EXCLUDED-MINOR CHARACTERIZATION OF APEX-OUTERPLANAR GRAPHS

GUOLI DING AND STAN DZIOBIAK

Abstract. The class of outerplanar graphs is minor-closed and can be characterized by two excluded minors: K_4 and $K_{2,3}$. The class of graphs that contain a vertex whose removal leaves an outerplanar graph is also minor-closed. We provide the complete list of 57 excluded minors for this class.

1. Introduction

A graph is *outerplanar* if it can be embedded in the plane (with no edges crossing) with all vertices incident to one common face. We say that a graph G is *apex-outerplanar* if there exists $v \in V(G)$ such that $G - v$ is outerplanar. Such a vertex, if it exists, is called an *apex* vertex of G. We let $\mathcal{O}$ and $\mathcal{O}^*$ denote the classes of outerplanar and apex-outerplanar graphs, respectively.

Given graphs H and G, H is a *minor* of G, denoted by $H \leqslant_m G$, or $G \geqslant_m H$, if H can be obtained from a subgraph of G by contracting edges. A class $\mathcal{C}$ of graphs is *minor-closed* if for every $G \in \mathcal{C}$ all the minors of G are also in $\mathcal{C}$. Examples of minor-closed classes are: planar graphs, outerplanar graphs, series-parallel graphs, graphs embeddable in a fixed surface, and graphs of tree-width bounded by a fixed constant.

Let $\mathcal{C}$ be a minor-closed class of graphs, and let $\mathcal{C}^*$ be the class of graphs that contain a vertex whose removal leaves a graph in $\mathcal{C}$. Hence, clearly $\mathcal{C} \subseteq \mathcal{C}^*$, and it is easy to check that $\mathcal{C}^*$ is also minor-closed, thus in particular $\mathcal{O}^*$ is minor-closed.

It is a landmark result of Robertson and Seymour (see [7]) that every proper minor-closed class of graphs $\mathcal{C}$ can be characterized by its finite set of *excluded minors*, or *obstructions*, that is, minor-minimal graphs not in $\mathcal{C}$. We call this set *obstruction set* of $\mathcal{C}$, and denote it by $\mathbf{ob}(\mathcal{C})$. For example, it is a well-known fact that $\mathbf{ob}(\mathcal{O}) = \{K_4, K_{2,3}\}$. Equivalently, G is outerplanar if and only if it does not contain a subdivision of K_4 nor a subdivision of $K_{2,3}$ as a subgraph. This equivalence follows from the known fact that if $H \leqslant_m G$ and $\Delta(H) \leqslant 3$, then G contains an H-subdivision.

Let $\mathcal{S}$ be the set of graphs in Figure 1, $\mathcal{T}$ be the set of graphs in Figure 4, $\mathcal{G}$ be the set of graphs in Figure 5, $\mathcal{J}$ be the set of graphs in Figure 6, $\mathcal{H}$ be the set of graphs in Figure 7, and $\mathcal{Q}$ be the set of graphs in Figure 8.

The following is our main result.

Theorem 1.1. *A graph is apex-outerplanar if and only if it does not contain any of the* 57 *graphs in the set* $\mathcal{S} \cup \mathcal{T} \cup \mathcal{G} \cup \mathcal{J} \cup \mathcal{H} \cup \mathcal{Q}$ *as a minor. Equivalently,* $\mathbf{ob}(\mathcal{O}^*) = \mathcal{S} \cup \mathcal{T} \cup \mathcal{G} \cup \mathcal{J} \cup \mathcal{H} \cup \mathcal{Q}$.

The reader should be confident that the 57 graphs in Theorem 1.1 are indeed pairwise non-isomorphic members of $\mathbf{ob}(\mathcal{O}^*)$. We have checked this several times. In this paper, we will show that there are no more graphs in $\mathbf{ob}(\mathcal{O}^*)$ other than the 57, that is, the list is complete.

Our result can be regarded as a test approach to the long-standing open problem of finding the complete list of excluded minors for the class of apex-planar graphs, which plays an important role in Graph Theory (for example, see [8]). Significant progress on this problem has already been made by A. Kezdy [6] and his team since our work was completed and announced in [4]. For instance, they have found all of the obstructions of connectivity 0, 1, and 2, and many of the ones of connectivity 3, 4, and 5, altogether 396 obstructions.

While working on the problem we did not use a computer, the 57 obstructions were found "by hand". We believe that this was an advantage, since we were able to control and understand the way in which the obstructions were being generated, and in which the proof should be organized. After we found $\mathbf{ob}(\mathcal{O}^*)$ and proved its completeness, G.E. Turner [9] kindly informed us that the 57 graphs had been known to him, since he had found them with the aid of a computer. However, he did not know whether his list was complete.

We now present an outline of the rest of the paper, which constitutes the proof of Theorem 1.1. In Section 2, we provide a starting set of seven obstructions $\mathcal{S} \subseteq \mathbf{ob}(\mathcal{O}^*)$, and prove a key lemma (Lemma 2.2), which together allow us to conclude that any obstruction in $\mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$ is planar and of connectivity 2 or 3. The search for the remaining obstructions begins.

The connectivity-three case is presented in Section 6. Here, we rely on the existence of contractible edges in 3-connected graphs and the minor-minimality of the obstructions to prove that there are no 3-connected obstructions in $\mathbf{ob}(\mathcal{O}^*)$ other then the ones already in our starting set $\mathcal{S}$.

Most of the work is in the connectivity-two case. Our key lemma (Lemma 2.2) splits the proof of this case into five major subcases, presented in Sections 3, 4, and 5. The cases are split based on the complexity of each side of a 2-separation in $G \in \mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$, as indicated by Lemma 2.2. In the following outline of the case structure, all of the 2-separations refer to 2-separations (L, R) in G over vertices $\{x, y\}$. Also, P_2 and C_4 are as drawn in Figure 3, with vertices $\{x, y\}$ as labelled in the Figure.

Case 1: There exists a 2-separation such that both $L \notin \mathcal{O}$ and $R \notin \mathcal{O}$ (Section 3);
Case 2: For each 2-separation, $L = P_2$ or C_4 (Sections 4 and 5);
Subcase 2.1: There exists a 2-separation such that $L = C_4$ (Proposition 4.1);
Subsubcase 2.1.1: There exists a 2-separation such that $L = C_4$ and $G - \{x, y\} \notin \mathcal{O}$;
Subsubcase 2.1.2: There exists a 2-separation such that $L = C_4$ and for every such 2-separation $G - \{x, y\} \in \mathcal{O}$;
Subcase 2.2: For each 2-separation, $L = P_2$ (Proposition 5.1);
Subsubcase 2.2.1: There exists a 2-separation such that $L = P_2$ and $G - \{x, y\} \notin \mathcal{O}$;
Subsubcase 2.2.2: For each 2-separation, $L = P_2$ and $G - \{x, y\} \in \mathcal{O}$.

Note that organizing the case analysis in this way restricts the structure of G more and more as we proceed through the cases. An outline of each case will be given at the beginning of the corresponding section.

2. Starting List and the Key Lemma

In this section, we provide a starting set of seven obstructions $\mathcal{S} \subseteq \mathbf{ob}(\mathcal{O}^*)$, and prove the key Lemma 2.2, which narrows down the structure of the remaining obstructions.

For two graphs G_1 and G_2, we let $G_1|G_2$ denote their disjoint union.

Let $\mathcal{S} := \{K_5, K_{3,3}, Oct, Q, 2K_4, K_4|K_{2,3}, 2K_{2,3}\}$ be the set of graphs in the figure below.

