Modular forms (Math 7280), Spring 2004 H. Verrill   Homework 1 : Solutions

For several of the problems, there is more than one way to find the solution. Here I give some improvements or alternatives to methods students used, plus some comments about the problems.

1. Prove that for $g \in \mathrm{GL}_2(\mathbf{R})$ and $z \in \mathbf{C}$, the imaginary parts of z and $g(z)$ are related by, for $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$

$$\mathrm{Im}(g(z)) = \frac{\det(g)\mathrm{Im}(z)}{|cz+d|^2}.$$

(This shows that $\mathrm{SL}_2(\mathbf{R})$ maps $\mathfrak{h}$ to itself.)

Solution: This is a straightforwards computation. Note that this means $\mathrm{Im}(z)$ is a real valued function that acts a bit like a modular form, but preserved under a modified action, which we can define by

$$f^g(z) = \frac{|cz+d|^2}{\det(g)} f(g(z)).$$

This is, in fact an action on functions on $\mathfrak{h}$, but it preserves real valued functions rather than holomorphic functions.

2. (a) Find a matrix in $\mathrm{PGL}_2(\mathbf{C})$ mapping the real line to the unit circle $\{z \in \mathbf{C} \mid |z| = 1\}$.
   (b) What is the set of all matrices in $\mathrm{PGL}_2(\mathbf{C})$ which map the disc $\{z \in \mathbf{C} \mid |z| < 1\}$ to itself?

Solution:

(a) If g maps 0, 1 and ∞ to any three points on the unit circle, then since Mobius transformations map circles and straight lines to circles and straight lines, (as discussed in the first week), g must map the real line (plus the point at infinity) to the unit circle. For instance, map 0 and ∞ to 1 and -1 via

$$g = \left(\begin{smallmatrix} -1 & \mu \\ 1 & \mu \end{smallmatrix}\right),$$

for some μ, then for $g(1) = i$, we need $-1 + \mu = i(1+\mu)$, which implies $\mu = i$
Note that since $\mathbf{R} \cap \{\infty\}$ is preserved by any element of $h = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \mathrm{GL}_2(\mathbf{R})$, any matrix

$$gh = \left(\begin{smallmatrix} -1 & i \\ 1 & i \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = \left(\begin{smallmatrix} -a+ic & -b+id \\ a+ic & b+id \end{smallmatrix}\right) = i\left(\begin{smallmatrix} c+ia & d+ib \\ c-ia & d-ib \end{smallmatrix}\right)$$

for $a, b, c, d \in \mathbf{R}, ad - bc \neq 0$, also maps $\mathbf{R}$ into the unit circle. In other words, the matrices

$$\begin{pmatrix} \lambda\alpha & \lambda\beta \\ \lambda\overline{\alpha} & \lambda\overline{\beta} \end{pmatrix},$$

for any $\alpha, \beta, \lambda \in \mathbf{C}$ with $\lambda(|\alpha|^2 + |\beta|^2) \neq 0$. So, this question does not have a single right answer, but your matrix should have this form.

(b) g in part (a) maps the real line to the circle. Since $g(i) = 0$, this shows that g maps the upperhalf plane to the unit disc. As discussed in class, the set of matrices in $\mathrm{PGL}_2(\mathbf{C})$ which preserve the upper half plane is $\mathrm{PGL}_2^+(\mathbf{R})$ (which equals $\mathrm{PSL}_2(\mathbf{R})$, since positive reals have roots in $\mathbf{R}$, so a matrix with positive determinant can be scaled to one with unit determinant). Thus the set of matrices fixing the unit circle is given by

$$g\,\mathrm{PSL}_2(R)g^{-1}$$

A typical element is given by $\left(\begin{smallmatrix} -1 & i \\ 1 & i \end{smallmatrix}\right)\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\left(\begin{smallmatrix} -1 & i \\ 1 & i \end{smallmatrix}\right)^{-1}$. After multiplying out, and a change of variables, we end up that elements of $g\,\mathrm{PSL}_2(R)g^{-1}$ are those of the form

$$\begin{pmatrix} \alpha & \overline{\beta} \\ \beta & \overline{\alpha} \end{pmatrix},$$

for any $\alpha, \beta \in \mathbf{C}$, with $|\alpha|^2 - |\beta|^2 > 0$.

3. (a) Prove that $\overline{\Gamma_1(N)}$ has index $\frac{1}{2}N\prod_{p|N}(1-1/p)$ in $\overline{\Gamma_0(N)}$, for $N \neq 2$. (Hint: see week 2 key points handout.)
   (b) Find a set of coset representatives for $\overline{\Gamma_1(5)}$ in $\overline{\Gamma_0(5)}$.
   (c) Write the cosets representatives just found in terms of the matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$.

Solution: This is mostly a straightforwards application of what was said in class, plus hint on the example sheet. Note however, after proving that $[SL_2(\mathbf{Z}) : \Gamma_1(N)] = N\prod_{p|N}(1-1/p)$, to pass to the projection either, replace the map $\Gamma_0(N) \to (\mathbf{Z}/NZ)^\times$ by the following map, which has kernel exactly $\overline{\Gamma_1(N)}$:

$$\begin{array}{ccc} \overline{\Gamma_0(N)} & \to & (\mathbf{Z}/NZ)^\times/\{\pm\} \\ \left(\begin{smallmatrix} a & b \\ Nc & d \end{smallmatrix}\right) & \mapsto & [d] \end{array}$$

Here $[d] = \{d, -d\}$ in the quotient group. When $N = 2$, $\{\pm 1\} = (\mathbf{Z}/2\mathbf{Z})^\times = \{1\}$, but otherwise, this is a subgroup of order two, so the the index is halved in passing to the projectivization.

