Appendix: Proof of Lemma 6.2

In *Theory of Linear and Integer Programming* (by Schrijver), there is a characterization (Corollary 22.13d) of TDI systems of the form $A\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$, where A is nonnegative and integral. The same proof also shows the following similar result.

Corollary 22.13d'. *Let A be a nonnegative integral matrix with no zero rows, and let $\boldsymbol{b}$ be a rational vector. Then the system $A\boldsymbol{x} \geq \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$ is TDI iff for each $\{0,1\}$-vector $\boldsymbol{y}$, there exists an integral vector $\boldsymbol{z} \geq \boldsymbol{0}$ with $\boldsymbol{z}^T A \leq \lceil \boldsymbol{y}^T A/2 \rceil$ and $2\boldsymbol{z}^T\boldsymbol{b} \geq \boldsymbol{y}\boldsymbol{b}$.*

This implies the following characterization of Mengerian clutters: Let $\mathcal{H} = (V, E)$ be a clutter (we assume $E \neq \emptyset$ and $\emptyset \notin E$). Then $\mathcal{H}$ is Mengerian iff

($*$) for every subset F of E, there exists a multisubset F' of E with $2|F'| \geq |F|$ and $d_{F'}(v) \leq \lceil d_F(v)/2 \rceil$ for all $v \in V$,

where $d_F(v)$ stands for the degree of v in the clutter $\mathcal{F} = (V, F)$. Note that "for every F" can be replaced by "for every F of odd cardinality".

Algorithm 1. The above theorem allows a brute force algorithm for deciding if a clutter $\mathcal{H} = (V, E)$ is Mengerian. Suppose we have a solver ILP$(A, \boldsymbol{c})$ for $\max\{\boldsymbol{y}^T\boldsymbol{1} : \ \boldsymbol{y}^T A \leq \boldsymbol{c}^T;\ \boldsymbol{y} \geq \boldsymbol{0} \text{ is integral}\}$.

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Let A_{m×n} be the E × V incidence matrix of H.
Let M be the set of k × n submatrices of A for all k ∈ {3, 5, ..., 2⌈m/2⌉ − 1}.
    while M ≠ ∅
        take any M ∈ M
        let k be the number of rows of M and let ȳ = ILP(A, ⌈1^T M/2⌉)
        if ȳ^T 1 < k/2, Return[ H is not Mengerian ]; else M := M\{M}
    end
Return[ H is Mengerian ]
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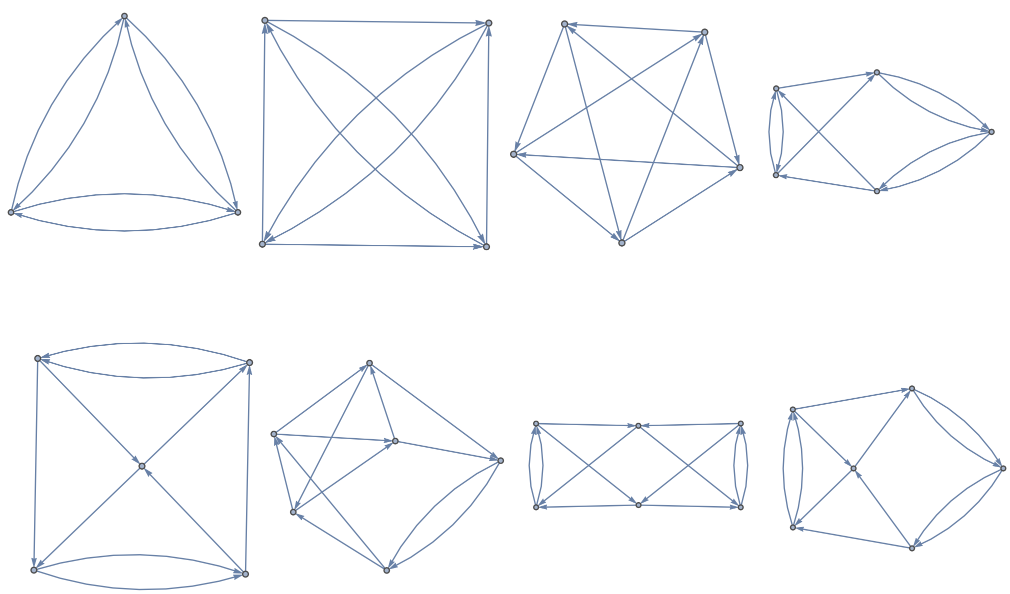
Lemma 6.2. *Tournament G_1 is cycle Mengerian.*

Proof. Our proof is an implementation of Algorithm 1 and thus is computer-assisted. In the following we explain how the proof goes. Tournament G_1 has vertices 1, 2, 3, 4, 5, 6 and arcs 12, 23, 34, 45, 51, 13, 35, 52, 24, 41, 16, 26, 63, 64, 65.

