A chain theorem for 3^+-connected graphs

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Abstract

A 3-connected graph is 3^+-connected if it has no 3-separation that separates a "large" fan or $K_{3,n}$ from the rest of the graph. It is proved in this paper that, except for K_4, every 3^+-connected graph has a 3^+-connected proper minor that is at most two edges away from the original graph. This result is used to characterize Q-minor-free graphs, where Q is obtained from the Cube by contracting an edge.

1 Introduction

A *chain theorem* for a class $\mathcal{G}$ of graphs is a result asserting the existence of a number t such that, if $G \in \mathcal{G}$ is not a minor minimal member of $\mathcal{G}$, then G has a proper minor $H \in \mathcal{G}$ with $|E(G)| - |E(H)| \leq t$. The best known chain theorem is the following result of Tutte [2], which says that $t = 2$ if $\mathcal{G}$ is the class of 3-connected simple graphs.

Theorem 1.1 (Tutte). *If a 3-connected simple graph G is not a wheel then G has an edge e such that either $G\backslash e$ or G/e is simple and 3-connected.*

Since a chain theorem provides a very useful induction tool, a lot of efforts have been made by different researchers on other connectivities, most of which are different variations of 4-connectivity, see [8, 9, 5, 4, 3, 1]. In this paper we prove a chain theorem for a slightly better 3-connectivity.

Throughout this paper, by a graph we always mean a loopless graph. A *separation* of a graph $G = (V, E)$ is a pair of subgraphs (G_1, G_2), where $G_i = (V_i, E_i)$ $(i = 1, 2)$, such that (E_1, E_2) is a partition of E, $V_1 \cup V_2 = V$, and $V_1 - V_2 \neq \emptyset \neq V_2 - V_1$. We will refer $V_1 \cap V_2$ as the *cut set* induced by the separation, and $V_1 - V_2, V_2 - V_1$ as the *interior* vertices of the two parts. If $|V_1 \cap V_2| = k$, then the separation is also called a *k-separation*. For an integer $k \geq 0$, G is called *k-connected* if $|V| > k$ and G has no k'-separations for any $k' < k$. A 3-connected simple graph is 3^+-*connected* if it has no 3-separation as illustrated in Figure 1.1. If a graph does have such a separation, then the part on the right will be referred to as the *special part*. In case the special part is a fan (the second graph in Figure 1.1), the vertex in the cut set that is adjacent to all three internal vertices of the special part will be called its *center vertex*.

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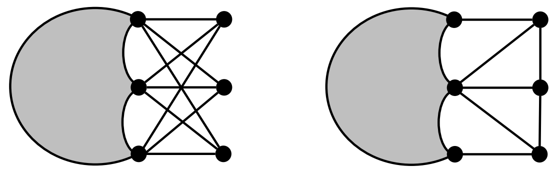

Figure 1.1: Forbidden 3-separations

It is not difficult to see that a 3-connected simple graph is 3^+-connected if and only if none of its 3-separations separates either $K_{3,n}$ ($n \geq 3$) or a fan (with ≥ 3 spokes) from the rest of the graph. This connectivity is stronger than 3-connectivity, but it is quite far away from 4-connectivity. This connectivity is very useful in dealing with graph properties that are unaffected when a large $K_{3,n}$ or fan is replaced by a smaller one. For instance, if Q is obtained by contracting an edge of the Cube, then being Q-free is such a property (as we shall see in Section 4). The following is our chain theorem.

Theorem 1.2 *Every 3^+-connected graph G has a 3^+-connected proper minor H with $|E(G)| - |E(H)| \leq 2$, unless $G = K_4$.*

In fact, what we can prove is stronger. We say that an induced cycle $x_1x_2...x_{3k}x_1$ of G *spans a ring* if $k \geq 2$ and there is a circular sequence $y_1y_2...y_ky_1$ of (not necessarily distinct) vertices such that

(i) $d(x_{3i-2}) = 4$ and $d(x_{3i-1}) = d(x_{3i}) = 3$, for $i = 1, 2, ..., k$;

(ii) $y_i \neq y_{i+1}$, for $i = 1, 2, ..., k-1$, and $y_k \neq y_1$;

(iii) $y_ix_{3i+j} \in E$, for $i = 1, 2, ..., k$ and $j = -2, -1, 0, 1$, where $x_{3k+1} = x_1$.

The subgraph formed by all edges incident with $x_1, x_2, ..., x_{3k}$ will be called a *ring*. Figure 1.2 shows a ring where all y_i's are distinct.

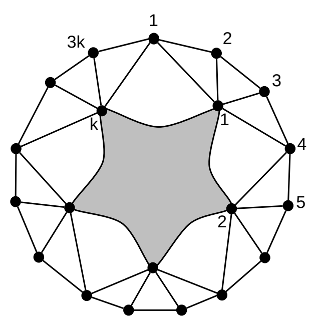

Figure 1.2: A ring with $k = 5$.

Let e be an edge of a ring. Since e is in a triangle, G/e is not simple. On the other hand, if e is incident with a cubic vertex, then $G\backslash e$ is not 3-connected; if e is not such an edge, then $G\backslash e$ contains a forbidden 3-separation. Therefore, no edge of a ring can be deleted or contracted to maintain 3^+-connectivity. The following is our main result, where W_k is the wheel with k spokes.

