r^2 - 9r + 20 = \cancel{(r-10)(r-1)}$
$= (r-5)(r-4)$

$\Rightarrow$ general sol^n $a_n = c5^n + d4^n$

$a_0 = c5^0 + d4^0 = c + d = 0 \Rightarrow c = -d$

$a_1 = c5^1 + d4^1 = 5c + 4d = 5$ & substitute

$5c - 4c = 5$

$c = 5$

so $a_n = 5 \times 5^n - 5 \cdot 4^n$
$= 5(5^n - 4^n)$

b) $a_n = 7a_{n-1} - 12a_{n-2}$ $n \geq 2$

$a_0 = 2, a_1 = 5$

char poly: $r^2 - 7r + 12 = (r-4)(r-3) = 0 \Rightarrow a_n = c4^n + d3^n$

$a_0 = c + d = 2 \Rightarrow c = 2 - d$

$a_1 = 4c + 3d = 5$ sub $c = 2-d$: $4(2-d) + 3d = 5$

$= 8 - 4d + 3d = 5$

$8 - d = 5$

$-d = -3 \Rightarrow d = 3$

$d = 3$ & $c = 2 - d \Rightarrow c = -1$

so $a_n = -4^n + 3 \cdot 3^n = 3^{n+1} - 4^n$

c) $a_n = 2a_{n-1} + 2a_{n-2}$ $n \geq 2$ $a_0 = 2, a_1 = 1$

char poly: $r^2 - 2r - 2 = 0 \Rightarrow r = \frac{2 \pm \sqrt{4+8}}{2} = \frac{1 \pm \sqrt{3}}{2} \Rightarrow a_n = c\left(\frac{1+\sqrt{3}}{2}\right)^n + d\left(\frac{1-\sqrt{3}}{2}\right)^n$

$a_0 = c + d = 2$

$a_1 = c\left(\frac{1+\sqrt{3}}{2}\right) + d\left(\frac{1-\sqrt{3}}{2}\right) = 1$

solved by $c = d = 1$

so $a_n = \left(\frac{1+\sqrt{3}}{2}\right)^n + \left(\frac{1-\sqrt{3}}{2}\right)^n$

d) $a_n = -a_{n-1} + a_{n-2}$, $n \geq 2$, $a_0 = 4, a_1 = 2$

char poly: $r^2 + r - 1 = 0 \Rightarrow r = \frac{-1 \pm \sqrt{1+4}}{2} = \frac{-1 \pm \sqrt{5}}{2} \Rightarrow a_n = c\left(\frac{-1+\sqrt{5}}{2}\right)^n + d\left(\frac{-1-\sqrt{5}}{2}\right)^n$

$c + d = 4$ $\quad$ $c\left(\frac{-1+\sqrt{5}}{2}\right) + d\left(\frac{-1-\sqrt{5}}{2}\right) = 2$

$c = 2 - d$ $\quad$ $2\left(\frac{-1+\sqrt{5}}{2}\right) + d\left(\frac{1-\sqrt{5}}{2} - \frac{-1-\sqrt{5}}{2}\right) = 4$

get $c = 2 + 4/\sqrt{5}$ $\quad$ $-1 + \sqrt{5} + d\sqrt{5} = 4$

$d = 2 - 4/\sqrt{5}$ $\quad$ $d\sqrt{5} = 5 - \sqrt{5} \Rightarrow d = \sqrt{5} - 1$

So $a_n = \left(2 + \frac{4}{\sqrt{5}}\right)\left(\frac{-1+\sqrt{5}}{2}\right)^n + \left(2 - \frac{4}{\sqrt{5}}\right)\left(\frac{-1-\sqrt{5}}{2}\right)^n$

(i) examples.

$$
\begin{array}{ccccccccccccccc}
 & & & & & & & 1 & & & & & & & \\
 & & & & & & 1 & & 1 & & & & & & \\
 & & & & & 1 & & 2 & & 1 & & & & & \\
 & & & & 1 & & 3 & & 3 & & 1 & & & & \\
 & & & 1 & & 4 & & 6 & & 4 & & 1 & & & \\
 & & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & & \\
 & 1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1 & \\
1 & & 7 & & 21 & & 35 & & 35 & & 21 & & 7 & & 1
\end{array}
$$

1, 1, 2, …, $\to 8$, $\to 13$

why does sum $F(n) + F(n+1)$ give $F(n+2)$?

$$\underbrace{1+4+3}_{F(5)} + \underbrace{1+5+6+1}_{F(6)}$$

$$= 1 + 1+5 + 4+6 + 3+1$$

$$= \underbrace{1 + 6 + 10 + 4}_{=F(7)}$$

$$F(n) = \binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} \cdots\cdots + \binom{n-[n/2]}{[n/2]}$$

when $r > n$, $\binom{n}{r}$ would be zero,

for $\binom{n-r}{r}$ non zero,

$n - r \geq r$

$\Rightarrow n \geq 2r$

$\Rightarrow r \leq n/2$

so max r is $[n/2]$ = integer part of $n/2$.

e.g.

$$F(4) = \binom{4}{0} + \binom{4-1}{1} + \binom{4-2}{2}$$

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
$$

use the rule

$$\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$$

$$
\begin{array}{ccccc}
 & & \binom{n-2}{0} & & \binom{n-2}{1} \\
 & \binom{n-1}{0} & & \binom{n-1}{1} & & \binom{n-1}{2} \\
\binom{n}{0} & & \binom{n}{1} & & \\
\binom{n+1}{0} & & \binom{n+1}{1} & &
\end{array}
$$

Proof that the $F(n)$ satisfy same relⁿ as Fibonacci #s:

$$F(n-1) + F(n)$$

$$= \underbrace{\binom{n-1}{0} + \binom{n-2}{1} + \binom{n-3}{2} + \binom{n-4}{3} + \cdots}_{F(n-1)}$$

$$\underbrace{\binom{n}{0} + \binom{n-1}{1} + \binom{n-2}{2} + \binom{n-3}{3} + \binom{n-4}{4} + \cdots}_{F(n)}$$

$$= \binom{n}{0} + \left(\binom{n-1}{0} + \binom{n-1}{1}\right) + \left(\binom{n-2}{1} + \binom{n-2}{2}\right) + \left(\binom{n-3}{2} + \binom{n-3}{3}\right) + \cdots$$

$\left(\text{using } \binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}\right)$

$$= \binom{n}{0} + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \cdots$$

$$= 1 + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \cdots$$

use $\binom{k}{0} = 1$ for all k

$$= \binom{n+1}{0} + \binom{n}{1} + \binom{n-1}{2} + \binom{n-2}{3} + \cdots$$

$$= F(n+1)$$

$F(0) = \binom{0}{0} = 1$, $F(1) = \binom{1}{0} = 1$

so since $F(0) = f_1$ & $F(1) = f_2$ & $F(n) = F(n-1) + F(n-2)$

$F(n)$ are Fibonacci numbers (shifted by 1)