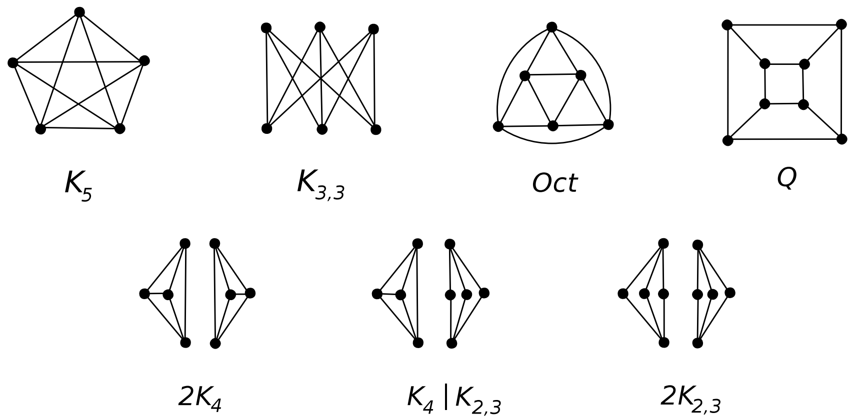


Figure 1. Starting list of excluded minors for $\mathcal{O}^*$

It is easy to check that $\mathcal{S} \subseteq \mathbf{ob}(\mathcal{O}^*)$.

Definition 2.1. *Let G be a graph and $x, y \in V(G)$. A* 1-separation *of G* over *x (or* across *x) (respectively, a* 2-separation *of G* over *$\{x, y\}$ (or* across *$\{x, y\}$)) is a pair $S = (L, R)$ of induced subgraphs L and R of G, called the* sides *of S, such that the following holds*

(1) *$E(L) \cup E(R) = E(G)$;*

(2) *$V(L) \cup V(R) = V(G)$ and $V(L) \cap V(R) = \{x\}$ (respectively, $V(L) \cap V(R) = \{x, y\}$);*

(3) *$V(L) - V(R) \neq \varnothing$ and $V(R) - V(L) \neq \varnothing$.*

Note that in definition 2.1, we require that L and R to be *induced* subgraphs, and that x is necessarily a cut-vertex of G (respectively, $\{x, y\}$ is a 2-cut of G). Also, if $S = (L, R)$ is a 2-separation of G over $\{x, y\}$, then we often denote L and R by $L(x, y)$ and $R(x, y)$, respectively, for emphasis.

We define a *K-graph* to be a graph that contains a K_4- or $K_{2,3}$-subdivision (both of which we call *K-subdivisions*) as a subgraph. Equivalently, K-graphs are precisely non-outerplanar graphs. It is a known fact that if G is 2-connected and contains a K-subdivision, then $G = K_4$ or G contains a $K_{2,3}$-subdivision.

Lemma 2.2. *If $G \in \mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$, then G is planar and of connectivity 2 or 3. Moreover, if the connectivity of G is 2, then for every 2-separation $S = (L, R)$ of G over vertices $\{x, y\}$ the following holds:*

(1) *If no side of S is in $\mathcal{O}$, then one side of S is L_1, L_2, L_3, L_4, or L_5 with prescribed vertices x and y, as shown in Figure 2.*

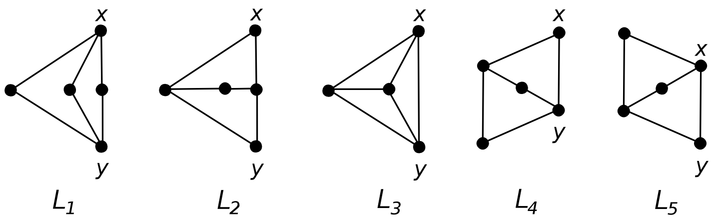


FIGURE 2. K_4 and $K_{2,3}$'s with prescribed vertices x and y

(2) *If one side of S is in $\mathcal{O}$, then $xy \notin E(G)$ and that side is P_2 or C_4, where P_2 is a path on two edges with endpoints x and y, and C_4 is a cycle on four edges with x and y non-adjacent, as shown in Figure 3.*

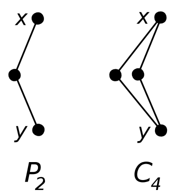


FIGURE 3. P_2 and C_4

Proof. Since $G \not\geqslant_m K_5$ and $G \not\geqslant_m K_{3,3}$, it follows that G is planar.

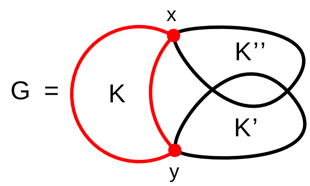
First, suppose that G is disconnected, and let G be a union of two disjoint (not necessarily connected) graphs G_1 and G_2. If one of them, say G_1 is outerplanar, then by the minor-minimality of G, $G_2 = G - G_1 \in \mathcal{O}^*$, hence G_2 has a vertex v such that $G_2 - v \in \mathcal{O}$. Then, $G_1|(G_2 - v) \in \mathcal{O}$, hence v is an apex vertex in G, a contradiction. Therefore, both G_1 and G_2 are not outerplanar, and so each contains K_4 or $K_{2,3}$ as a minor. Hence G contains one of $2K_4, K_4|K_{2,3}, 2K_{2,3}$ as a minor, a contradiction. Thus G is connected.

Now, suppose that G has a cut-vertex x and let (L, R) be the 1-separation across x. By the same argument as above, both L and R are not outerplanar, hence they both contain K_4 or $K_{2,3}$ as a minor. This implies that both $R - x$ and $L - x$ are outerplanar (for otherwise, G would contain one of $2K_4, K_4|K_{2,3}, 2K_{2,3}$ as a minor). Hence $G - x \in \mathcal{O}$, and so $G \in \mathcal{O}^*$, a contradiction. Therefore, G is 2-connected.

Now, suppose that G is 4-connected. Then $\delta(G) \geqslant 4$, and so by the theorem of Halin and Jung from [5], which says that G contains a K_5- or Oct-minor whenever $\delta(G) \geqslant 4$, it follows that the assumption that G is 4-connected is not true, because K_5 and Oct are in $\mathcal{S}$. Therefore the connectivity of G is 2 or 3.

Proof of (1). Suppose now that the connectivity of G is 2 and that no side of S, neither L nor R, is in $\mathcal{O}$. Note that $G - \{x, y\} \in \mathcal{O}$, for otherwise G would contain two disjoint K-graphs (for instance, L and $R - \{x, y\}$) which cannot happen because G does not contain $2K_4, K_4|K_{2,3}, 2K_{2,3}$ as a minor. Since $G \notin \mathcal{O}^*$, none of its vertices is apex. In particular, since x is not apex in G and y is a cut-vertex in $G - x$, it follows that $L - x$ or $R - x$, say $R - x$, contains a K-subdivision, call it K', which contains y (since $R - \{x, y\}$ is outerplanar). Similarly, $R - y$ contains a K-subdivision

K'' (not $L-y$, because such a K-subdivision would be disjoint from K'), which contains x. K' and K'' must intersect, otherwise G would contain two disjoint K-graphs. Also, $L-x \in \mathcal{O}$ since it is disjoint from K'', and $L-y \in \mathcal{O}$ since it is disjoint from K'. Hence, G must have the following structure:



Note that, as long as $L \notin \mathcal{O}$, a graph with the above structure does not belong to $\mathcal{O}^*$. This is because none of its vertices is apex: x is not apex, because of K'; y is not apex, because of K''; if $v \in L - \{x,y\}$, then v is not apex, because of K' (or K''); finally if $v \in R - \{x,y\}$, then v is not apex, because of L. Therefore, if $L \notin \{K_4, K_{2,3}\}$, then since $L \notin \mathcal{O}$, it follows that L contains an edge $e \neq xy$ such that either $L\backslash e \notin \mathcal{O}$, or $L/e \notin \mathcal{O}$. Hence, either $G\backslash e \notin \mathcal{O}^*$ or $G/e \notin \mathcal{O}^*$, a contradiction since G is minor-minimal not in $\mathcal{O}^*$. Therefore $L \in \{L_1, L_2, L_3, L_4, L_5\}$ with x, y as prescribed in Figure 2.

Proof of (2). Without loss of generality, suppose that $L \in \mathcal{O}$. Since $G \notin \mathcal{O}^*$, none of its vertices are apex. In particular, since x is not apex, it follows that $R-x$ contains a K-subdivision. Similarly, $R-y$ contains a K-subdivision. Since G is 2-connected, it follows that L is connected. We have two cases based on the number of blocks of L.

Case 1. L has exactly one block.

Note that $L \neq K_2$, for otherwise (L,R) is not a 2-separation. Hence L is 2-connected.

