$x = p_1^{m_1} \cdot p_2^{m_2} \cdots p_r^{m_r}$ → $x = p_1 \times (p_1^{m_1-1} \cdot p_2^{m_2} \cdots p_r^{m_r})$

$x = p_1 \times (p_1^{m_1-1} \cdot p_2^{m_2} \cdots p_r^{m_r})$ → $p_1 | x$

$p_1 | x$ and $x = q_1^{n_1} \cdot q_2^{n_2} \cdots q_s^{n_s}$, with q_i primes → $p_1 = q_j$ for some j with $1 \leq j \leq s$

Let p_i, q_i, n_i, m_i be as in the statement of the theorem.
Let p_1 and j be as in a previous step, so that $q_j = p_1$.
Then there are 3 possible cases: either $m_1 < n_j$, or $m_1 > n_j$, or $m_1 = n_j$

Assume the hypothesis and notation of the statement of the theorem

Assume j is as in a previous step of this proof.
If $m_1 < n_j$ then
we can divide x by $p^{m_1} = q^{m_1}$, to get $p_2^{m_2} \cdot p_3^{m_3} \cdots p_r^{m_r} = q_1^{n_1} \cdots q_j^{n_j - m_1} \cdots q_s^{n_s}$.
Since $n_j - m_1 > 0$, this is divisible by p_1. If $q^j = p_1$ divides $q_1^{n_1} \cdots q_j^{n_j - m_1} \cdots q_s^{n_s}$, then $p_1 = q_j$ must be equal to one of the q_i with $1 \leq i \leq s$ and $i \neq j$. But this can not be the case, since we are assuming that all the q_i are different from each other (this should be one of the hypothesis).
Since this is a contradiction, it can not be true that $m_1 < n_j$.

Assume the hypothesis and notation of the statement of the theorem

Assume j is as in a previous step of this proof.
If $m_1 > n_j$ then
we can divide x by $p^{n_j} = q^{n_j}$, to get
$p_1^{m_1 - n_j} \cdot p_2^{m_2} \cdot p_3^{m_3} \cdots p_r^{m_r} = q_1^{n_1} \cdots q_{j-1}^{m_{j-1}} \cdot q_j^0 \cdot q_{j+1}^{m_{j+1}} \cdots q_s^{n_s}$.
Since $m_1 - n_j > 0$, this is divisible by q_j. If $q^j = p_1$ divides $p_2^{m_2} \cdot p_3^{m_3} \cdots p_r^{m_r}$, then p_1 must be equal to one of $p_2, \ldots, p_r$. But this can not be the case, since we are assuming that all the p_i are different from each other (this should be one of the hypothesis). Since this is a contradiction, it can not be true that $m_1 > n_j$.

Since we can't have $m_1 < n_j$ or $m_1 > n_j$, we must have $m_1 = n_j$.

Putting together the results so far: For some j, $q_j = p_1$, and $m_1 = n_j$.

In the same way, for each i with $1 \leq i \leq r$, there is some $j = j(i)$, an integer depending on i, with $1 \leq j(i) \leq s$, such that $q_{j(i)} = p_j$, and $m_i = n_{j(i)}$.
This means that each of the p_is is equal to one of the q_js, and the corresponding exponents agree.

In the same way, each of the q_is is equal to one of the p_js.

With set up as in the hypothesis of the theorem:
If each p_i equals a q_j, since p_is and q_js are all different from each other, $r \leq s$.

With set up as in the hypothesis of the theorem:
If each q_i equals a p_j, since p_is and q_js are all different from each other, $s \leq r$.

$s \leq r$ and $r \leq s$ → $r = s$

From what we have seen, the set of p_i is the same as the set of q_j,
and if $p_i = q_j$, then $m_i = n_j$.