Theorem 1.3 *Suppose G is 3^+-connected and neither $G\backslash e$ nor G/e is 3^+-connected, for every $e \in E(G)$. Then either $G \in \{W_3, W_4, W_5\}$ or G has a ring.*

We will see in Section 3 that if a 3^+-connected graph G has a ring (as labeled above), then $G\backslash x_1y_1/x_1x_2$ is 3^+-connected. Thus Theorem 1.2 follows from Theorem 1.3 immediately.

It is natural to ask if, in Theorem 1.3, "G has a ring" can be replaced with "every edge belongs to a ring". Unfortunately, this is not true, as shown by the following examples, where e cannot be deleted or contracted yet they do not belong to any ring.

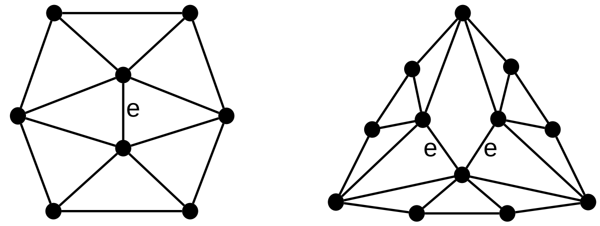

Figure 1.3: Graphs R_2 and R_3

On the positive side, our proof does imply that in Theorem 1.3, "G has a ring" can be replaced with "every edge incident with a cubic vertex belongs to a ring". Although this result does not completely characterize all "critical" graphs, it is good enough for many applications. We prove the theorems in the next two sections and we use them in the last section to characterized Q-minor-free graphs, where $Q = Cube/e$.

2 Deletion and contraction

In this section we present a few lemmas on deletion and contraction. We remark that a 3-connected graph may have parallel edges but not any loops. Our first lemma is a result of Seymour [10].

Lemma 2.1 (Seymour). *Let e be an edge of a 3-connected simple graph G on five or more vertices. Then either G/e is 3-connected (which may not be simple) or $G\backslash e$ is a subdivision of a 3-connected simple graph.*

The next is a characterization of "deletable" edges.

Lemma 2.2 *Let $e = xy$ be an edge of a 3-connected graph such that $G\backslash e$ has three internally vertex-disjoint xy-paths. Then $G\backslash e$ is 3-connected.*

Proof. If $G\backslash e$ is not 3-connected, then $G\backslash e$ has a 2-separation (G_1, G_2). Since G is 3-connected, (G_1, G_2) cannot be extended into a 2-separation of G, it follows that x is an internal vertex of some G_i and y is an internal vertex of G_j with $j \neq i$. But this is impossible since $G\backslash e$ has three internally vertex-disjoint xy-paths. Thus $G\backslash e$ is 3-connected. ■

The next lemma is about "contractible" edges. In particular, part (i) is due to Halin [2].

Lemma 2.3 *Let x be a cubic vertex of a 3-connected graph G and let x_1, x_2, x_3 be its three neighbors. Then*

(i) G/xx_i is 3-connected for at least one i;

(ii) G/xx_1 is 3-connected if $G - \{x, x_1\}$ has a cycle containing both x_2 and x_3;

(iii) G/xx_1 is 3-connected if x_2x_3 is an edge of G.

Proof. We only prove (ii) and (iii). Suppose G/xx_1 is not 3-connected. Then G/xx_1 has two vertices whose deletion disconnects the graph. Since G is 3-connected, one of these two vertices must be the vertex obtained by contracting xx_1. Let y be the other vertex. Thus G has a 3-separation (G_1, G_2) with $\{x, x_1, y\}$ being the corresponding cut set. Since x is cubic and $G - \{x_1, y\}$ is connected, neither G_1 nor G_2 contains both x_2 and x_3. That is, x_2 and x_3 are contained in different components of $G-\{x, x_1, y\}$, which contradicts the assumptions in both (ii) and (iii), and thus the lemma is proved. ■

Our last lemma is about forbidden 3-separations.

Lemma 2.4 *Let xy be an edge of a 3^+-connected graph G. Suppose G/xy is simple and has a forbidden 3-separation (G_1, G_2), where G_2 is the special part. Then*

(i) the new vertex is in the cut set;

(ii) each of x and y is adjacent (in G) to at least one interior vertex of G_2;

(iii) the new vertex is adjacent to all three interior vertices of G_2.

Proof. Let z be the new vertex obtained by contracting xy. Since G is 3-connected and G/xy is simple, z must have degree at least four, which implies that z is not an interior vertex of G_2. On the other hand, if z is an interior vertex of G_1, then uncontracting z would result in a forbidden 3-separation of G, which is impossible, so (i) is proved. If one of x and y is not adjacent (in G) to any interior vertex of G_2, then $(E(G_1) \cup \{xy\}, E(G_2))$ defines a forbidden 3-separation of G, which is impossible, and thus (ii) is proved. Finally, since z is adjacent to either one or three interior vertices of G_2, (iii) follows from (ii) immediately. ■

3 Proving the main theorems

Proof of Theorem 1.2 (using Theorem 1.3). If $G = W_k$ ($k = 4, 5$), then $H = W_{k-1}$ satisfies the requirement. By Theorem 1.3, we may assume that G has a ring whose vertices are labeled as in the definition in Section 1. We prove that $H = G\backslash x_1y_1/x_1x_2$ satisfies the requirement. Since G/x_1y_1 is not 3-connected, by Lemma 2.1, $G\backslash x_1y_1$ is simple and 3-connected. Similarly, since $G\backslash x_1y_1\backslash x_1x_2$ is not a subdivision of a simple graph, H is simple and 3-connected.