Since L is 2-connected and outerplanar, it follows that L is a cycle C with chords, which has a unique planar embedding such that all the vertices and edges of C are incident with the outer face, and all the chords lie in the interior of the disk bounded by C. We now show that L has no chords. So, suppose that L does have a chord e. Let s be an apex vertex in $G\backslash e$. Then, since $R-x$ and $R-y$ contain K-subdivisions, it follows that $s \in V(R-\{x,y\})$. Assume that $(G\backslash e)-s \in \mathcal{O}$ is embedded in the plane so that all of its vertices are incident with the outer face. Then this embedding, restricted to the subgraph $L\backslash e$, is such that all the vertices and edges of C are incident with the outer face. Therefore, by putting the chord e back in, we obtain an embedding of $G-s$ with all of its vertices still incident with the outer face, hence $G-s$ is outerplanar, a contradiction. Hence, we have shown that L has no chords, therefore $L = C$.

Now, suppose that x and y are consecutive vertices of C, that is $xy \in E(C)$. Let s be an apex vertex in $G\backslash xy$. Then, again we have that $s \in V(R-\{x,y\})$. Assume that $(G\backslash xy)-s \in \mathcal{O}$ is embedded in the plane so that all of its vertices are incident with the outer face. Since all the vertices of $C-\{x,y\}$ have degree $=2$ in $(G\backslash xy)-s$, it follows that all the edges of C except for xy are incident with the outer face. Therefore, by putting the edge xy back in, we obtain an embedding of $G-s$ with all of its vertices still incident with the outer face, a contradiction.

Therefore, x and y are non-consecutive, which implies that the length of C is at least four. In fact $C = C_4$, for suppose that $C = C_n$ with $n \geqslant 5$. Then one of the two paths from x to y in C must have length at least three. Let f be an edge on that path with endpoints different from x and

y. Let s be an apex vertex in G/f. Then, again $s \in V(R - \{x,y\})$. Assume that $(G/f) - s \in \mathcal{O}$ is embedded in the plane so that all of its vertices are incident with the outer face. Since all the vertices of $(C/f) - \{x,y\}$ have degree $= 2$ in $(G/f) - s$, it follows that all the edges of C/f are incident with the outer face. Therefore, by uncontracting edge $f \in E(C)$, we obtain an embedding of $G - s$ with all of its vertices still incident with the outer face, hence $G - s$ is outerplanar, a contradiction. Hence, we have shown that $L = C = C_4$.

Therefore, we have shown that if L has only one block, then L is 2-connected, and in fact $L = C_4$ with x and y non-adjacent. Now, we consider the more general case.

Case 2. L has at least two blocks.

Let B_x and B_y be two distinct blocks containing x and y, respectively. Then the block tree of L is, in fact, a path from B_x to B_y, for otherwise G would contain a cut-vertex. Every block on this path is either K_2 or is 2-connected. If L contains a block B that is 2-connected, then let $s, t \in V(B)$ be the two cut-vertices in L (or in the case of B_x and B_y the associated pair is given by the corresponding cut-vertex, and x or y, respectively). Then since G has a 2-separation (B, R') over $\{s,t\}$, it follows by the previous argument that $B = C_4$. Therefore, every block of L (which is a path) is either K_2 or C_4.

Now suppose that L contains a block $B = C_4$, and let B' be any other block. Denote by G/B' the graph obtained by contracting all the edges of B'. Again, let s be an apex vertex in G/B'. Then again $s \in V(R - \{x,y\})$. Assume that $(G/B') - s \in \mathcal{O}$ is embedded in the plane so that all of its vertices are incident with the outer face. Since two of the non-adjacent vertices of B have degree $= 2$ in $(G/B') - s$ and since all the blocks are either K_2 or C_4, it follows that all the edges of B and, in fact, all the edges of L/B' are incident with the outer face. Therefore, by uncontracting block B', we obtain an embedding of $G - s$ with all of its vertices still incident with the outer face, a contradiction. Hence, we have shown that L does not contain a block $B = C_4$, and therefore all the blocks of L are K_2's, or equivalently L is an induced path of length at least two from x to y.

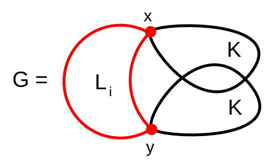
Then, in fact, $L = P_2$, for suppose that $L = P_n$ with $n \geqslant 3$. Let f be an edge in $L = P_n$ with endpoints different from x and y. Let s be an apex vertex in G/f. Then, again $s \in V(R - \{x,y\})$. Assume that $(G/f) - s \in \mathcal{O}$ is embedded in the plane so that all of its vertices are incident with the outer face. Since all the vertices of $(L/f) - \{x,y\}$ have degree $= 2$ in $(G/f) - s$, it follows that all the edges of L/f are incident with the outer face. Therefore, by uncontracting edge f, we obtain an embedding of $G - s$ with all of its vertices still incident with the outer face, a contradiction. Hence, we have shown that $L = P_2$. This proves (2). □

3. Connectivity 2: No Side in $\mathcal{O}$

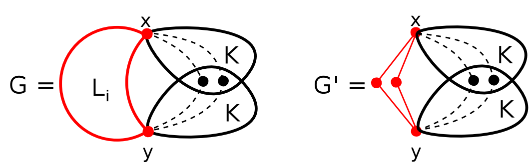
In this section, we focus on Case 1 of the outline given in the Introduction. Namely, we prove Proposition 3.1, which says that if an obstruction $G \in \mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$ has a 2-separation both sides of which are not outerplanar, then $G \in \mathcal{T}$.

Proposition 3.1. *If $G \in \mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$ is of connectivity 2 and has a 2-separation no side of which is in $\mathcal{O}$, then G is a member the family $\mathcal{T}$.*

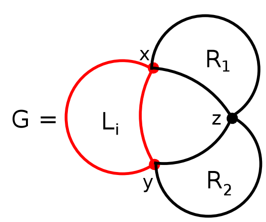
Proof. Let $S = (L, R)$ be a 2-separation of G over $\{x, y\}$ no side of which is in $\mathcal{O}$. Since (R, L) is also a 2-separation of G with the same property, we may assume without loss of generality that $L \in \{L_1, L_2, L_3, L_4, L_5\}$ (see Lemma 2.2). Note that $R - \{x, y\}$ is outerplanar, for otherwise G contains two disjoint K-graphs. Since $G \notin \mathcal{O}^*$, none of its vertices is apex. In particular, since x is not apex, $R - x$ contains a K-subdivision, which contains y (since $R - \{x, y\}$ is outerplanar). Similarly, $R - y$ contains a K-subdivision, which contains x. These two K-subdivisions must intersect, otherwise G would contain two disjoint K-graphs. Hence, G must have the following structure:



Note that each of the L_i ($i = 1, \ldots, 5$) contains C_4 as a minor (with the vertices x and y preserved). Let G' be the graph obtained from G by reducing L (under the minor operation) to C_4, so that (C_4, R) is a 2-separation of G' over $\{x, y\}$. Note that G' is a proper minor of G, hence by the minor-minimality of G, it follows that $G' \in \mathcal{O}^*$. If there are at least two internally disjoint paths in R from x to y, then G' has no apex vertex, a contradiction.



Hence, R has a cut-vertex z. Note that $R - z \in \mathcal{O}$, otherwise R contains two disjoint K-graphs.



