Modular forms (Math 7280), Spring 2004. First example sheet

- Review basic results in the theory of complex functions: (Next week we will quickly go over this in class, but it would be a good idea to look this over in advance.)

  For a subset $U \subset \mathbb{C}$, a function $f : U \to \mathbb{C}$ is *analytic* or *holomorphic* on U if

  1. The complex derivative of f exists at all points in U
  2. $\iff$ f satisfys the Cauchy Riemann relations
  3. $\iff$ f has a Taylor series expansion at all points of U, i.e., in a neighbourhood of $a \in U$,

  $$f(z) = \sum_{n=0}^{\infty} \frac{1}{n!} f^{(n)}(a)(z-a)^n$$

  Facts: holomorphic functions are infinitely differentiable; composition of holomorphic functions is holomorphic.

  Exercise: verify that the choice of *branch* of $\sqrt{z}$, defined by $\sqrt{z} = \sqrt{r}\exp(i\theta/2)$, where $z = r\exp(i\theta)$, with $r \in \mathbb{R}^+ := \{x : x \in \mathbb{R} | x \geq 0\}$ and $0 \leq \theta \leq 2\pi$ is holomorphic on $\mathbb{C} \setminus \mathbb{R}^+$.

  Reference: See, e.g., *Complex Analysis*, Newman and Bak, sections 3.1 and 6.1.

- Have a go with the program "pari" to see examples of modular forms: The Dedekind eta function is a multivalued function on $\mathfrak{H}$ given by

  $$\eta(\tau) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n),$$

  where $q = \exp(2\pi i \tau)$. This is not a modular form, but many modular forms, such as $\Delta = \eta^{24}$ are given in terms of η. Pari leaves out the $q^{1/24}$, so we have to add it back in.

  To start pari, type "gp". For help, type "?". Lines starting with "?" are input, and lines starting with "%" are output.

```
? \ps40
   seriesprecision = 40 significant terms
? D=q*eta(q)^24
%2 = q - 24*q^2 + 252*q^3 - 1472*q^4 + 4830*q^5 - 6048*q^6 - 16744*q^7 + 84480*q^8 - 113643*q^9
- 115920*q^10 + 534612*q^11 - 370944*q^12 - 577738*q^13 + 401856*q^14 + 1217160*q^15 + 987136*q^
- 6905934*q^17 + 2727432*q^18 + 10661420*q^19 - 7109760*q^20 + O(q^21)
?
? \ps200
   seriesprecision = 200 significant terms
? eta(q)
%5 = 1 - q - q^2 + q^5 + q^7 - q^12 - q^15 + q^22 + q^26 - q^35 - q^40 + q^51 + q^57 - q^70
 - q^77 + q^92 + q^100 - q^117 - q^126 + q^145 + q^155 - q^176 - q^187 + O(q^200)
?
? \ps15
   seriesprecision = 15 significant terms
? ellj(q)
%3 = q^-1 + 744 + 196884*q + 21493760*q^2 + 864299970*q^3 + 20245856256*q^4
 + 333202640600*q^5 + 4252023300096*q^6 + 44656994071935*q^7 + 401490886656000*q^8
 + 3176440229784420*q^9 + 22567393309593600*q^10
 + 146211911499519294*q^11 + 874313719685775360*q^12 + O(q^13)
?
? \p40
   realprecision = 48 significant digits (40 digits displayed)
? exp(Pi*sqrt(163))
%9 = 262537412640768743.9999999999992500725972
?
```