Suppose H is not 3^+-connected. Then H has a forbidden 3-separation (H_1, H_2), where H_2 is the special part. Using the same argument as the one used in proving Lemma 2.4(i) we conclude that the new vertex x_0 belongs to the cut set. Since x_0 has only two cubic neighbors x_2 and x_{3k}, it follows that H_2 has to be a fan and x_0 must start a path on five vertices with all three internal vertices being cubic. However, since neither y_1 nor y_k is cubic in H, no such path exists, which proves that H is 3^+-connected. ■

In the rest of this section we prove Theorem 1.3. We divide the whole proof into a sequence of lemmas.

Lemma 3.1 *If G is 3^+-connected and has minimum degree ≥ 4, then G has an edge e such that either $G\backslash e$ or G/e is 3^+-connected.*

Proof. The degree condition implies that G is not a wheel. By Theorem 1.1, G has an edge e such that a member H of $\{G\backslash e, G/e\}$ is simple and 3-connected. It follows that H has at most two cubic vertices and thus H is 3^+-connected. ■

Let x be a cubic vertex of G. We call x a type-I vertex if at most one pair of neighbors of x are adjacent; we call x a type-II vertex if at least two pairs of neighbors of x are adjacent.

Lemma 3.2 *If G is 3^+-connected and has a type-I vertex x, then G has an edge e such that either $G\backslash e$ or G/e is 3^+-connected.*

Proof. By (i) and (iii) of Lemma 2.3, x has a neighbor x' such that G/xx' is simple and 3-connected. Thus we may assume that G/xx' has a forbidden 3-separation (H_1, H_2), where H_2 is the special part. By

Lemma 2.4(i), the new vertex x^* must belong to $V(H_1) \cap V(H_2)$. Let $V(H_1) \cap V(H_2) = \{x^*, y, z\}$ and let $V(H_2) - V(H_1) = \{u, v, w\}$. Naturally, the pair $(E(H_1) \cup \{xx'\}, E(H_2))$ induces a 4-separation (G_1, G_2) of G such that $V(G_1) \cap V(G_2) = \{x, x', y, z\}$ and $V(G_2) - V(G_1) = \{u, v, w\}$. By Lemma 2.4(ii), it is routine to verify that G is one of the four graphs in Figure 3.1. We distinguish among three cases.

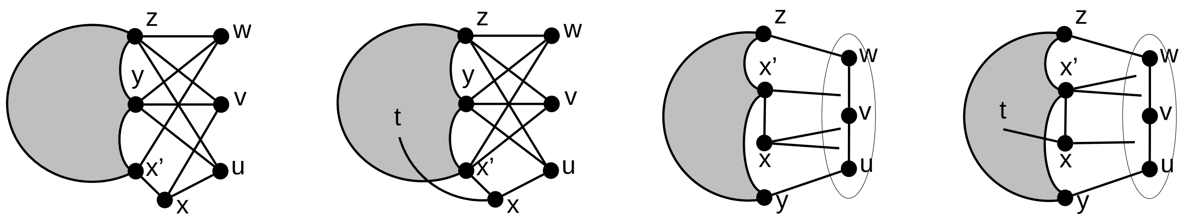

Figure 3.1: Uncontracting xx'.

Case 1. H_2 is $K_{3,3}$, and so G is one of the first two graphs in Figure 3.1.

Claim 1. *If G/ux is simple, then G/ux is 3^+-connected.*

By Lemma 2.3(ii), G/ux is 3-connected since u is cubic and $G - \{u, x\}$ has a cycle $yvzw$. If G/ux is not 3^+-connected, it has a forbidden 3-separation (G_1', G_2') such that G_2' is the special part. By Lemma 2.4, the new vertex belongs to both G_1' and G_2', and it is adjacent to all three interior vertices of G_2'. Therefore, at least three of the four neighbors of $\{x, u\}$ are cubic, which implies that at least one of y, z is cubic, contradicting the 3-connectivity of G.

By Claim 1 we may assume that G is the second graph in Figure 3.1 and t is y or z, say y.

Claim 2. If $yz \in E(G)$ then $G \backslash yz$ is 3^+-connected.

By Lemma 2.2, $G \backslash yz$ is simple and 3-connected. Suppose $G \backslash yz$ has a forbidden 3-separation (G_1', G_2'), where G_2' is the special part. Then at least one of y, z must be an interior vertex of G_2', for otherwise $(G_1' + yz, G_2')$ would be a forbidden 3-separation of G. However, this is impossible since every interior vertex of G_2' has degree 3 yet $d_G(y), d_G(z) \geq 5$.

By Claim 2 we may further assume that $yz \notin E(G)$. To complete Case 1 we prove that G/vy is 3^+-connected. By Lemma 2.3 (ii), G/vy is 3-connected since $G - \{v, y\}$ has a cycle $zwx'xu$. Notice that $x'y \notin E(G)$ since $t = y$ and G/xx' is simple. Thus G/vy is simple. If G/vy has a forbidden 3-separation, by Lemma 2.4(ii), at least one of the interior (cubic) vertices of the special part is adjacent to v. However, it is easy to see from the second graph in Figure 3.1 that all three neighbors of v have degree at least four, which proves that G/vy is 3^+-connected and thus Case 1 is settled.

Since Case 1 is settled, we may assume in the following that if edge e has a cubic end and G/e is simple and 3-connected, then in every forbidden 3-separation of G/e the special part is a fan.

Case 2. G is the third graph in Figure 3.1.

