Relations on sets

Writing $3 \equiv 8 \mod 5$ is an example of an **equivalence** relation. The relation here is “equivalence modulo 5”, which we defined to mean the remainder is the same when divided by 5.

The concept of relationships between objects is very important in mathematics. Sometimes the relationships between mathematical objects is more important than the objects themselves.

One of the most important kind of relationships in mathematics is the equivalence relationship. This comes up again and again whenever mathematicians study a particular kind of object.

1. Relationships
   Write either aRb or $(a, b) \in R$ to mean a and b satisfy a relationship R.
   Some authors also write $a \sim b$, or $a \sim_R b$.
   Kinds of relationships:

   - Transitive: aRb and $bRc \Rightarrow aRb$. E.g., $<$ on $\mathbf{Z}$.
   - Reflexive: aRa for all possible a. E.g., “divides” on $\mathbf{Z}$.
   - Symmetric: $aRb \Rightarrow bRa$.
     E.g., “is married to” on the set of people. Or “is the mirror image of” for figures in the plane. Or “are two of the side lengths of an integer sided right angle triangle”, on $\mathbf{Z}$.
   - Antisymmetric: aRb and $bRa \Rightarrow a = b$. E.g., $\leq$ on $\mathbf{Z}$.

2. Equivalence relationships:

   - An *equivalence* relation is transitive, symmetric, and reflexive. E.g., $\equiv \mod 5$ on $\mathbf{Z}$.
   - An equivalence relation on a set A divides it into **equivalence classes**. The set of all equivalence classes is called $[A]_R$ or A/R.
   - For each equivalence class A_i we pick a **representative** element $a_i \in A_i$ for that class, and write $A_i = [a_i]$ (or $A_i = [a_i]_R$).
   - The important thing about an equivalence relation is that it divides a set up into disjoint equivalence classes. It’s a way of grouping elements of a set into different types of objects. This is called a **partition** of the set.

3. Order relations

   (a) A relationship which is transitive, reflexive, and antisymmetric is called a **partial ordering**

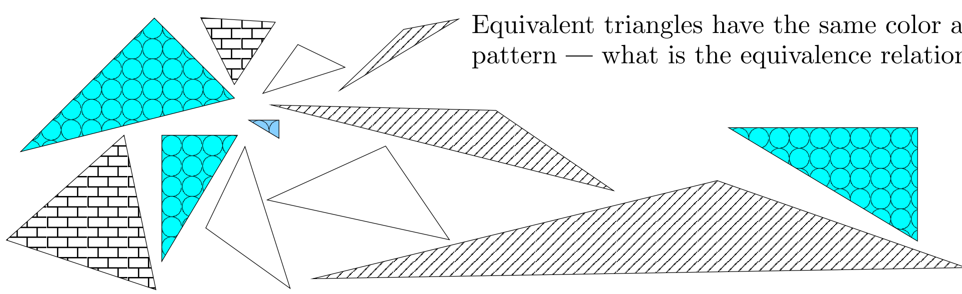
   (b) A relationship R on a set S which is a partial ordering, and satisfies the following:

   $$\text{For all } a, b \in S, \text{ either } aRb \text{ or } bRa$$

   is called a **total ordering**

4. Examples

   - Elements of $\mathbf{Z}/5\mathbf{Z}$ are equivalence classes in $\mathbf{Z}$ under the equivalence relation $\equiv \mod 5$.
   - The subset relation $\subseteq$ on sets is a partial ordering, but not a total ordering.
   - The integers are totally ordered by the relation $\leq$.
   - Exercise:



Equivalent triangles have the same color and pattern — what is the equivalence relation?