Some facts about Pascals Δ & Binomials $\binom{n}{r} = \frac{n!}{r!(n-r)!}$

1.) $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

for all $n \geq 0$

$0 \leq r \leq n-1$

ex.) $n = 9 \quad r = 3$

$\binom{9}{3} = 84 \qquad \binom{9}{4} = 126$

$84 + 126 = 210 = \binom{10}{4}$

2.) $\sum_{K=0}^{n} \binom{n}{K} = 2^n$ (row sums are powers of 2)

3.) $\sum_{K=0}^{n} \binom{n}{K}^2 = \binom{2n}{n}$ (sums of squares in row is $\binom{2n}{n}$)

We'll give combinatical proof of these facts ie "counting proofs"

<u>Proof</u> of $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$

$\binom{n+1}{r+1}$ = # of ways of taking r objects from n objects.

e.g. $\binom{5}{3} = 10$

1	2	3	4	5
X	X	X		
X	X		X	
X	X			X
X		X	X	
X			X	X
	X	X	X	
	X	X		X
	X		X	X
		X	X	X

If we have 3 objects from 5, either the 3 include the 1st object & 2 of the remaining 4.

OR

these 3 do not include the 1st object, & so we choose 3 from the remaining 4 objects.

i.e. we have $\binom{4}{2} + \binom{4}{3}$ ways of choosing 3 objects from 5

<u>Generally</u> to choose r objects from n, we choose 1st object & r-1 of remaining n-1

ie $\binom{n}{r} = \binom{n-1}{r-1} + \binom{n-1}{r}$

<u>OR</u> we don't choose 1st so we choose all r from the n-1 object

Holds for all n & all r $0 \leq r \leq n$, sub $n \to n+1$, $r \to r+1$ to get $\binom{n}{r} + \binom{n}{r+1} = \binom{n+1}{r+1}$ $\square$ QED

#1.) on W.S.

\# of lines: $\binom{8}{2} = 28$
\# of Δ's: $\binom{8}{3} = 56$
\# of quadratarals: $\binom{8}{4} = 70$

<u>Proof</u> of fact #2

(More General) <u>Theorem</u>: $\sum_{K=0}^{r} \binom{n}{K}\binom{m}{r-K}$ (r, n, m are fixed integers) $= \binom{n+m}{r}$

eg. $\sum_{K=0}^{3} \binom{4}{K}\binom{5}{3-K} = \binom{4}{0}\binom{5}{3} + \binom{4}{1}\binom{5}{2} + \binom{4}{2}\binom{5}{1} + \binom{4}{3}\binom{5}{0}$
$= \binom{9}{3} = \binom{4+5}{3}$

<u>Proof</u>: to choose r objects from n+m objects

$$\underbrace{|\times|\times|\times|\times|\ |}_{n=5}\underbrace{|\ |\ |\ |\ |\ |\ |}_{m=6} \quad \text{eg. } r = 4$$

Suppose K objects are choosen from the 1st n, then must choose r − K from remaining m objects

There $\binom{n}{K} + \binom{m}{r-K}$ ways of choosing K from the 1st n
K is from 0 to r so,

$\binom{n+m}{r} = \sum_{K=0}^{r} \binom{n}{K}\binom{m}{r-K}$ $\square$ QED

From the Theorem, by taking $m = n$ & $r = n$

then, $\sum_{K=0}^{n} \binom{n}{K}\binom{n}{n-K} = \binom{2n}{n}$

Since $\binom{n}{n-K} = \binom{n}{K}$ this implies that $\sum_{K=0}^{n} \binom{n}{K}\binom{n}{K} = \binom{2n}{n}$

note: $\binom{n}{n-K} = \binom{n}{K}$ because $\binom{n}{n-K} = \frac{n!}{(n-K)!(n-(n-K))!}$
$= \frac{n!}{(n-K)!K!} = \binom{n}{K} = \frac{n!}{K!(n-K)!}$

Theorem $\sum_{k=0}^{n} \binom{n}{k} = 2^n$

This follows from the result $\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n$

by taking $x=1$ get $\sum_{k=0}^{n} \binom{n}{k} = (1+1)^n = 2^n$

by taking $x=-1$ get $\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$

eg. $(1+x)^4 = x^4 + 4x^3 + 6x^2 + 4x + 1$

In general $(1+x)^n = 1 + \binom{n}{1}x + \binom{n}{2}x^2 + \binom{n}{3}x^3 + \dots$
$\binom{n}{n-1}x^{n-1} + \binom{n}{n}x^n$

ie the coefficient of x^r in the expression of $(1+x)^n$ is $\binom{n}{r}$

Why? $\underbrace{(1+x)(1+x)(1+x)\dots(1+x)}_{n \text{ times}}$

to get x^r we must choose r factors $(1+x)$ from which we take x, take one from remaining factors, so get $\binom{n}{r}$ ways to get x^r factors $\square$ QED

Another way to prove above

$\sum \binom{n}{k} = 2^n$ # of subsets of a set of size n