By symmetry we assume $xu \in E(G)$. We prove that G/uy is 3^+-connected. Since y is adjacent to neither x nor v, it follows that G/uy is simple. On the other hand, G/uy is 3-connected by Lemma 2.3 (ii) since u is cubic and $G - \{u, y\}$ has a cycle on $\{v, w, x, x'\}$. Suppose G/uy has a forbidden 3-separation. Then the special part is a fan and, by Lemma 2.4(iii), the new vertex is the center. Let P be the path formed by the interior vertices of the special part. Since w is adjacent to neither u nor y, w is not on P. Since P contains at least one neighbor of u and at least one neighbor of y, P must contain x'. It follows that $d_G(x') = 3$ and $yx' \in E(G)$, which implies that $G - \{y, z\}$ is disconnected, a contradiction.

Case 3. G is the last graph in Figure 3.1.

By symmetry we assume $wx' \in E(G)$. Since u, v do not have common neighbors, G/uv is simple. Since G/xx' is 3-connected, it has a tz-path P avoiding x^* and y. Clearly, P is also a tz-path of G avoiding x, x', y. Hence, by Lemma 2.3 (ii), G/uv is 3-connected since v is cubic and $G - \{u, v\}$ has a cycle $zPtxx'w$. Since both u, v are cubic, any forbidden 3-separation of G/uv induces a 4-separation of G as illustrated by the third graph in Figure 3.1. Therefore, we deduce from the Case 2 that G/e is 3^+-connected for some edge e, which completes our proof of the lemma. ∎

In the following we analyze cubic vertices of type-II. A *triple* $(x; y, z)$ consists of three vertices of G such that x is of type-II, xyz is a triangle, and $d(y) + d(z) \geq d(y') + d(z')$ for every triangle $xy'z'$.

Lemma 3.3 *Let G be a 3^+-connected graph other than W_3, W_4, W_5. If $(x; y, z)$ is a triple of G, then $G\backslash yz$ is 3-connected.*

Proof. Suppose $G\backslash yz$ is not 3-connected. Since G/yz is not 3-connected, by Lemma 2.1, one of y, z, say z, is cubic. Let u be the third neighbor of x. Then u and z are not adjacent, for otherwise $G - \{u, y\}$ is disconnected. Since x is of type-II, uy must be an edge of G. From $d(y) + d(z) \geq d(y) + d(u)$ we deduce that u is also cubic. Let z', u' be the other neighbor of z, u, respectively. Since $G \neq W_4$, we must have $z' \neq u'$, for otherwise G is not 3-connected. Similarly, since $G \neq W_5$, G has more than six vertices. Thus the cut set $\{y, z', u'\}$ defines a forbidden 3-separation, a contradiction. ∎

The next is our key lemma on type-II vertices.

Lemma 3.4 *Let $G \notin \{W_3, W_4, W_5\}$ be 3^+-connected. Suppose G has no type-I vertices and suppose, for every $e \in E(G)$, neither $G\backslash e$ nor G/e is 3^+-connected. Let $(x; y, z)$ be a triple. Then G has triangles zpq, zqx, vys, vst, vtw, such that (cf. Figure 3.3) $d(q) = d(s) = d(t) = 3$, $d(y) = 4$, and $d(v), d(z) \geq 5$, where all vertices are distinct, except that p could be v or w.*

Proof. By Lemma 3.3, $G\backslash yz$ has a forbidden 3-separation (H_1, H_2), where H_2 is the special part. Let $\{u, v, w\}$ be the cut set induced by the separation and let r, s, t be the interior vertices of H_2. Clearly, $\{y, z\} \cap \{r, s, t\} \neq \emptyset$, so we assume $y \in \{r, s, t\}$ and thus $x \in V(H_2)$.

If H_2 is $K_{3,3}$, then none of u, v, w is cubic in $G\backslash yz$ (since $G\backslash yz$ is 3-connected and H_1 has at least one interior vertex), which leaves no room for x, a contradiction. Thus H_2 is a fan. Let us assume that $urstw$ is a path and v is adjacent to r, s, t, as shown in Figure 3.2 below.

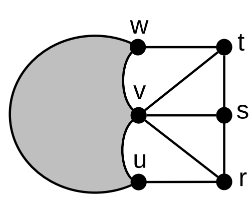

Figure 3.2: Graph $G\backslash yz$.

Claim 1. $z \notin \{r, s, t\}$.

Suppose otherwise. Then $\{y, z\} = \{r, t\}$ and $x = s$. Since $(x; y, z)$ is a triple, $d_G(r) + d_G(t) \geq d_G(r) + d_G(v)$, which implies $d_G(v) = 4$. Thus $uv \notin E(G)$ and so G/ur is simple. By Lemma 2.1, G/ur is also 3-connected since $\{v, w\}$ is a cut set of $G\backslash ur$. Therefore, G/ur has a forbidden 3-separation (H_1', H_2'), where H_2' is the special part. By Lemma 2.4(i-ii), all interior vertices of H_2' are also vertices of G, and r is adjacent to at least one of them. Since these interior vertices are cubic and r has only one cubic neighbor s (other than u), s has to be an interior vertex of H_2'. Note that s has no cubic neighbors in G/ur, so $H_2' = K_{3,3}$ and the cut set defined by (H_1', H_2') must consist of the three neighbors of s. It follows that t, one of these three neighbors, is adjacent to all three interior vertices of H_2', which means that t has three cubic neighbors in G/ur and in G. This is impossible and thus Claim 1 is proved.

Claim 2. $y \neq s$.

