I. INTRODUCTION

The general problem we will address is the following: we seek to find certain unknown functions $x_1(t), \ldots, x_n(t)$ of a single real variable t. The variable t frequently represents *time*; the quantities $x_1, \ldots, x_n$ typically represent quantities asociated with some system—perhaps physical, chemical, or biological—under study, such as the position or velocity of a particle, the current in an electric circuit, or the population of some species in a specific environment. We are given a system of equations which these functions must satisfy; equations which involve not only the functions but also their derivatives, and are therefore called *differential equations*. We will always assume that these equations have been written in the form

$$\begin{aligned} x_1'(t) &= f_1(t, x_1, \ldots, x_n) \\ &\vdots \\ x_n'(t) &= f_n(t, x_1, \ldots, x_n) \end{aligned} \tag{1.1}$$

Remark 1.1 below). We refer to t as the *independent* variable and $x_1, \ldots, x_n$ as the *dependent* variables. Because there is only one independent variable, the equations (1.1) are called *ordinary* differential equations (o.d.e.'s); because only first derivatives appear, they are *first order* equations. When there are several equations, that is, when $n \geq 2$, we speak of a *system* of differential equations. Simple examples convince us that (1.1) will in general have many solutions; thus $x(t) = Ae^{rt}$ is a solution of $x' = rx$ for any value of A. To obtain a unique solution we must supplement (1.1) with an *initial condition* which specifies the values of the functions at some initial time:

$$x_i(t_0) = x_i^0, \qquad i = 1, \ldots, n. \tag{1.2}$$

We usually express (1.1) and (1.2) in vector notation, either regarding elements of $\mathbb{R}^n$ as column vectors, and thus writing

$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}, \qquad x' = \begin{bmatrix} x_1' \\ \vdots \\ x_n' \end{bmatrix}, \qquad \text{and} \qquad f = \begin{bmatrix} f_1 \\ \vdots \\ f_n \end{bmatrix}, \tag{1.3}$$

or using the notation

$$x = (x_1, \ldots, x_n), \qquad x' = (x_1', \ldots, x_n'), \qquad \text{and} \qquad f = (f_1, \ldots, f_n). \tag{1.4}$$

(In most cases the distinction between these notations is unimportant, but it has significance when we realize linear mappings via matrix multiplication.) In either case, (1.1) and (1.2) then form an *initial value problem*

$$\begin{aligned} x' &= f(t, x), \\ x(t_0) &= x^0. \end{aligned} \tag{1.5}$$

Typically the function f is defined and continuous in some open set $D \subset \mathbb{R} \times \mathbb{R}^n$.

Example 1.1: Population models. (a) Let $x(t)$ denote the population, at time t, of a certain species in a certain geographical region. At first approximation the rate of change of such a population will be proportional to the population (that is, a fixed fraction of individuals will die or give birth in each fixed time interval), suggesting the differential equation $x' = rx$, with r a constant *growth rate.* More generally, this growth rate may vary:

$$x' = r(t,x)x.$$

One standard model, the *logistic equation*, takes $r(t,x) = r_0(1 - x/x_0)$, where r_0 and x_0 are (constant) parameters. Here for small populations ($x \ll x_0$) the growth rate satisfies $r(t,x) \simeq r_0$, but environmental effects (crowding, limited food supply, etc.) cause the growth rate to decrease as the population increases; the growth rate is zero when $x = x_0$ and the population actually decreases when $x > x_0$. x_0 is called the *carrying capacity* of the environment.

(b) We may write similar equations to model the populations of two or more species. For example, the *predator-prey* model of Lotka and Volterra describes a predator species, whose population is denoted y, and a prey species, whose population is x. The equations are

$$\begin{aligned} x' &= r(x,y)x \equiv r_0(1 - x/x_0 - \alpha y)x, \\ y' &= s(x,y)y \equiv -s_0(1 - \beta x)y. \end{aligned} \tag{1.6}$$

Note that, in the absence of predators ($y = 0$), the prey population obeys the logistic equation described in (a), while predators decrease the growth rate of the prey. On the other hand, predators in the absence of prey have a negative growth rate (they are starving to death) which becomes positive in the presence of sufficient numbers of prey. The model, which we will discuss in more detail later, predicts a cyclic oscillation in the populations of the two species.

Remark 1.1: The initial value problem (1.5) may be generalized in several ways.

