Modular forms (Math 7280), Spring 2004 H. Verrill

Homework 1 January 26 (due February 9)

1. Prove that for $g \in \mathrm{GL}_2(\mathbf{R})$ and $z \in \mathbf{C}$, the imaginary parts of z and $g(z)$ are related by

$$\mathrm{Im}(g(z)) = \frac{\det(g)\mathrm{Im}(z)}{|cz+d|^2}.$$

   (This shows that $\mathrm{SL}_2(\mathbf{R})$ maps $\mathfrak{h}$ to itself.)

2. (a) Find a matrix in $\mathrm{PGL}_2(\mathbf{C})$ mapping the real line to the unit circle $\{z \in \mathbf{C} \mid |z| = 1\}$.
   (b) What is the set of all matrices in $\mathrm{PGL}_2(\mathbf{C})$ which map the disc $\{z \in \mathbf{C} \mid |z| < 1\}$ to itself?

3. (a) Prove that $\overline{\Gamma_1(N)}$ has index $\frac{1}{2}N\prod_{p|N}(1-1/p)$ in $\overline{\Gamma_0(N)}$, for $N \neq 2$. (Hint: see week 2 key points handout.)
   (b) Find a set of coset representatives for $\overline{\Gamma_1(5)}$ in $\overline{\Gamma_0(5)}$.
   (c) Write the cosets representatives just found in terms of the matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

4. (a) Let $\mathcal{F}$ be any fundamental domain for $\mathrm{PSL}_2(\mathbf{Z})$ acting on $\mathfrak{h}$. Prove that there is only one point in $\mathcal{F}$ such that isotropy subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ at z has order 2. (The isotropy subgroup is also called the stabilizer, denoted $\mathrm{Stab}_{\mathrm{PSL}_2(\mathbf{Z})}(z) \subset \mathrm{PSL}_2(\mathbf{Z})$)
   (b) Find the set of all elements $z \in \mathfrak{h}$ such that the isotropy subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ at z has order 2.

5. Find an element in $\mathrm{PSL}_2(\mathbf{R})$ which has order 4.

6. Show that if $g, h \in \mathrm{PSL}_2(\mathbf{R})$ have a common fixed point, then $ghg^{-1}h^{-1}$ is either parabolic or the identity. (Hint: first consider the case that this point is ∞.)

7. Prove that for a prime p, there are exactly 2 equivalence classes of cusps under the action of $\Gamma_0(p)$ on $\mathfrak{h}^*$.

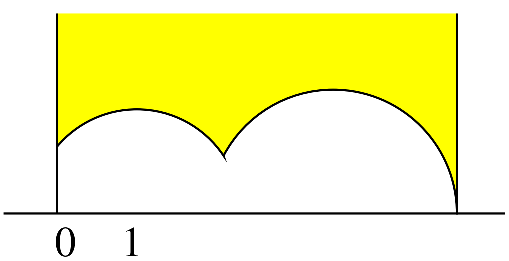
Examples using Magma

Magma can work with $\mathrm{PSL}_2(\mathbf{Z})$ and congruence subgroups, and illustrate action on $\mathfrak{h}$. Start magma from the command line by typing "magma".

First define the upper half complex plane H, together with the points $i = \sqrt{-1}$ and $r = \sqrt[3]{-1}$:

```
> H<i,r>:=UpperHalfPlane();
> // Now you can define any polygon in H, e.g.:
> poly:=[H|0,i,r+2,6,Infinity()];
> // And then look at a picture of it:
> DisplayPolygons(poly,"pic1.ps":Show:=true,Size:=[0,6,3,25]);
```

(The [0,6,3,50] means the image will show the area of $\mathfrak{h}$ with $0 \le \mathrm{Re}(x) \le 6$ and $0 \le \mathrm{Im}(x) \le 3$, and the scale is 50 pixels per unit.) When you type this command, up pops a window with the following picture:



Now define some elements of $\mathrm{PSL}_2(\mathbf{Z})$, e.g., $U := \left(\begin{smallmatrix} 1 & 0 \\ 1 & 1 \end{smallmatrix}\right)$, and look at the image of the polygon "poly" under the action of powers of U:

```
> U:=Gamma0(1)![1,0,1,1];
> DisplayPolygons([(U^n)*poly : n in [-3..4] | n ne 0], "pic2.ps":Show:=true,Size:=[-1.5,1,0.6,150]);
```

The above will display the images of the polygon under the powers of U^n for $-3 \le n \le 4$, $n \neq 0$. Make sure you understand why the picture looks the way it does.

More examples of magma commands to produce pictures in the upper half plane can be found at:
http://www.math.lsu.edu/~verrill/fundomain/magmaFD.html