Suppose $y = s$. Since $xy \in E(G)$ and x is cubic, x must belong to $\{r, t\}$, say $x = r$, and then $z = u$. Since t is cubic of type-II, we must have $vw \in E(G)$, which implies that $d_G(v) \geq 5$. Moreover, $d_G(u) \geq 4$ since $G\backslash us$ is 3-connected. By Lemma 2.1, $G\backslash vs$ is 3-connected. It follows that $G\backslash vs$ has a forbidden 3-separation (H_1', H_2'), where H_2' is the special part. Since s is cubic in $G\backslash vs$ yet v is not, and since at least one of v, s is an interior vertex of H_2', we deduce that s is an interior vertex of H_2'. Notice that s has cubic neighbors in $G\backslash vs$, it follows that H_2' is a fan. Since rsu is the only triangle in $G\backslash vs$ that contains s and $d_{G\backslash vs}(r) = 3 < d_{G\backslash vs}(u)$, r is an interior vertex of H_2' and u is the center of the fan H_2'. It follows that the third interior vertex of H_2' is adjacent to both r and u, which is impossible since the only potential vertex is v, which is not cubic in $G\backslash vs$. Thus Claim 2 is proved.

By Claim 2 we assume that $y = r$. Then, by Claim 1, t is a cubic vertex of type-II, which implies that $vw \in E(G)$ and thus $d_G(v) \geq 5$.

Claim 3. $z \neq w$.

Suppose $z = w$. Since xyz is a triangle and x is cubic, we must have $x = u$ and $uw \in E(G)$. It follows that $d_G(w) \geq 5$. By Lemma 2.2, $G\backslash vw$ is 3-connected, which implies that $G\backslash vw$ is 3^+-connected since $d_G(v) \geq 5$ and $d_G(w) \geq 5$. This contradiction proves Claim 3.

By Claim 3, z is an interior vertex of H_1. Thus x has to be u. Let q be the third neighbor of x, in addition to y and z.

Claim 4. q is an interior cubic vertex of H_1 and $qz \in E(G)$.

By Lemma 2.1, $G\backslash rv$ is 3-connected. Thus $G\backslash rv$ has a forbidden 3-separation (H_1', H_2'), where H_2' is the special part. Since $d_G(v) \geq 5$, r must be an interior vertex of H_2'. Since r has cubic neighbors in $G\backslash rv$, H_2' must be a fan. Note that ruz is the only triangle of $G\backslash rv$ that contains r, one of u, z is an interior vertex of H_2' and the other is the center. Since u is cubic and the center is not cubic, z has to be the center and u is the interior vertex. It follows that the third interior vertex is a cubic vertex adjacent to both u and z, which implies that this vertex is q. Since q is cubic in $G\backslash rv$, q cannot be v or w, so q is an interior vertex of H_1', which proves Claim 4.

Let the third neighbor of q be p, in addition to u and z (see Figure 3.3). Since q is of type-II, $pz \in E(G)$. Since $\{p, r\}$ is not a cut set of G, we must have $d_G(z) \geq 5$. Thus the lemma is proved. ■

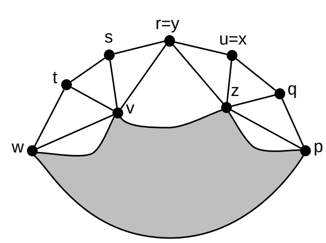

Figure 3.3: Every triple $(x; y, z)$ can be extended into part of a ring.

Finally, we prove the following result, which is slightly stronger than Theorem 1.3.

Theorem 3.5 *Suppose G is 3^+-connected and neither $G\backslash e$ nor G/e is 3^+-connected, for every $e \in E(G)$.*

Then G has cubic vertices and, either $G \in \{W_3, W_4, W_5\}$ or every edge incident with a cubic vertex belongs to a ring.

Proof. By Lemma 3.1 and Lemma 3.2, G has cubic vertices and all of which are of type-II. Suppose $G \notin \{W_3, W_4, W_5\}$. Then Lemma 3.4 implies that every cubic vertex x is contain in a subgraph as illustrated in Figure 3.3. If we apply the lemma again to triple $(q; p, z)$, where we are using the same notation used in Figure 3.3, then we conclude that $d(p) = 4$ (so $p \neq v$) and p is the end of another fan. By repeatedly applying this lemma we generate a sequence of fans. When the process terminate, the end of the last fan must be w, which creates a ring that contains all edges incident with x. Thus the theorem is proved. ■

4 Excluding $Cube/e$

Let Q denote the graph obtained from the Cube by contracting an edge, which is illustrated in Figure 4.1. In this section we characterize Q-free graphs.

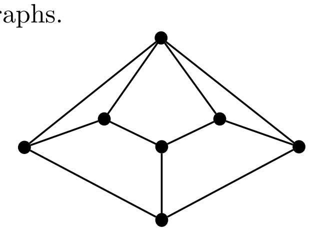

Figure 4.1: Graph Q

In the literature there are many result on excluding a single graph. The best known are the result of Hall on $K_{3,3}$-free graphs and the result of Wagner on K_5-free graphs [11]. On the other hand, the problems of characterizing K_6-free graphs and Petersen-free graphs are still open. Note that both K_6 and the Petersen graph have fifteen edges. In fact, no complete characterization is known for excluding any single graph with thirteen or more edges. Therefore, it is desirable to understand G-free graphs for all "small" graphs G, since these results could lead to a better understanding of K_6-free and Petersen-free graphs. In a separate paper, the authors of this paper studied this problem systematically. They characterized G-free graphs for every 3-connected G, except for Q, with eleven or fewer edges that have not yet been studied in the literature. Graph Q is different from all other small graphs in the way that none of the known splitter theorems is good enough to produce a complete characterization. Part of the reason is that Q has a nontrival 3-separation. It turns out that 3^+-connectivity is the right connectivity for Q.