Let R_1 and R_2 be the two sides of the 1-separation of R across z, such that $x \in R_1$ and $y \in R_2$. By applying Lemma 2.2 to the 2-separation in G over $\{x, z\}$, and to the 2-separation in G over $\{y, z\}$, we conclude that both $R_1, R_2 \in \{L_1, L_2, L_3, L_4, L_5\}$. Therefore, G is one of the 30 graphs $\{T_1, T_2, \ldots, T_{30}\}$ listed in Figure 4. It is straightforward to verify that each T_i is minor-minimal $\notin \mathcal{O}^*$ satisfying the hypothesis of Case 1. Hence $T_i \in \mathbf{ob}(\mathcal{O}^*)$ for $i = 1, \ldots, 30$. □

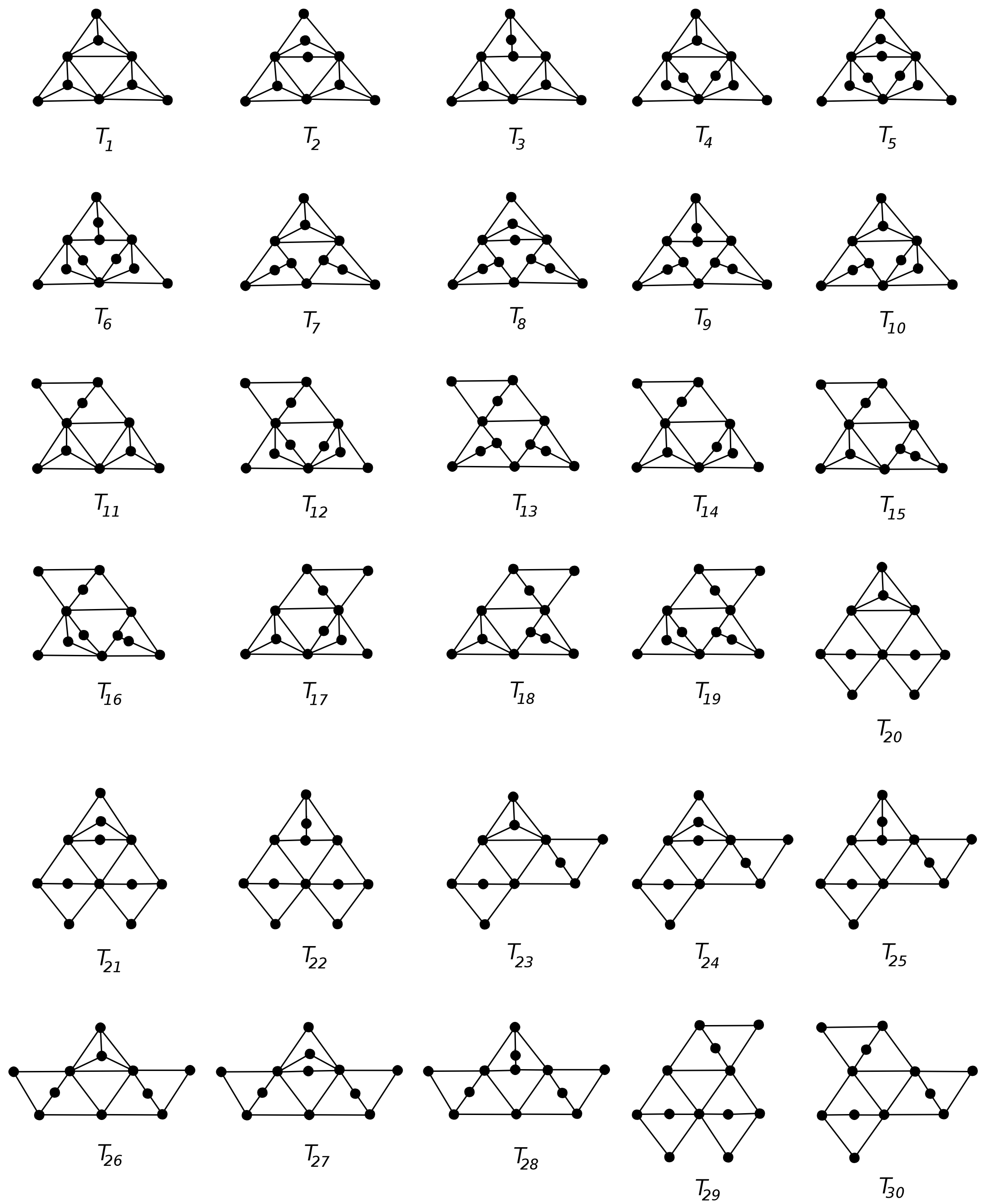


Figure 4. $\mathcal{T}$ family

4. Connectivity 2: At Least One Side C_4

In this section and the next (Section 5) we focus on Case 2 of the outline given in the Introduction. Namely, we assume that every 2-separation of $G \in \mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$ has one side that is outerplanar, which by Lemma 2.2 implies that that side is P_2 or C_4. In this section, we focus on the case that G has a 2-separation one side of which is C_4 (Subcase 2.1 of the outline given in the Introduction).

We prove Proposition 4.1, which says that in this case $G \in \mathcal{G} \cup \mathcal{J}$. In the next section, we analyze the case that every 2-separation of G has one side that is P_2 (Subcase 2.2).

Before we state and prove Proposition 4.1, we introduce some necessary terminology and notation. If $P = u_1, u_2, \dots u_n, u_{n+1}$ is a path on n vertices, then we define its *length* to be n, and denote P by P_n. We call the set $\{u_2, u_3, \dots, u_n\}$ the *interior* of P and denote it by $int(P)$. Two paths P and Q are said to be *internally disjoint* if their interiors are disjoint. If $C = u_1, u_2, \dots u_n, u_1$ is a cycle, then its *length* is n, and we denote C by C_n. An edge $e \notin E(C)$ with both endpoints in $V(C)$ is called a *chord* of C. If $C = u_1, u_2, \dots u_n, u_1$ is a cycle embedded in the plane with vertices listed in the clockwise order around C, then we denote by $C[u_i, u_j]$ the set $\{u_i, u_{i+1}, \dots, u_j\}$ if $i \leqslant j$, or the set $\{u_i, u_{i+1}, \dots, u_n, u_1, \dots, u_j\}$ if $i > j$. Similarly, $C[u_i, u_j) := C[u_i, u_j] - \{u_j\}$, $C(u_i, u_j] := C[u_i, u_j] - \{u_i\}$, and $C(u_i, u_j) := C[u_i, u_j] - \{u_i, u_j\}$. Also, if $P = u_1, u_2, \dots u_n$ is a path, then we define $P[u_i, u_j]$, $P[u_i, u_j)$, $P(u_i, u_j]$, and $P(u_i, u_j)$ analogously, and so $int(P) = P(u_1, u_n)$.

Proposition 4.1. *If $G \in \mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$ is of connectivity 2 and one side of every 2-separation of G is in $\mathcal{O}$ and, moreover, if G has a 2-separation S over $\{x, y\}$ one side of which is C_4, then following holds true:*

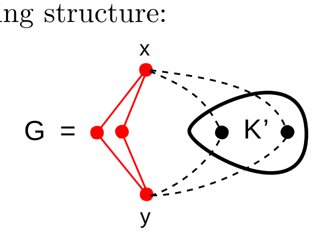
(1) *If $G - \{x, y\} \notin \mathcal{O}$ for some such S, then G is a member of the family $\mathcal{G}$;*

(2) *If $G - \{x, y\} \in \mathcal{O}$ for every such S, then G is a member of the family $\mathcal{J}$.*

4.1. **Proof of (1).** Let $S = (L, R)$ be a 2-separation G over $\{x, y\}$ such that one side of it, say, L, is C_4, and let $G - \{x, y\} \notin \mathcal{O}$. Then $R - \{x, y\} \notin \mathcal{O}$ and hence $R - \{x, y\}$ contains a K-subdivision, call it K'. Note that if R does not have at least two internally disjoint paths from x to y, then R has a cut-vertex z separating x and y, and hence G has a 2-separation (L', R') over $\{x, z\}$ or over $\{y, z\}$ with the property that $R' \notin \mathcal{O}$, and either $L' \notin \mathcal{O}$ (violating the the hypothesis that one side of every 2-separation of G is in $\mathcal{O}$) or $L' \in \mathcal{O}$ but with L' different from P_2 and C_4 (violating Lemma 2.2), a contradiction. Hence,

1. R has at least two internally disjoint paths from x to y.

Also, note that R does not have a path P from x to y disjoint from K', for otherwise G would contain two disjoint K-graphs (namely K' and the $K_{2,3}$-subdivision formed from the union of L and P). Therefore G has the following structure:

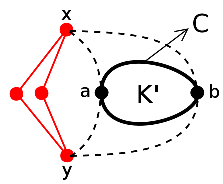


Note that,

2. A graph with the above structure does not belong to $\mathcal{O}^*$.

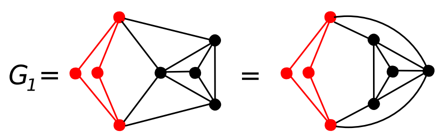
This is because none of its vertices is apex: if $v \in V(G) - V(K')$, then v is not apex, because of K'; and if $v \in V(K')$, then $R - v$ has a path from x to y, which along with L forms a $K_{2,3}$-subdivision in $G - v$, hence v is not apex.

