Modular forms (Math 7280), Spring 2004 H. Verrill

Homework 2 February 11 (due March 1)

1. Prove that the action of $\mathrm{GL}_2^+(\mathbf{R})$ on functions on $\mathfrak{h}$ defined by

$$f|_k g(z) = (\det(g))^{k-1}(cz+d)^{-k} f(gz)$$

for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbf{R})$, is indeed an action.

2. Define a Riemann surface structure on $Y_0(1) := \mathrm{SL}_2(\mathbf{Z})\backslash\mathfrak{h}$, the quotient of the upper half complex plane under the action of $\mathrm{SL}_2(\mathbf{Z})$.

3. Show that cubics over $\mathbf{R}$ and $\mathbf{C}$ have infinitly many points.

4. (a) Show if $f(z)$ is a modular form for $\mathrm{SL}_2(\mathbf{Z})$ of weight not divisible by 4, then $f(i) = 0$.

   (b) Show if $f(z)$ is a modular form for $\mathrm{SL}_2(\mathbf{Z})$ of weight not divisible by 6, then $f(\frac{1}{2} + \frac{i\sqrt{3}}{2}) = 0$.

   Hint: consider the action on modular forms of matrices fixing the points in question.

5. Show that if f and g are modular forms for $\mathrm{SL}_2(\mathbf{Z})$ of weight m and n, then fg is a modular form of weight $m+n$.

6. Let

$$E_4(\tau) = 1 + 240\sum_n \sigma_3(n)q^n$$

and

$$E_8(\tau) = 1 + 480\sum_n \sigma_7(n)q^n,$$

where $q = \exp(2\pi i\tau)$, and where $\sigma_k(n) = \sum_{d|n} d^k$. You may assume that these are modular forms of weight 4 and 8 respectively. Given that the space of modular forms of weight 8 for $\mathrm{SL}_2(\mathbf{Z})$ is one dimensional, prove that $E_4^2 = E_8$. Use this to show that

$$\sigma_7(n) = \sigma_3(n) + 120\sum_{d=1}^{n} \sigma_3(d)\sigma_3(n-d).$$

7. Define a "differential", as usual, to satisfy the usual rule:

$$d(F(z)) = \left[\frac{\partial}{\partial z}(F(z))\right] dz.$$

Show that if $f(z)$ is a modular form of weight $2k$ for $\mathrm{SL}_2(\mathbf{Z})$, then $f(z)(dz)^k$ is invariant under the action induced on such differential forms by $z \mapsto gz$. (I.e., show that $f(g(z))(d(gz))^k = f(z)(dz)^k$.)