It should be pointed out that Maharry [7] has characterized Cube-free graphs. Since the Cube is internally 4-connected, Maharry's result requires the operation of 3-sum, which introduces a major obstacle in its application to Q-free graphs, as being Q-free is not preserved under 3-sums.

The rest of this section is arranged as follows. We first explain how a Q-free graph can be constructed from 3^+-connected Q-free graphs. Then we use Theorem 1.3 to determine all these building blocks. For this part, some of the case are proved using computer.

4.1 Reductions

In the remainder of this paper we only consider simple graphs. Let G_1, G_2 be two graphs. Then their 0-*sum* is their disjoint union, their 1-*sum* is obtained by identifying one vertex of G_1 with one vertex of G_2, and their 2-*sum* is obtained by identifying one edge of G_1 with one edge of G_2, where the identified edge may or may not be deleted after the identification. The following result says that if G is 3-connected, then to characterize G-free graphs one only needs to characterize 3-connected G-free graphs. Since the result is well-known we omit its proof.

Lemma 4.1 *Let G be 3-connected. Then G-free graphs are precisely those that are constructed by* 0-, 1-, 2-*sums starting from K_1, K_2, K_3, and 3-connected G-free graphs.*

An *augmentation* of a graph G is obtained by replacing a smaller $K_{3,n}$ or a fan with a larger one. To be precise, let (G_1, G_2) be a 3-separation of G such that $V(G_1) \cap V(G_2) = \{u, v, w\}$, $V(G_2) - V(G_1) = \{x, y\}$, and $E(G_2) = \{xu, xv, xw, yu, yv, yw\}$ or $\{xy, xu, xv, yv, yw\}$. Then an *augmentation* of G (with respect to this 3-separation) is the graph obtained by adding a new vertex z and, either adding edges zu, zv, zw in the first case or replacing xy with zx, zy, zv in the second case.

Lemma 4.2 *Suppose H is an augmentation of G. Then*

(i) G is 3-connected if and only if H is 3-connected;

(ii) G is Q-free if and only if H is Q-free.

Proof. In this proof we follow the notation used in the definition of augmentation. Note that H is obtained by either adding a new vertex z and three edges from z to G, or adding a parallel edge xv and then splitting vertex x. Since these operations preserve 3-connectivity, so if G is 3-connected, then H is also 3-connected. Conversely, suppose H is 3-connected. If G_2 is a fan, then G is 3-connected by Lemma 2.3(iii); if G_2 is $K_{2,3}$, then G is 3-connected by Lemma 2.3(i) and Lemma 2.2. Thus (i) is proved.

The "if" part of (ii) is clear since G is a minor of H. To prove the "only if" part we assume that H has a Q-minor and we prove that G has a Q-minor as well. Since we may assume that G, H are connected, we may further assume $Q = H \backslash F_1 / F_2$, where F_1, F_2 are disjoint subsets of $E(H)$. Let E_0 denote the set of edges of H that are incident with at least one vertex in $\{x, y, z\}$. We consider the two cases separately.

Suppose G_2 is a fan. Let e_1, e_2, e_3 be the three edges in E_0 that are incident with v. If $e_i \in F_1$ for some i, then G has a Q-minor since $H \backslash e_i$ is a subdivision of G. So we assume $e_i \notin F_1$ for all i. Since every e_i is in a triangle with another e_j, it follows that $e_i \notin F_2$ for all i. Therefore, $zx, zy \notin F_1 \cup F_2$, which implies that Q has a two triangles with a common edges, a contradiction.

Next, suppose G_2 is $K_{2,3}$. Observe that at least two vertices in $\{x, y, z\}$ are incident with edges in F_2 because otherwise, since the minimum degree of Q is three, at least two of $\{x, y, z\}$ are not incident with any edge in $F_1 \cup F_2$, which implies that Q has two cubic vertices with the same set of neighbors, a contradiction. We may assume that no two edges of $F_2 \cap E_0$ are incident with a common vertex in $\{u, v, w\}$ because if the opposite happens, say uy, uz are two such edges, then G/uy can be obtained from $H/\{uy, uz\}$ by deleting parallel edges, which implies that G/uy has a Q-minor. If all three vertices in $\{x, y, z\}$ are incident with edges of F_2, then we may assume $xu, yv, zw \in F_2$. It follows that $G/\{xu, yv\}$ can be obtained from $H/\{xu, yv, zw\}$ by deleting parallel edges, and thus $G/\{xu, yv\}$ has a Q-minor. Therefore, exactly two vertices in $\{x, y, z\}$, say y, z, are incident with edges in F_2, and so x is not adjacent with any edge in $F_1 \cup F_2$. Let $e_y, e_z \in F_2$ be incident with y, z, respectively. Since for every cubic vertex of Q, its three neighbors do not form a triangle, $H/\{e_y, e_z\}$ can be simulated by G/yt, for some $t \in \{u, v, w\}$, and thus G has a Q-minor. ∎

The last Lemma immediately implies the following.

Lemma 4.3 *Every 3-connected Q-free graphs is obtained from a 3^+-connected Q-free graph by a sequence of augmentations.*

Now it is clear that Q-free graphs are completely characterized by Lemma 4.1, Lemma 4.3, and the following theorem, whose proof will occupy the rest of the paper.

