Math 2020 1-25

Home work Hints:

Q2 How many ways can a factorial be written as a difference of 2 squares?

Strategies: 1) try examples of the problem (eg. use a particular value

2) Try and solve a similar, but easier problem.

eg. How can you write 12 as a difference of 2 squares

3) For homework for a particular class, try and use something from the class

eg. $4! = 5^2 - 1^2$

factoring $4! = 4 \times 3 \times 2 \times 1 = 2 \times 2 \times 3 \times 2 = 2^3 \times 3$

$5^2 - 1^2$?

Recall $x^2 - 9 = (x-3)(x+3)$

$a^2 - b^2 = (a-b)(a+b)$

then

$5^2 - 1^2 = (5-1)(5+1) = 4 \cdot 6 = 2^2 \cdot 2 \cdot 3 = 2^2 \times 3$

eg $60 = 6 \times 10$

$= (a+b) \times (a-b)$

$= (8+2) \times (8-2)$

$= 8^2 + 2^2 = 64 - 4 = 60$

Strategy 4) try to find a pattern

Q3 What is the smallest integer...

Factorization fact:

if $n = p_1^{n_1} \times \dots p_m^{a_m}$ eg. $12 = 2^2 \times 3$

then $n^m = p_1^{m n_1} \times \dots p_m^{m a_m}$ eg. $12^7 = 2^{14} \times 3^7$

because $n^m = \underbrace{n \times n \times \ldots \times n}_{m} = (p_1^{a_1} \times \ldots \times p_m^{a_m}) \times \ldots \times (p_1^{a_1} \times \ldots p_m^{a_m})$

$= (\underbrace{p_1^{a_1} p_1^{a_1} \ldots p_1^{a_1}}_{m}) \ldots (\underbrace{p_m^{a_m} \ldots p_m^{a_m}}_{m}) = p_1^{ma_1} \cdots p_m^{ma_m}$

Fact $(a^m)^n = a^{mn}$ $\quad (2^3)^5 = 8^5 = 2^{15}$

Ex. if $\frac{1}{2}$ of a is a cube (a is a positive whole number)
if $\frac{1}{5}$ of a is a seventh power
what is the smallest possible value of a?

this question means: $\frac{1}{2}a = b^3$
$\frac{1}{5}a = c^7$

VOTE 4 PEDRO

$a = 2b^3$ and $a = 5c^7$

so assume $a = 2^r 5^s$ then $\frac{a}{2} = 2^{r-1} \times 5^s = b^3$
and $\frac{a}{5} = 2^r \times 5^{s-1} = c^7$

$\frac{a^b}{a^c} = a^{b-c}$

let $b = 2^m 5^n$
then $b^3 = 2^{3m} 5^{3n}$
$\Rightarrow 2^{3m} 5^{3n} = 2^{r-1} 5^s$
$\Rightarrow 3m = r-1$ and $3n = s$

from $\frac{a}{5} = c^7$
we deduce that
$r = 7m_1$, $\quad m, n$
$s-1 = 7n_1$ $\quad$ some integer

So $r-1$ is a mult. of 3
r is a mult. of 7

$r = 7$ $\quad$ b/c $\quad 7-1 = 6 = 3 \times 2$

8. $s-1$ is a mult. of 7
s is a mult. of 3

$s = 15$

So, can take $a = 2^7 5^{15}$

Check $a/2 = 2^6 5^{15} = (2^2 \times 5^5)^3$
$a/5 = 2^7 5^{14} = (2 \times 5^2)^7$

<u>Next lecture</u> Mersenne primes eg.
largest known prime $= 2^{24036583} - 1$

A <u>proposition</u> - (in logic) is a sentence which is true or false. eg. "Hello!" is not a proposition.
"$5+7=49$" is a proposition

propositions are also called statements.
these have a <u>truth value</u>, which is <u>true</u> or <u>false</u>.

propositions are joined by <u>connectives</u>:

not	$\neg$
and	$\wedge$
or	$\vee$
if ... then	$\longrightarrow$
if and only if	$\longleftrightarrow$

eg. if p is the statement
"P is a red polygon"
if q is the statement
"P has 3 sides and is a polygon"

$P \wedge Q$ P is a red triangle
$Q \wedge (\neg P) =$ P is a triangle which is not red.
$P \wedge (\neg Q) =$ P is a red polygon which is not a triangle.
$P \longrightarrow Q =$ "If P is a red Polygon, then P is a triangle.

Proofs - Direct proof -

eg. Theorem - the sum of two odd numbers is even

Proof: Let x and y be 2 odd numbers,

so: $x = 2r+1$, $y = 2s+1$

so $x+y = 2(r+s+1)$

$\therefore$ $x+y$ is even.

Note this Theorem uses a definition.

Definition - an even number is a multiple of 2.
an odd number is not a number divisible by 2.

We also use the following result:

lemma: if x is a positive number then $x = 2r+1$ for some positive integer.

Proof: let r be the largest integer with $2r < x \Rightarrow 2(r+1) > x$

$2r < x < 2r+2 \longrightarrow x = 2r+1$ QED = $\square$

In a proof by contradiction, we assume what we want to prove is false & draw an obvious contradiction, so the result must be true

Ex: lemma - a positive number has at most one prime factor bigger than it's square root, unless the number itself is prime.

Proof: Suppose n is not prime & n is a positive integer.
say $p \mid n$ & $q \mid n$ & $p, q > \sqrt{n}$, p & q 2 different primes

$\Rightarrow p \times q \mid n \Rightarrow pq \leq n$ $pq > \sqrt{n} \times \sqrt{n} = n$

contradiction, so the result is true QED