4/5

Recurrence relation

Recall problem: how many trains of length n, t_n are made of cars of length 1 + 2 with 2 kinds of cars of length 2.

We saw that $t_n = t_{n-1} + 2t_{n-2}$, this a recurrence relation for the t_n

<u>aim</u>: find a formula for t_n which doesn't depend on previous t_n.

n	t_n	
1	1	
2	3	(1), (2A), (2B)
3	5	
4	11	
5	21	
6	43	
7	85	
8	171	
9	341	

$n^2 - (n+1)$

<u>Conjecture</u>

$t_n = 2^{n-1} + t_{n-2} \quad \geq 3$

<u>Proof:</u> (by induction)

1) this is true for $n=3$? Yes, b/c $t_3 = 5$ and $2^{n-1} + t_{n-2} = 2^2 + t_1 = 4 + 1 = 5$

2) induction: assume $t_k = 2^{k-1} + t_{k-2}$.

3) to show: $t_{k+1} = 2^k + t_{k-1}$

By definition $t_{k+1} = t_k + 2t_{k-1}$. Use* $t_{k+1} = 2^{k-1} + t_{k-2} + 2t_{k-1}$. If we assume inductively that $t_n = 2^{n-1} + t_{n-2}$ for all $n \leq k$, then by taking $n = k-2$, we get $t_{k-2} = 2^{k-3} + t_{k-4}$. Also, by taking $n = k-1$, $t_{k-1} = 2^{k-2} + t_{k-3}$. Substituting into * gives

$$
\begin{aligned}
t_{k+1} &= 2^{k-1} + t_{k-2} + 2(2^{k-2} + t_{k-3}) \\
&= 2^{k-1} + t_{k-2} + 2^{k-1} + 2t_{k-3} \\
&= 2^{k-1} + 2^{k-1} + \underbrace{t_{k-2} + 2t_{k-3}}_{t_{k-1}} \\
&= 2 \cdot 2^{k-1} + t_{k-1} \\
&= 2^k + t_{k-1} \Rightarrow t_{k+1} = 2^k + t_{k-1}
\end{aligned}
$$

i.e. Result is true for $k=3$, $k=4$, + we've shown that if result holds for $n=k-1$ + $n=k$ □.

- Our t_n satisfy $t_n = t_{n-1} + 2t_{n-2}$ for all $n \geq 3$
  $t_n = 2^{n-1} + t_{n-2}$ for all $n \geq 3$
  $t_1 = 1,\ t_2 = 3$

$t_{16} = 2^{15} + 2^{13} + 2^{11} + \cdots + 2^3 + 2^1 + 1$

Note on geometric series:
a sum of the form $s = b + ba + ba^2 + ba^3 + \cdots + ba^m$
is a sum of a geometric series

$$as = ba + ba^2 + ba^3 + \cdots + ba^{m+1}$$
$$s - as = b - ba^{m+1}$$
$$s(1-a) = b(1-a^{m+1})$$
$$s = \frac{b(1-a^{m+1})}{(1-a)}$$

$b = 2^3$
$a = 4$
$m = 6$

$$t_{16} = 3 + (2^3 + 2^3 \times 4 + 2^3 \times 4^2 + \cdots + 2^3 \times 4^6)$$
$$= 3 + \frac{2^3(1-4^7)}{1-4} = 3 + 8\left(\frac{4^7-1}{4-1}\right)$$

so $t_{16} = 3 + 2^3\left(\frac{(2^2)^7 - 1}{3}\right) = 3 + 2^3\left(\frac{2^{14}-1}{3}\right)$

$$= 3 + \frac{2^{17} - 2^3}{3}$$
$$= \frac{9 + 2^{17} - 8}{3} = \boxed{\frac{2^{17}+1}{3}}$$

<u>Conjecture</u>: $t_n = \frac{2^{n+1}+1}{3}$

n	t_n
1	$5/3$
2	3
3	$17/3$
4	11
5	$65/3$

$t_n = 2^{n+1} + 1/3$ seems to work for n even

$t_n = 2^{n+1} - 1/3$ for n odd

i.e. $t_n = \frac{2^{n+1} + (-1)^n}{3}$ for all $n \geq 1$

$$t_n = \frac{2^{n+1} - (-1)^{n+1}}{3}$$

e.g. $t_{15} = \frac{2^{16}-1}{3} = 21845$

<u>Conjecture</u>: $t_n = \frac{2^{n+1} + (-1)^n}{3}$

We've checked this for $n=16$. Exercise: prove by induction or by using $t_n = 2^{n-1} + t_{n-2}$ + geom. series.

$t_n = t_{n-1} + 2t_{n-2}$ we have $t_1 = 1$, $t_2 = 3$ then

n	t_n
1	1
2	3
3	5
4	11
5	21
6	43
7	85

now assume $t_n = t_{n-1} + 2_{n-2}$

assume $t_1 = 1$

let $t_2 = a$

n	1	2	3	4	5	6	7	8	9	10
$a=1$	1	1	3	5	11	21				
$a=2$	1	2	4	8	16	32	64			
$a=0$	1	0	2	2	6	10	22	42	86	170
$a=4$	1	4	6	14	26					

if $t_1 = 1$

$t_2 = -1$

then $t_3 = t_2 + 2t_1 = 1$

$t_4 = t_3 + 2t_2 = -1$

$t_5 = 1$

$t_6 = -1$

$t_7 = 1$

$t_8 = -1$

Given that $t_n = t_{n-1} + 2t_{n-2}$

given $t_1 = 1$

what does t_2 have to be so that this is a geometric series?

i.e. want to choose a so that $t_1 = 1$, $t_2 = a$, $t_3 = a^2$, $t_4 = a^3$, $t_5 = a^5$.

If this is true, then $t_n = a^{n-1}$

since $t_n = t_{n-1} + 2t_{n-2}$

$t_n = a^{n-1}$

$t_{n-1} = a^{n-2}$

$t_{n-2} = a^{n-3}$

so we would have $a^{n-1} = a^{n-2} + 2a^{n-3}$

eliminate n: divide by a^n to get $a^{-1} = a^{-2} + 2a^{-3}$

multiply by a^3 to get $a^2 = a + 2$

$$\Rightarrow a^2 - a - 2 = 0$$

$$\Rightarrow (a-2)(a+1) = 0 \Rightarrow a = 2 \text{ or } a = -1$$

if $a = 2$ or -1, then $a^{n-1} = a^{n-2} + 2a^{n-3}$

so if $t_n = a^{n-1}$, does satisfy the relation $t_n = t_{n-1} + 2t_{n-2}$

i.e. for $t_n = t_{n-1} + 2t_{n-2}$ $t_1 = 1$

one solution is

$t_1 = 1$, $t_2 = 2$, $t_3 = 4$ ... $t_r = 2^{n-1}$

another is

$t_1 = 1$, $t_2 = -1$, ..., $t_n = (-1)^{n-1}$

Recall, we conjecture that if $t_1 = 1$, $t_2 = 3$

then $t_n = \dfrac{2^{n+1} - (-1)^{n+1}}{3}$