Fix a planar embedding of G. Let C be the outer cycle of K'. Let $S_x \subseteq V(C)$ and $S_y \subseteq V(C)$ be the sets of vertices of C from which there is a path to x, or respectively to y, that doesn't contain other vertices of C. It follows, by **1**, that $|S_x| \geqslant 2$ and $|S_y| \geqslant 2$, hence $|S_x \cup S_y| \geqslant 2$. However, if $|S_x \cup S_y| = 2$ (see the following figure), then let $\{a, b\} := S_x = S_y$, and note that G has a 2-separation (L'', R'') over $\{a, b\}$, where $L'' = K' \notin \mathcal{O}$ and R'' contains a subdivision of $K_{2,4}$, hence $R'' \notin \mathcal{O}$, a contradiction because one side of (L'', R'') must be in $\mathcal{O}$.



Hence, $|S_x \cup S_y| \geqslant 3$. Also note that, by **2**, the paths from S_x to x and S_y to y are actually simple edges, for otherwise we could perform a contraction along such a path, and by **2**, the resulting graph would still be outside of $\mathcal{O}^*$, contradicting the minor-minimality of G.

Since K' is a subdivision of either K_4 or $K_{2,3}$, it follows that actually $K' = K_4$ or K' is a subdivision of $K_{2,3}$. If $K' = K_4$, then in view of all the observations above, G is the following graph:

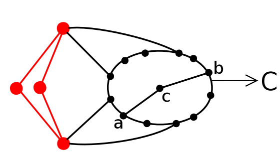


It is easy to verify that the above graph is minor-minimal $\notin \mathcal{O}^*$ satisfying the hypothesis of (1) and the initial hypothesis of Proposition 4.1. We label it G_1, and so $G_1 \in \mathbf{ob}(\mathcal{O}^*)$.

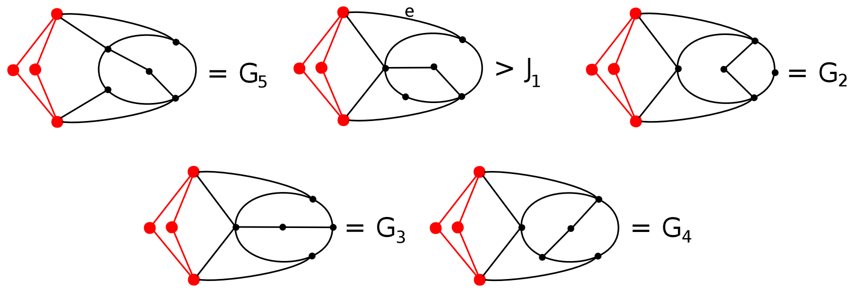
So now, $K' \neq K_4$, and so K' is a subdivision of $K_{2,3}$. Therefore K' consists of the outer cycle C and a path Q of length at least 2 connecting two non-adjacent vertices of C. Note that Q has length exactly 2, for otherwise we could perform a contraction along Q, and by **2**, the resulting graph would still be outside of $\mathcal{O}^*$, contradicting the minor-minimality of G. Let $Q = a, c, b$, so that $a, b \in V(C)$. Then since K' is a subdivision of $K_{2,3}$, we have:

3. There is at least one vertex in $C(a, b)$ and at least one in $C(b, a)$.

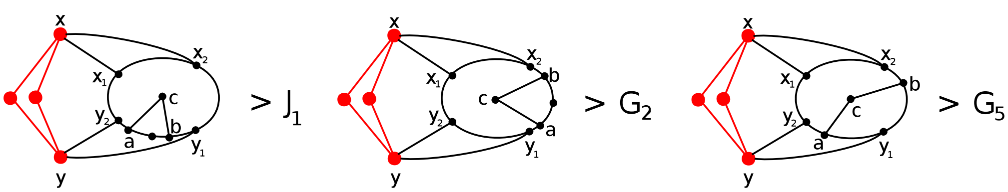
Thus, G has the following structure:



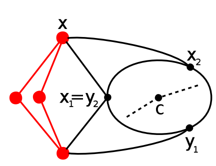
It is straightforward to verify that the following graphs are minor-minimal $\notin \mathcal{O}^*$ satisfying the hypothesis of (1) and the initial hypothesis of Proposition 4.1 (except the second one, which is minor-minimal after contracting e; the resulting graph is $J_1 \in \mathcal{J}$ from Figure 6). We label them G_2, G_3, G_4, G_5. Hence $J_1, G_i \in \mathbf{ob}(\mathcal{O}^*)$ for $i = 1, \ldots, 5$.



In the remainder of the proof, we assume furthermore that $G \notin \{J_1, G_1, G_2, G_3, G_4, G_5\}$. Let $x_1, x_2 \in S_x$ and $y_1, y_2 \in S_y$ in the clockwise order x_1, x_2, y_1, y_2 around C. First, assume that all four can be chosen so that they are all distinct. Then, if $a, b \in C[x_1, x_2]$ or $a, b \in C[y_1, y_2]$, then by **3**, $G \geqslant_m J_1$, a contradiction. If $a, b \in C[x_2, y_1]$ or $a, b \in C[y_2, x_1]$, then by **3**, $G \geqslant_m G_2$, a contradiction. Finally, if a and b are in distinct segments among $C(x_1, x_2)$, $C(y_1, y_2)$, $C(x_2, y_1)$, $C(y_2, x_1)$, or if $\{a, b\} = \{x_1, y_1\}$ or if $\{a, b\} = \{x_2, y_2\}$, then $G \geqslant_m G_5$, a contradiction.



Therefore x_1, x_2, y_1, y_2 cannot be chosen to be all distinct. Since $|S_x| \geqslant 2$ and $|S_y| \geqslant 2$, and $|S_x \cup S_y| \geqslant 3$, it follows that $|S_x \cup S_y| = 3$. Hence, we let $x_1 = y_2$ and $x_2 \neq y_1$, as in the figure below.



Now, if a is in one of $C(x_1, x_2)$ or $C(y_1, x_1)$, say $C(y_1, x_1)$, then: if $b \in C[y_1, x_1]$, then by **3**, $G \geqslant_m J_1$, a contradiction; if $b \in C(x_1, x_2]$, then $G \geqslant_m G_4$; finally, if $b \in C(x_2, y_1)$, then $G \geqslant_m G_3$. Hence, we have shown that neither a nor b can be in $C(x_1, x_2) \cup C(y_1, x_1)$. If $a = x_1$, then if $b = x_2$ or y_1, then by **3**, $G \geqslant_m J_1$, a contradiction; and if $b \in C(x_2, y_1)$, then $G \geqslant_m G_3$, a contradiction. So finally, both a and b must be in $C[x_2, y_1]$. But, then it follows by **3** that $G \geqslant_m G_2$, a contradiction. This concludes the proof of (1) of Proposition 4.1.

Figure 5 shows slightly different embeddings of the G_i's from the ones above.

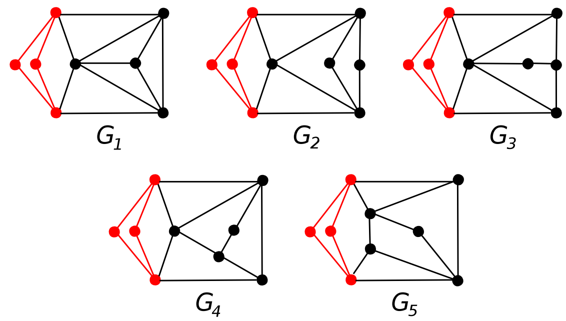


FIGURE 5. $\mathcal{G}$ family

4.2. **Proof of (2).** It is straightforward to verify that the graphs in Figure 6 are minor-minimal $\notin \mathcal{O}^*$ satisfying the hypothesis of (2) and the initial hypothesis of Proposition 4.1. We label them J_1, J_2, J_3, J_4, J_5. Hence $J_i \in \mathbf{ob}(\mathcal{O}^*)$ for $i = 1, \ldots, 5$.