(a) We could consider differential equations of the form

$$F_k(t, x', x) = 0, \qquad k = 1, \ldots, n. \tag{1.7}$$

Here we still have n first order equations for the n unknown functions. If we could solve the equations $F_k(t,y,x) = 0$ for y, that is, find functions $f(x,t)$ such that $F(t,y,x) = 0$ if and only if $y = f(t,x)$, then (1.7) would take precisely the form (1.1). (We emphasize that we are here solving algebraic, not differential, equations.) The implicit function theorem assures us that if t_0, y^0, and x^0 satisfy $F(t_0, y^0, x^0) = 0$, and if moreover the n by n matrix

$$\frac{\partial F_k}{\partial y_j}(t_0, y^0, x^0), \qquad k, j = 1, \ldots, n$$

is not singular, then we can in fact solve for y as $y = f(t,x)$ in some neighborhood of (t_0, y^0, x^0), so that locally (1.7) reduces to (1.1). (This is particularly relevant when we are imposing the initial condition $x(t_0) = x^0$.) We will not pursue this question further.

(b) We might consider differential equations containing derivatives of higher than first order. For example, consider n equations of order m, of the form

$$x_i^{(m)} = F_i(t, x^{(m-1)}, x^{(m-2)}, \ldots, x', x) = 0, \qquad i = 1, \ldots, n. \tag{1.8}$$

There is a standard trick to reduce such equations to first order ones: introduce new variables $z_1(t), \ldots z_{m-1}(t)$ (each representing an n-vector of unknowns) and require that x, z_1, $\ldots z_{m-1}$ satisfy

$$\begin{aligned} x' &= z_1, \\ z_1' &= z_2, \\ z_{m-2}' &= z_{m-1} \\ z_{m-1}' &= F(t, z_{m-1}, z_{m-2}, \ldots, z_1, x). \end{aligned} \tag{1.9}$$

of equations (1.9) is clearly equivalent to (1.8), in the sense that any solution to one furnishes a solution to the other, but (1.9) is now a first order system of the form discussed in (a).

(c) Differential equations encountered in studying physical systems often have the form $x' = f(t, x, \mu)$, where the additional variables $\mu = (\mu_1, \ldots, \mu_m)$ are called *parameters*. The quantities α, β, x_0, and r_0 in the Lotka-Volterra equations (1.6) are parameters. From one point of view the presence of such parameters makes little difference; they do not vary with t and may therefore be ignored in determining the solution of the differential equations. But the solution we find does depend on the value assigned to the parameters, and we will ask whether this dependence is continuous, differentiable, etc. For example, the initial value problem $x' = kx$, $x(0) = x^0$ for a single dependent variable x has solution $x(t) = x^0 e^{kt}$ which is a differentiable function of the parameter k.

Example 1.2: *Newton's equations.* Consider a physical system composed of N point particles, free to move in three dimensions; we let $\vec{x}_k = (x_{k1}, x_{k2}, x_{k3})$ denote the position of the k^{th} particle. This particle has a mass m_k and is subject to a force $\vec{F}_k$ which depends on the positions of all the particles and possibly also on the time. In this context, Newton's equations may be written as

$$m_k x_{k\alpha}'' = F_{k\alpha}(t, x), \qquad k = 1, \ldots, n, \quad \alpha = 1, 2, 3.$$

This is an example of a second-order system of equations as considered in Remark 1.1(b). If we introduce the velocities $\vec{v}_k$ of the particles as new variables, we reduce to a first order system of the form (1.1):

$$\begin{aligned} x_{k\alpha}' &= v_{k\alpha} \\ v_{k\alpha}' &= m_k^{-1} F_{k\alpha}(t, x). \end{aligned} \tag{1.10}$$

In these equations the variables m_k play the role of parameters.

Remark 1.2: We will pay particular attention to two special classes of equations:

(a) *Linear* differential equations have the form

$$x' = A(t)x + u(t), \tag{1.11}$$

where A is an n by n matrix and we are using the the column vector notation (1.3); in terms of components, (1.11) is written

$$x_i' = \sum_{j=1}^{n} A_{ij}(t)x_j + u_i(t).$$

The form (1.11) is so special that we can say much more about the properties of the solutions than we can for the general equation (1.5).