Theorem 4.4 *A 3^+-connected graph is Q-free if and only if it is a 3^+-connected minor of K_6, R_2, or Γ_i $(i = 1, 2, ..., 6)$ shown in Figure 4.2.*

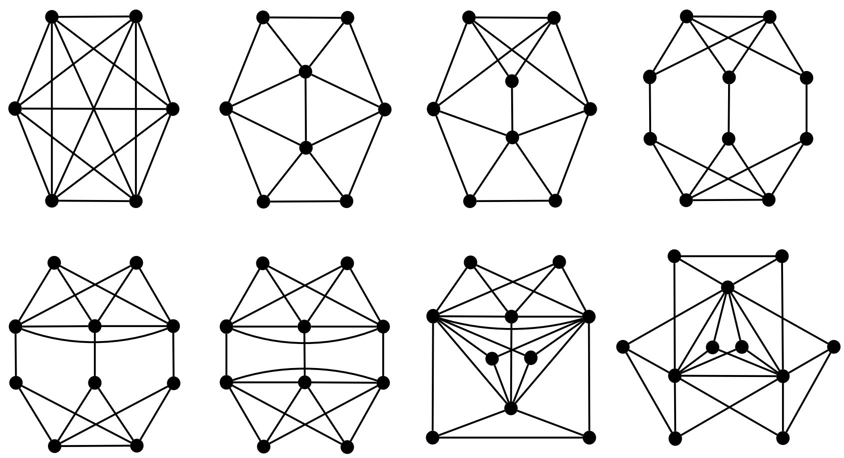

Figure 4.2: Maximal 3^+-connected Q-free graphs

4.2 Proof Outline

Let G be a 3^+-connected graph. By Theorem 1.3, there is a sequence $G_1, G_2, ..., G_m$ of 3^+-connected graphs such that $G_1 \in \{W_3, W_4, W_5\}$, $G_m = G$, and $G_{i-1} \in \{G_i \backslash e, G_i/e, G_i \backslash e/f\}$ for some $e, f \in E(G_i)$ $(i = 2, 3, ..., m)$. Moreover, if $G_{i-1} = G_i \backslash e/f$, then G_i has a ring that contains both e and f. In this situation, we call the operation $G_{i-1} \to G_i$ a *ring-completion*. Using this language, Theorem 1.3 is equivalent to: every 3^+-connected graph can be constructed from W_3, W_4, W_5 by a sequence of undeletions, uncontractions, and ring-completions, such that all the intermediate graphs are also 3^+-connected. We will follow this procedure to generate all 3^+-connected Q-free graphs.

Since a ring-completion requires the presence of almost an entire ring, it is understandable that this operation is not used very often. In fact, the following lemma says that this operation can be avoided for small Q-free graphs. Let R_2, R_3 be the two graphs in Figre 1.3.

Lemma 4.5 *Every 3^+-connected Q-free graph G with 27 or fewer edges can be generated from W_3, W_4, W_5, R_2 by undeletions and uncontractions.*

Proof. Let $G_1, G_2, ..., G_m$ be as define above. Suppose $G_{i-1} = G_i \backslash e/f$ and G_i has a k-ring. Then $k \leq 3$ because otherwise G_i would have ≥ 28 edges. In case $k = 3$, we claim that G has an R_3-minor. This is clear if G_i has no vertices other than those in the 3-ring since we need at least two extra edges to make G_i 3-connected. If G_i does have another vertex x, then G_i has three paths from x to the 3-ring such that they are vertex-disjoint, except at x. Thus an R_3-minor can be obtained from the union of these paths and the 3-ring, which proves the claim. However, it is not difficult to find a Q-minor in R_3, so k can only be two. Now notice that G_i has no vertices other than those in the 2-ring since G_i is 3-connected. It follows that $G_i = R_2$ and thus G is obtained from R_2 by only undeletions and uncontractions. ∎

Because of this Lemma, when generating 3^+-connected Q-free graphs, we don't need to worry about ring-completions before the graphs reach 26 edges. However, as we will see, the process terminates when the graphs reach 24 edges, so we never need to consider ring-completions.

In summary, to prove Theorem 4.4, the only thing we need to do is to repeatedly construct, starting from W_3, W_4, W_5, R_2, all 3^+-connected Q-free undeletions and uncontractions.

4.3 Using computer

Since the expansion process is routine and laborsome, we use computer to handle this tedious work. For a set $\mathcal{G}$ of 3^+-connected Q-free graphs, let $\Phi(\mathcal{G})$ be the set of 3^+-connected Q-free graphs that are obtained from graphs in $\mathcal{G}$ by a single undeletion or a single uncontraction, and let $\Psi(\mathcal{G})$ be the set of graphs in $\mathcal{G}$ that are not a proper minor of any graph in $\Phi(\mathcal{G})$.

Clearly, $\Psi(\mathcal{G})$ will capture the maximal graphs that we are looking for. Notice that, after $\Phi(\mathcal{G})$ is computed, $\Psi(\mathcal{G})$ can be easily obtained by $|\mathcal{G}| \cdot |\Phi(\mathcal{G})|$ minor-testings. As for $\Phi(\mathcal{G})$, we compute it as follows:

(i) obtain all undeletions of members of $\mathcal{G}$ and only keep those that are Q-free;

(ii) obtain all uncontractions of members of $\mathcal{G}$ and only keep those that are 3^+-connected and Q-free.