Claim 1. G_1 has 32 cycles (directed): 1-9-10, 2-7-8, 3-6-10, 4-8-9, 5-6-7, 5-11-15, 8-12-15, 10-11-14, 1-2-3-10, 1-2-5-7, 1-4-5-9, 1-5-12-15, 1-10-12-14, 2-3-4-8, 3-4-5-6, 3-10-11-13, 4-5-11-14, 4-8-12-14, 5-7-11-13, 7-8-12-13, 1-2-3-4-5, 1-3-10-12-13, 1-4-5-12-14, 1-5-7-12-13, 3-4-5-11-13, 3-4-8-12-13, 6-7-8-9-10, 8-9-10-11-15, 1-3-4-5-12-13, 2-3-8-10-11-15, 6-7-8-10-12-14, 7-8-9-10-11-13, where each number i represents the ith arc of G_1.

Claim 2. Each digraph in the figure can be expressed as the arc-disjoint union of ≥ 3 cycles.



Claim 3. Let $B = \{\{5,30\}, \{6,31\}, \{9,27\}, \{9,31\}, \{9,32\}, \{10,27\}, \{10,28\}, \{10,30\}, \{10,31\}, \{10,32\}, \{11,27\}, \{11,28\}, \{11,30\}, \{11,31\}, \{11,32\}, \{12,27\}, \{12,28\}, \{12,30\}, \{12,31\}, \{12,32\}, \{13,28\}, \{13,30\}, \{13,32\}, \{14,27\}, \{14,31\}, \{14,32\}, \{15,27\}, \{15,28\}, \{15,30\}, \{15,31\}, \{15,32\}, \{16,31\}, \{17,27\}, \{17,28\}, \{17,30\}, \{17,31\}, \{17,32\}, \{18,28\}, \{18,30\}, \{18,32\}, \{19,30\}, \{19,31\}, \{20,30\}, \{21,27\}, \{21,28\}, \{21,30\}, \{21,31\}, \{21,32\}, \{22,27\}, \{22,28\}, \{22,30\}, \{22,31\}, \{22,32\}, \{23,27\}, \{23,28\}, \{23,30\}, \{23,31\}, \{23,32\}, \{24,27\}, \{24,28\}, \{24,30\}, \{24,31\}, \{24,32\}, \{25,27\}, \{25,28\}, \{25,30\}, \{25,31\}, \{25,32\}, \{26,27\}, \{26,28\}, \{26,30\}, \{26,31\}, \{26,32\}, \{27,29\}, \{27,30\}, \{28,29\}, \{28,31\}, \{29,30\}, \{29,31\}, \{29,32\}, \{30,31\}, \{30,32\}, \{31,32\}\}$. Then for each pair $\{i,j\}$ in B, the arc-disjoint union of the ith and jth cycles of G_1 is a subdivision of a digraph in the above figure. It follows that if F is a minimal set of cycles that violates $(*)$, then $F \not\supseteq \{c_i, c_j\}$ for all $\{i,j\} \in B$.

Let A be the cycle-arc incidence matrix of G_1. Let $\mathtt{R} = \{1, 2, ..., 32\}$, which is the index set for rows of A. For each $\mathtt{r} \subseteq \mathtt{R}$, let $A_{\mathtt{r}}$ denote the submatrix of A formed by its rows indexed by elements of $\mathtt{r}$. The following is our implementation of Algorithm 1.

```
r = ∅;
For i = 1, i < 2^32, i++,
    r = NextSubset [ r, R ];
    if |r| is even or r contains a member of B
        do nothing;
    else
        c = ⌈1^T A_r/2⌉;
        ỹ = ILP(A, c);
        ȳ = ⌈ỹ⌉;
        if ȳ or c−ȳ^T A has a negative coordinate
            Return[the solver has a bug];
        else
            if ȳ^T 1  < |r|/2
                return[G_1 is not Mengerian];
            else
                do nothing;
            end;
        end;
    end;
 end;
 Return[G_1 is Mengerian];
```

This implementation contains two refinements over Algorithm 1. By Claim 2, we may skip sets $\mathtt{r}$ if it contains a member of B. This speeds up the program by a factor of 10. For each solution $\tilde{\boldsymbol{y}}$ produced by the solver, we do not assume the optimality or even the feasibility of $\tilde{\boldsymbol{y}}$, since we do not want to rely on the accuracy of the solver. We take the corresponding rounding integral vector $\bar{\boldsymbol{y}}$ and we verify that $\bar{\boldsymbol{y}}$ satisfies $\bar{\boldsymbol{y}} \geq \mathbf{0}$, $\bar{\boldsymbol{y}}^T\mathbf{1} \geq |\mathtt{r}|/2$, and $\boldsymbol{c}^T \geq \bar{\boldsymbol{y}}^T A$. This extra step eliminates potential errors caused by the solver. Therefore, the only operations we need the computer to perform are addition, subtractions, and multiplications of integers.

We implemented this program under Mathematica. It took an ordinary desktop 83 hours to complete the computation. The final result shows that G_1 is cycle Mengerian. ∎