(b) *Autonomous* differential equations have the form

$$x' = f(x); \tag{1.12}$$

that is, the right hand side has no explicit t dependence. The logistic and predator-prey equations discussed in Example 1.1 are autonomous, as are Newton's equations (1.10) when the force function F is independent of time. In practice, an autonomous equation indicates that the system being studied (physical, chemical, or biological) is isolated from the rest of the world, that is, that its behaviour depends only on its own state (as determined by the values of the quantities x). For such a system, the initial time t_0 is irrelevant to the behavior, since if $x(t)$ is any solution of (1.12), then so is $\tilde{x}(t) \equiv x(t - t_1)$, for any t_1.

There is again a standard trick by which any system can be converted into an autonomous system, at the price of introducing an extra dependent variable. Suppose that we are given a (possibly nonautonomous) initial value problem for $x = (x_1(t), \ldots, x_n(t))$:

$$x' = f(t, x); \qquad x(t_0) = x^0. \tag{1.13}$$

We introduce a new dependent variable $u(t)$ and a new problem

$$u' = 1, \qquad x' = f(u, x); \qquad u(t_0) = t_0, \qquad x(t_0) = x^0. \tag{1.14}$$

If we first solve (1.14) for $u(t)$ we obtain $u(t) = t$, and the remaining equations in (1.14) are then clearly equivalent to (1.13). (This trick, although clever, is not often very useful, since the new problem is likely to be as difficult to analyze as the original one.)

Remark 1.3: There is an familiar graphical interpretation of the initial value problem which is easiest to visualize when $n = 1$. Suppose for simplicity that $f(t, x)$ is defined in all of $\mathbb{R}^2$, and imagine drawing a graph of the solution $x(t)$. The equation $x'(t) = f(t, x)$ tells us what the slope of the solution curve should be at the point (t, x), if the curve goes through this point. We say that the equation specifies a *direction field*. To visualize the field, we may draw a short line segment of slope $f(t, x)$ at many points (t, x), as in Figure 1.1 A solution curve of the o.d.e., also called an *integral curve* of the direction field, is a curve which is everywhere tangent to these line segments. To solve the initial value problem is to find an integral curve which passes through the point (t_0, x^0). Similar pictures are very usuful in analyzing autonomous systems with two dependent variables, such as the Lotka-Volterra system (1.6), as we shall see in Chapter 4.

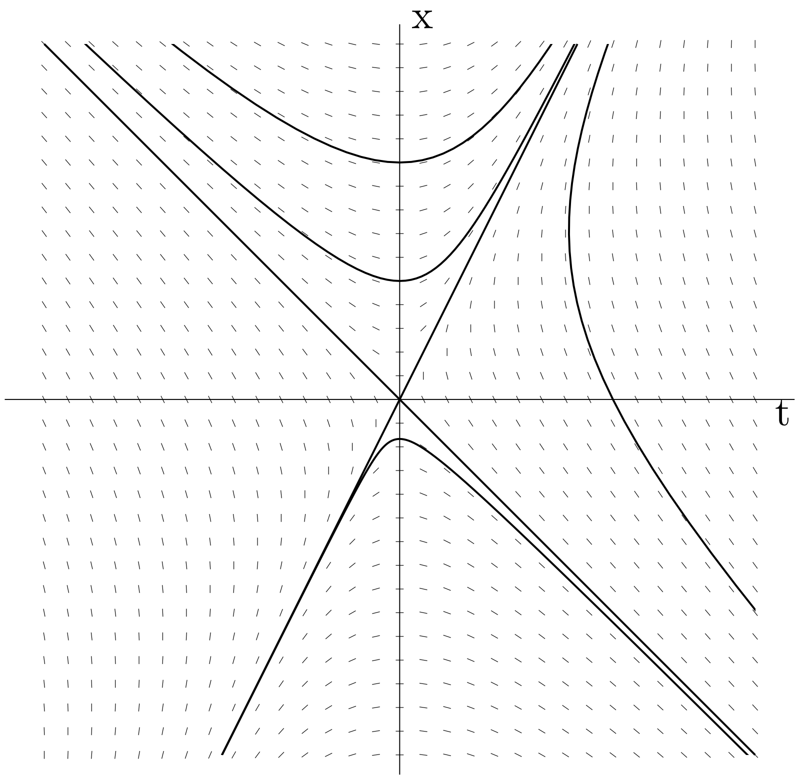

Figure 1.1
Direction field and integral curves for $x'(t) = \dfrac{2t}{x-t}$