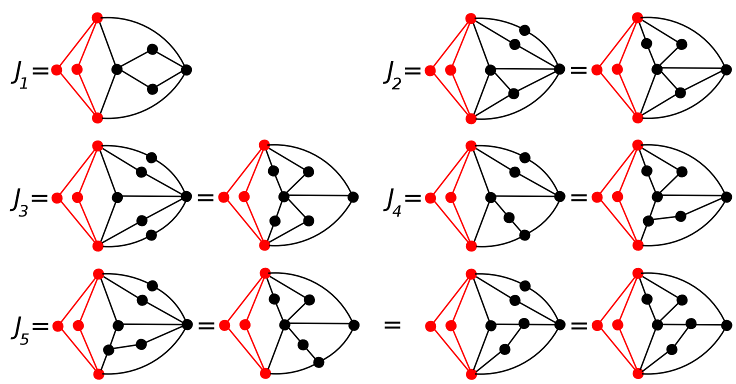
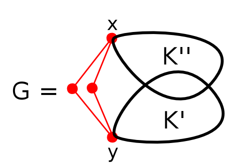


FIGURE 6. $\mathcal{J}$ family

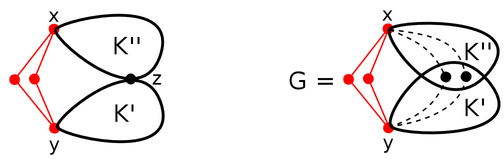
In the remainder of the proof, we assume that $G \notin \{J_1, J_2, J_3, J_4, J_5, Q_2\}$, where $Q_2 \in \mathcal{Q}$ from Figure 8. Since $R - \{x, y\} \in \mathcal{O}$, it follows by the same arguments as in the proof of Proposition 3.1, that G must have the following structure:



where K' is a K-subdivision contained in $R - x$ containing y (so that $K' - y \in \mathcal{O}$), and K'' is a K-subdivision contained in $R - y$ containing x (so that $K'' - x \in \mathcal{O}$). Note that,

1. R does not have a path P from x to y that is internally disjoint from $K' \cup K''$.
For otherwise, G would have a 2-separation (L', R') over $\{x, y\}$, with $L' = L \cup P \notin \mathcal{O}$ and $R' = R \notin \mathcal{O}$, contradicting the hypothesis the one side must be in $\mathcal{O}$.

Also, note that if R does not have at least two internally disjoint paths from x to y, then R has a cut-vertex z. Note that z lies at the intersection of K' and K'' (for otherwise K' and K'' would be disjoint, or $R - \{x, y\}$ would not be outerplanar). But, $R - z \in \mathcal{O}$ (for otherwise K' and K'' would be disjoint), therefore $G - z \in \mathcal{O}$, a contradiction. Hence, R has at least two internally disjoint paths from x to y.



Note that,

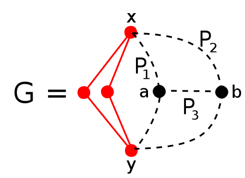
2. A graph with the above structure (on the right) does not belong to $\mathcal{O}^*$.

This is because none of its vertices is apex: if $v \in V(G) - V(K')$, then v is not apex, because of K'; if $v \in V(G) - V(K'')$, then v is not apex, because of K''; and if $v \in V(K') \cap V(K'')$, then $R - v$ has a path from x to y, which along with L forms a $K_{2,3}$-subdivision in $G - v$, hence v is not apex.

Fix a planar embedding of G with x and y incident with the outer face. Since R does not have a cut-vertex, it is 2-connected. Let C be the outer cycle of R, so that the rest of R is embedded in the closed disk bounded by C. Let P_1 and P_2 be the two internally disjoint paths from x to y whose union is C. Note that neither P_1 nor P_2 is a simple edge, since $xy \notin E(G)$. Note that,

3. There must be a path P_3 between $int(P_1)$ and $int(P_2)$ such that $V(P_3) \cap V(C) = \{a, b\}$, where $a \in int(P_1)$ and $b \in int(P_2)$ are the endpoints of P_3.

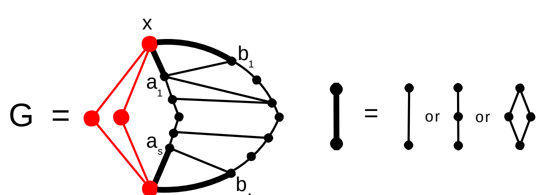
For otherwise, one of $int(P_1)$ or $int(P_2)$ would be vertex-disjoint from $K' \cup K''$, contradicting **1**.



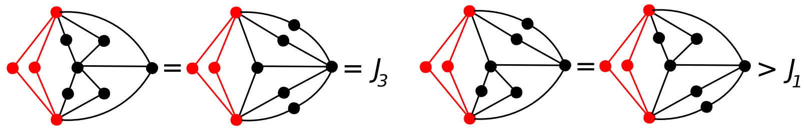
Let $\mathcal{P}$ be the set of paths with property **3**. By **3**, it follows that $\mathcal{P}$ is non-empty. Let $l(\mathcal{P})$ be the length of the longest path in $\mathcal{P}$.

We first suppose that $l(\mathcal{P}) = 1$. Then, all of the paths in $\mathcal{P}$ are simple edges. Let $a_1, a_2, \ldots, a_s \in int(P_1)$ be the left endpoints of the paths in $\mathcal{P}$ in the order of vertices in P_1 from x to y, and similarly let $b_1, b_2, \ldots, b_t \in int(P_2)$ be the right endpoints of the paths in $\mathcal{P}$ in the order of vertices in P_2 from x to y. Note that, for any $i = 1, \ldots, s-1$ (and for any $j = 1, \ldots t-1$), if $a_i a_{i+1}$ (or $b_j b_{j+1}$) is not a simple edge, then G has a 2-separation (L', R') over $\{a_i, a_{i+1}\}$ (or over $\{b_j, b_{j+1}\}$). By the initial hypothesis of Proposition 4.1 and (2) of Lemma 2.2, $L' = P_2$ or C_4. However, by the hypothesis of (2) of Proposition 4.1, $L' \neq C_4$, because $G - \{a_i, a_{i+1}\}$ (and $G - \{b_j, b_{j+1}\}$) contains a $K_{2,3}$-subdivision. Hence,

4. For $i = 1, \ldots, s-1$ and for $j = 1, \ldots t-1$, $a_i a_{i+1}$ and $b_j b_{j+1}$ are either simple edges or edges subdivided once.



Similarly, if xa_1, xb_1, ya_s, or yb_t is not a simple edge, then G has a 2-separation (L', R') over the corresponding 2-vertex set, and by the initial hypothesis of Proposition 4.1 and Lemma 2.2, $L' = P_2$ or C_4. If $L'(x, a_1) = C_4$ and $L'(y, a_s) = C_4$ (or $L'(x, b_1) = C_4$ and $L'(y, b_t) = C_4$), then $G \geqslant_m J_3$, a contradiction (see figure below). Similarly, $L'(x, a_1) = C_4$ and $L'(y, b_t) = C_4$ (or $L'(x, b_1) = C_4$ and $L'(y, a_s) = C_4$), then $G \geqslant_m J_1$, a contradiction (see figure below).



Therefore, for one of the sides, say the x-side, we must have that xa_1 and xb_1 are either simple edges, or edges subdivided once. Therefore, it follows by **4** that the vertex y is apex in G, a contradiction since $G \notin \mathcal{O}^*$. Thus we have proved that $l(\mathcal{P}) \geqslant 2$.

Let $P = p_0 p_1 \dots p_n$ be a path in $\mathcal{P}$ of length $n := l(\mathcal{P}) \geqslant 2$, with $p_0 \in int(P_1)$ and $p_n \in int(P_2)$. Since $G \not\geqslant_m J_1$, it follows that:

5a. For $i = 0, 1, \dots, n-2$, there is no path of length $\geqslant 2$ from p_i to $int(P_2)$ that is internally disjoint from $P \cup C$.

Note that, by choice of P, the same holds true for $i = n-1$. Similarly:

6a. For $i = 2, 3, \dots, n$, there is no path of length $\geqslant 2$ from p_i to $int(P_1)$ that is internally disjoint from $P \cup C$.