For example, $\Phi(\{W_3\}) = \emptyset$ since no undeletion or uncontraction applies to W_3. It then follows clearly that $\Psi(\{W_3\}) = \{W_3\}$. On the other hand, $\Phi(\{W_4\})$ consists of $K_5 \backslash e$, obtained by undeletion, and $K_{3,3}$ and the Prism, obtained by uncontrations. In this case $\Psi(\{W_4\}) = \emptyset$, meaning that every graph in $\{W_4\}$ extends to some graph in $\Phi(\{W_4\})$. In the following we report $|\Phi(\mathcal{G})|$ and $\Psi(\mathcal{G})$ of each iteration. A detailed list of $\Phi(\mathcal{G})$ can be found in [6]. We generate graphs according to their number of edges.

$$
\begin{array}{llll}
\mathcal{G}_{06} = \{W_3\} & \rightarrow \ |\Phi(\mathcal{G}_{06})| = 0 & \text{and} & \Psi(\mathcal{G}_{06}) = \{W_3\} \\
\mathcal{G}_{08} = \{W_4\} & \rightarrow \ |\Phi(\mathcal{G}_{08})| = 3 & \text{and} & \Psi(\mathcal{G}_{08}) = \emptyset \\
\mathcal{G}_{09} = \Phi(\mathcal{G}_{08}) & \rightarrow \ |\Phi(\mathcal{G}_{09})| = 3 & \text{and} & \Psi(\mathcal{G}_{09}) = \emptyset \\
\mathcal{G}_{10} = \Phi(\mathcal{G}_{09}) \cup \{W_5\} & \rightarrow \ |\Phi(\mathcal{G}_{10})| = 5 & \text{and} & \Psi(\mathcal{G}_{10}) = \emptyset \\
\mathcal{G}_{11} = \Phi(\mathcal{G}_{10}) & \rightarrow \ |\Phi(\mathcal{G}_{11})| = 12 & \text{and} & \Psi(\mathcal{G}_{11}) = \emptyset \\
\mathcal{G}_{12} = \Phi(\mathcal{G}_{11}) & \rightarrow \ |\Phi(\mathcal{G}_{12})| = 17 & \text{and} & \Psi(\mathcal{G}_{12}) = \emptyset \\
\mathcal{G}_{13} = \Phi(\mathcal{G}_{12}) & \rightarrow \ |\Phi(\mathcal{G}_{13})| = 24 & \text{and} & \Psi(\mathcal{G}_{13}) = \emptyset \\
\mathcal{G}_{14} = \Phi(\mathcal{G}_{13}) & \rightarrow \ |\Phi(\mathcal{G}_{14})| = 32 & \text{and} & \Psi(\mathcal{G}_{14}) = \emptyset \\
\mathcal{G}_{15} = \Phi(\mathcal{G}_{14}) \cup \{R_2\} & \rightarrow \ |\Phi(\mathcal{G}_{15})| = 33 & \text{and} & \Psi(\mathcal{G}_{15}) = \{K_6, R_2, \Gamma_1\} \\
\mathcal{G}_{16} = \Phi(\mathcal{G}_{15}) & \rightarrow \ |\Phi(\mathcal{G}_{16})| = 30 & \text{and} & \Psi(\mathcal{G}_{16}) = \emptyset \\
\mathcal{G}_{17} = \Phi(\mathcal{G}_{16}) & \rightarrow \ |\Phi(\mathcal{G}_{17})| = 26 & \text{and} & \Psi(\mathcal{G}_{17}) = \{\Gamma_2\} \\
\mathcal{G}_{18} = \Phi(\mathcal{G}_{17}) & \rightarrow \ |\Phi(\mathcal{G}_{18})| = 16 & \text{and} & \Psi(\mathcal{G}_{18}) = \emptyset \\
\mathcal{G}_{19} = \Phi(\mathcal{G}_{18}) & \rightarrow \ |\Phi(\mathcal{G}_{19})| = 11 & \text{and} & \Psi(\mathcal{G}_{19}) = \{\Gamma_3\} \\
\mathcal{G}_{20} = \Phi(\mathcal{G}_{19}) & \rightarrow \ |\Phi(\mathcal{G}_{20})| = 9 & \text{and} & \Psi(\mathcal{G}_{20}) = \emptyset \\
\mathcal{G}_{21} = \Phi(\mathcal{G}_{20}) & \rightarrow \ |\Phi(\mathcal{G}_{21})| = 4 & \text{and} & \Psi(\mathcal{G}_{21}) = \{\Gamma_4\} \\
\mathcal{G}_{22} = \Phi(\mathcal{G}_{21}) & \rightarrow \ |\Phi(\mathcal{G}_{22})| = 2 & \text{and} & \Psi(\mathcal{G}_{22}) = \emptyset \\
\mathcal{G}_{23} = \Phi(\mathcal{G}_{22}) & \rightarrow \ |\Phi(\mathcal{G}_{23})| = 1 & \text{and} & \Psi(\mathcal{G}_{23}) = \{\Gamma_5\} \\
\mathcal{G}_{24} = \Phi(\mathcal{G}_{23}) & \rightarrow \ |\Phi(\mathcal{G}_{24})| = 0 & \text{and} & \Psi(\mathcal{G}_{24}) = \{\Gamma_6\}
\end{array}
$$

4.4 Proof of Theorem 4.4

By Lemma 4.5 and the computation of the last subsection, we conclude that every 3^+-connected Q-free graph with ≤ 27 edges is a minor of K_6, R_2, or Γ_i $(i = 1, 2, ..., 6)$. Since the largest among all such graphs has only 24 edges, we deduce from Theorem 1.2 that there are no other 3^+-connected Q-free graphs, which proves the Theorem. ■

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