Alternatively, note that if g is a coset representative for $\Gamma_1(N)$ in $\Gamma_0(N)$, then since $-I \notin \Gamma_1(N)$ (unless $N = 2$), but $-I \in \Gamma_0(N)$, the cosets $\Gamma_1(N)g$ and $\Gamma_1(N)(-g)$ are distinct (since $g(-g)^{-1} = -I \notin \Gamma_1(N)$), so we can take a set of coset representatives of the form $g_i, -g_i, i = 1, \ldots n$. Passing to the projectivization, we now have $g_i(-g_i)^{-1} \in \overline{\Gamma_1(N)}$, but for $i \neq j$, $g_i(g_j)^{-1} \notin \overline{\Gamma_1(N)}$, since this can only be the case if $\pm(g_i g_j^{-1} \in \Gamma_1(N))$, but since we have written the cosets of $\Gamma_1(N)$ in $\Gamma_0(N)$ as $\pm g_i$, $\pm g_j$, and these are all distinct coset representatives, $\pm g_i g_j^{-1} \notin \Gamma_1(N)$. So g_i is a complete set of coset representatives for $\overline{\Gamma_1(N)}$ in $\overline{\Gamma_0(N)}$. I.e., the index is half of $[\Gamma_0(N) : \Gamma_1(N)]$.

4. (a) Let $\mathcal{F}$ be any fundamental domain for $\mathrm{PSL}_2(\mathbf{Z})$ acting on $\mathfrak{h}$. Prove that there is only one point in $\mathcal{F}$ such that isotropy subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ at z has order 2. (The isotropy subgroup is also called the stabilizer, denoted $\mathrm{Stab}_{\mathrm{PSL}_2(\mathbf{Z})}(z) \subset \mathrm{PSL}_2(\mathbf{Z})$)

   (b) Find the set of all elements in $z \in \mathfrak{h}$ such that the isotropy subgroup of $\mathrm{PSL}_2(\mathbf{Z})$ at z has order 2.

   Solution: Starting with the "standard" fundamental domain, (a) is reasonably straightforwards following the same arguments as were used to prove that this was in fact a fundamental domain. Since points of any other fundamental domain all have isotropy groups of exactly the same order, since if g fixes z, then hgh^{-1} fixes hz, so conjugation by h induces an isomorphism between the isotropy groups of z and hz. So, if there's only one point with isotropy subgroup of order 2 in one domain, this is true for all fundamental domains.

   (b) is similar — if $z \in \mathfrak{h}$ has an isotropy group of order 2 in $\mathrm{PSL}_2(Z)$, then by a similar argument to above, $z = gi$ for some $g \in \mathrm{PSL}_2(\mathbf{Z})$. One can go further, expanding gi, and writing down the form of all these points.

   An alternative method is to write down all the matrices of order 2 in $\mathrm{PSL}_2(\mathbf{Z})$. As in the classification given in the lectures, these are exactly the matrices with trace 0, so these have the form

$$\begin{pmatrix} a & b \\ c & -a \end{pmatrix}$$

   where $-a^2 - bc = 1$. The fixed point in $\mathfrak{h}$ of this matrix is a solution to $az + b = cz^2 - az$, i.e.,

$$\frac{a+i}{c},$$

   with the condition $c|(1 + a^2)$. Thus the points in $\mathfrak{h}$ with isotropy group of order 2 are all the points of this form.

5. Find an element in $\mathrm{PSL}_2(\mathbf{R})$ which has order 4.

   Solution: One can make use of the well known fact that for example, rotation matrices acting on $\mathbf{R}^2$ have the form $\begin{pmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{pmatrix}$ A matrix of order 4 in $\mathrm{PSL}_2(\mathbf{R})$ must have order 8 in $\mathrm{SL}_2(\mathbf{R})$, so we just take a rotation matrix of order 8, i.e., take $\theta = 2\pi/8$. One could then conjugate this to obtain other matrices of order 4 in $\mathrm{PSL}_2(\mathbf{R})$, since conjugation does not change order.

   However, in the spirit of the classification of the order of matrices in terms of their trace, as given in class for the order 2, 3 and ∞ case, we might attempt to find a similar condition for a matrix to have order 4.

   If a matrix in $\mathrm{SL}_2(\mathbf{R})$ has order 8, it's characteristic polynomial must divide $x^8 - 1$, or $x^4 + 1$, since we assume the order is not less than 8.

   Note that over $\mathbf{R}$, x^4+1 factors as $(x^2-\sqrt{2}x+1)(x^2+\sqrt{2}x+1)$, and so since matrices in $\mathrm{GL}_2(\mathbf{R})$ have characteristic polynomial given by

$$x^2 - \mathrm{Tr}(g)x + \det(g),$$

   and since $\det(g) = 1$ in $\mathrm{PSL}_2(\mathbf{R})$, we must have $\mathrm{Tr}(g) = \sqrt{2}$, and this is a necessary and sufficient condition for a matrix in $\mathrm{PSL}_2(\mathbf{Z})$ to have order 4. For example

$$\begin{pmatrix} \sqrt{2} & -1 \\ 1 & 0 \end{pmatrix}$$

   has order 8. Similar conditions can be found for other orders.

6. Questions 6. and 7. are straightforwards; keen students could try and find a formula for the number of inequivalent cusps for $\Gamma_0(N)$, for arbitrary N.