And, by choice of P, the above also holds true for $i = 1$. Therefore, equivalently:

5b. For $i = 0, 1, \dots, n-1$, all the paths from p_i to $int(P_2)$ that are internally disjoint from $P \cup C$ are simple edges.

6b. for $i = 1, 2, \dots, n$, all the paths from p_i to $int(P_1)$ that are internally disjoint from $P \cup C$ are simple edges.

Let P_{11} and P_{12} be the subpaths of P_1 from x to p_0, and from p_0 to y, respectively. Similarly, Let P_{21} and P_{22} be the subpaths of P_2 from x to p_n, and from p_n to y, respectively. Let C_x be the cycle formed from the union of the paths P, P_{11} and P_{21}, and let C_y be the cycle formed from the union of the paths P, P_{12} and P_{22}.

Again, since $G \not\geqslant_m J_1$, it follows that:

7. All the paths in $\mathcal{P}$ that are internally disjoint from P are simple edges.

It follows by **5b** and **6b**, that G does not have a non-trivial bridge (where by a *trivial* bridge, we understand a simple edge) with one foot in $int(P)$ and another in $int(P_1) \cup int(P_2)$. Also, if G has a non-trivial bridge with two feet in P, then if the feet are consecutive vertices of P, then this violates the choice of P; and if they are non-consecutive, then $G \geqslant_m J_1$, a contradiction. Therefore:

8a. The only non-trivial bridges of G that attach to $int(P)$ have exactly two feet: one in $int(P)$ and the other at x or y.

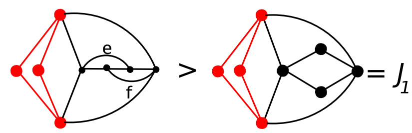
Let B be a non-trivial bridge that attaches to $int(P)$. Then, it follows by **8a** that B has one foot, call it p, in $int(P)$ and the other at x or y, say x. Then G has a 2-separation (L', R') over $\{x, p\}$, and it follows by the hypothesis of Proposition 4.1 and Lemma 2.2 that $L' = P_2$ or C_4. Hence, $B - \{x, p\}$ is a single vertex, or a pair of non-adjacent vertices. We call such a bridge a P_2-*bridge*, or a C_4-*bridge over* $\{x, p\}$, respectively. Thus we have shown:

8b. If B is a non-trivial bridge with one foot $p \in int(P)$ and the other at x (or y), then B is a P_2- or C_4-bridge over $\{x, p\}$ (over $\{y, p\}$ respectively).

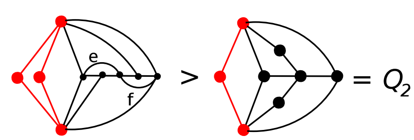
Let F_0 be the set of edges with one endpoint in $int(P_1) - \{p_0\}$ and the other in $int(P_2) - \{p_n\}$, and let F_1 be the set of edges whose both endpoints are non-consecutive vertices of P. Let F_2 be the set of edges with one endpoint in $\{p_0, p_1, \ldots, p_{n-2}\}$ and the other in $int(P_2) - \{p_n\}$, and let F_3 be the set of edges with one endpoint in $\{p_2, p_3, \ldots, p_n\}$ and the other in $int(P_1) - \{p_0\}$. Note that F_0, F_1, F_2, and F_3 are pairwise disjoint. Let $F := F_0 \cup F_1 \cup F_2 \cup F_3$ if $n \geq 3$. For shorthand, we will say that an edge or a vertex is *embedded in the top* or *in the bottom*, if it is embedded in the closed disk bounded by C_x or in the closed disk bounded by C_y, respectively. We now prove the following:

9. If $F \neq \varnothing$, then all edges of F can be embedded on one side: top or bottom.

Pf. First, suppose that the claim in **9** is not true due to two edges e and f of F_1. If the endpoints of $e = p_{i_0}p_{i_1}$ and $f = p_{i_2}p_{i_3}$ overlap, in the sense that $i_0 < i_2 < i_1 < i_3$, then $G \geqslant_m J_1$, a contradiction (see figure below).

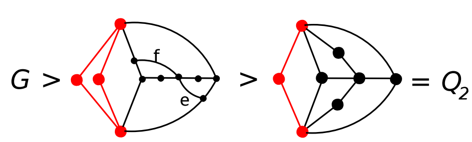


If the endpoints of e and f do not overlap (in the sense that $i_0 < i_1 < i_2 < i_3$) and, without loss of generality, e is in the top and f is in the bottom, then since G does not have a 2-separation over $\{p_{i_0}, p_{i_1}\}$ (by the initial hypothesis of Proposition 4.1 and (2) of Lemma 2.2), and since the vertices p_{i_0}, p_{i_1} are non-consecutive in P, there is a path from a vertex in $P(p_{i_0}, p_{i_1})$ to P_{12} (note that if the path is to a vertex in $int(P_2)$, then $G \geqslant_m J_1$ as in the overlapping case above; and similarly if the path is to a vertex $p_{i_4} \in P$ for some $i_4 < i_0$ or $i_4 > i_1$). Similarly, since G does not have a 2-separation over $\{p_{i_2}, p_{i_3}\}$, there is a path from a vertex in $P(p_{i_2}, p_{i_3})$ to P_{21}. Therefore $G \geqslant_m Q_2$, a contradiction (see figure below).



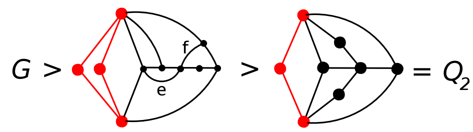
Second, suppose that the claim in **9** is not true due to two edges e and f of F_2 (the proof for F_3 is similar). Hence, both e and f have one endpoint in $\{p_0, p_1, \ldots, p_{n-2}\}$, however e has the other endpoint in $int(P_{21})$ and f in $int(P_{22})$. Then, $G - \{x, y\}$ contains a $K_{2,3}$-subdivision, contradicting the hypothesis that $G - \{x, y\}$ is in $\mathcal{O}$.

Third, suppose that the claim in **9** is not true due to an edge $e \in F_2$, embedded, say, in the bottom, and an edge $f \in F_3$ embedded in the top. Then G contains the following minor, which contains a Q_2-minor, a contradiction (see figure below).

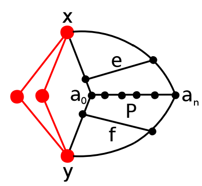


Fourth, suppose that the claim in **9** is not true due to an edge $e \in F_1$, embedded, say, in the bottom, and an edge $f \in F_2$ (the proof for $f \in F_3$ is similar) embedded in the top. Let $p_{i_0}q := f$

with $i_0 \in \{0, 1, \ldots, n-2\}$ and $q \in int(P_{21})$, and let $p_{i_1}p_{i_2} := e$ with $i_1 < i_2$. If $i_1 \geqslant i_0$, then $G - \{x, y\}$ contains a K_4-subdivision, a contradiction. Hence, $i_1 < i_0$. If $i_2 = n$, then since $i_0 \in \{0, 1, \ldots, n-2\}$, it follows that $G - \{x, y\}$ contains a $K_{2,3}$-subdivision, a contradiction. If $i_2 \in (i_0, n-1]$, then $G \geqslant_m J_1$ (as in the overlapping case), a contradiction. Therefore, $i_2 \leqslant i_0$ and since G does not have a 2-separation over $\{p_{i_1}, p_{i_2}\}$ (by the initial hypothesis of Proposition 4.1 and (2) of Lemma 2.2), there is a path from a vertex in $P(p_{i_1}, p_{i_2})$ to $P_{12} - \{p_0\}$, and thus G contains the following minor, which contains a Q_2-minor, a contradiction (see figure below).



Finally, suppose that the claim **9** is not true due to an edge $e \in F_0$, embedded, say, in the top, and an edge $f \in F_0 \cup F_1 \cup F_2 \cup F_3$ embedded in the bottom (the case $f \in F_0$ is illustrated below). Then, it can easily be checked that $G - \{x, y\}$ contains a K_4- or $K_{2,3}$-subdivision, a contradiction. This proves **9**.



As in the $l(\mathcal{P}) = 1$ case, let $a_1, a_2, \ldots, a_s \in int(P_1)$ be the left endpoints of the paths in $\mathcal{P}$ in the order of vertices on P_1 from x to y, and similarly let $b_1, b_2, \ldots, b_t \in int(P_2)$ be the right endpoints of the paths in $\mathcal{P}$ in the order of vertices on P_2 from x to y. Similarly to **4**, we have that:

10. For $i = 1, \ldots, s-1$ and for $j = 1, \ldots t-1$, a_ia_{i+1} and b_jb_{j+1} are either simple edges or edges subdivided once.

Similarly, if xa_1, xb_1, ya_s, or yb_t is not a simple edge, then G has a 2-separation (L', R') over the corresponding 2-vertex set, and by the initial hypothesis of Proposition 4.1 and Lemma 2.2, $L' = P_2$ or C_4. Thus:

11. If xa_1, xb_1, ya_s, or yb_t is not a simple edge, then $L'(x, a_1)$, $L'(x, b_1)$, $L'(y, a_s)$, $L'(y, b_t) \in \{P_2, C_4\}$, respectively (equivalently, G has a P_2- or C_4-bridge over $\{x, a_1\}$, $\{x, b_1\}$, $\{y, a_s\}$, or $\{y, b_t\}$, respectively).

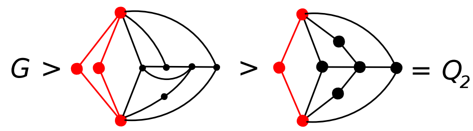
We now have two possibilities: either $F \neq \varnothing$ or $F = \varnothing$. We consider them below as Cases 1 and 2, respectively.

Case 1. $F \neq \varnothing$.

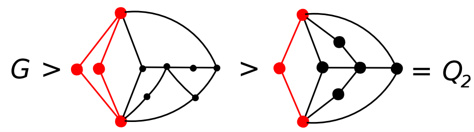
It follows from **9** that all the edges of F can be embedded, say, in the bottom (hence there are no edges of F embedded in the top). We will show that since G does not contain J_i-minor for $i = 1, \ldots, 5$, the vertex x will be apex in G, obtaining a contradiction. To do this, we first prove the following.

12. The only vertices embedded in the bottom are those lying on the cycle C_y.

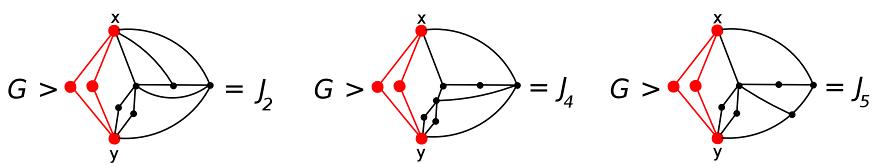
Pf. We prove this claim by showing that there are no non-trivial bridges embedded in the interior of the disk bounded by C_y. So assume that there is such a bridge B. First, if B has a foot in $int(P)$, then by **8a** and **8b**, it follows that the other foot of B is y. Since $F \neq \varnothing$, it contains an edge $e \in F_i$ for some $i = 0, 1, 2, 3$. Actually, $e \notin F_0$, for otherwise e would cross B, a contradiction. If $e \in F_1$, then G contains the following minor, which contains a Q_2-minor, contradiction (see figure below).



And if $e \in F_2$ (the proof for F_3 is similar), then G contains the following minor, which again contains a Q_2-minor, contradiction (see figure below).



Therefore B has its feet in $P_{12} \cup P_{22}$, but it cannot have a foot in P_{12} and another in P_{22}, because this would contradict either **5b**, **6b**, or **7**. Hence, B has all of its feet in P_{12} or all in P_{22}; by symmetry, we may assume that in P_{12}. Let p and q be the first and last feet of B in the order of vertices on P_{12}. Then G has a 2-separation (L', R') over $\{p, q\}$, and by the initial hypothesis of Proposition 4.1 and Lemma 2.2, $L' = P_2$ or C_4, so that B is a P_2- or C_4-bridge over $\{p, q\}$. Since $F \neq \varnothing$, it follows that $B \neq C_4$ for otherwise G would contain a J_2-, J_4-, or J_5-minor (see figure below).

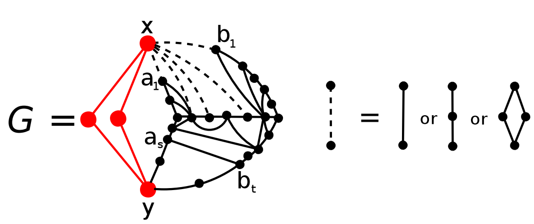


Hence, $B = P_2$, and so B is a subgraph of P_{12}. This proves **12**.

It follows by **12** that $L'(y, a_s) \neq C_4$ and $L'(y, b_t) \neq C_4$. Hence, ya_s and yb_t are either simple edges, or edges subdivided once. However, $L'(x, a_1)$ and $L'(x, b_1)$ could be either P_2 or C_4, or xa_1 and xb_1 could be simple edges.

By the fact that there are no edges of F in the top, and from **8a**, **8b**, **10**, and **11**, it follows that the only possible edges in the top are:

- edges from p_1 to P_{11};
- edges from p_{n-1} to P_{21};
- edges from $int(P)$ to x;
- edges that are part of the P_2- or C_4-bridges from $int(P)$ to x;
- edges that are part of the P_2- or C_4-bridges from a_1 or b_1 to x;
- edges of the cycle C_x;



Hence, the only possible vertices lying in the interior of the disk bounded by C_x are those from the P_2- or C_4-bridges from $int(P) \cup \{a_1, b_1\}$ to x. Hence, from this and **12** it follows that $G - x$ is outerplanar (i.e. x is an apex vertex of G), a contradiction.

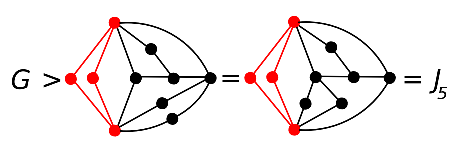
Case 2. $F = \varnothing$.

Again, by the fact F is empty, and from **8a**, **8b**, **10**, and **11**, it follows that the only possible edges in G are:

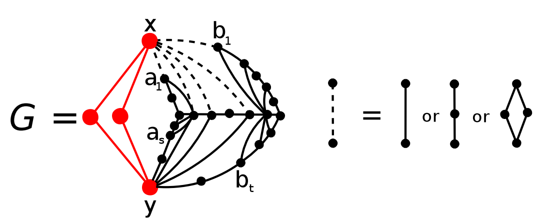
- edges from p_1 to P_1;
- edges from p_{n-1} to P_2;
- edges from $int(P)$ to x or to y;
- edges that are part of the $+P_2$- or C_4-bridges from $int(P)$ to x or to y;
- edges that are part of the P_2- or C_4-bridges from a_1 or b_1 to x, and from a_s or b_t to y;
- edges of the cycles C_x and C_y.

If there are no P_2- or C_4-bridges from $int(P)$ to x nor to y, then, just as in the proof of the $l(\mathcal{P}) = 1$ case, if $L'(x, a_1) = C_4$ and $L'(y, a_s) = C_4$ (or $L'(x, b_1) = C_4$ and $L'(y, b_t) = C_4$), then $G \geqslant_m J_3$. Similarly, if $L'(x, a_1) = C_4$ and $L'(y, b_t) = C_4$ (or $L'(x, b_1) = C_4$ and $L'(y, a_s) = C_4$), then $G \geqslant_m J_1$. Therefore, for one of the sides, say the x-side, we must have that xa_1 and xb_1 are either simple edges, or edges subdivided once. Hence, $G - y$ is outerplanar, a contradiction.

Hence, there is a P_2- or C_4-bridge from $int(P)$ to x or to y, but there cannot be such bridges to both x and y, for otherwise G would contain a Q_2-minor. Hence, there is a P_2- or C_4-bridge from $int(P)$ to, say x, but not to y. Then, $L'(y, a_s) \neq C_4$ and $L'(y, b_t) \neq C_4$, for otherwise $G \geqslant_m J_5$ (see figure below).



Therefore, ya_s and yb_t are either simple edges, or edges subdivided once. Hence, $G - x$ is outerplanar, a contradiction (see figure below).



This concludes the proof of (2) of Proposition 4.1.

5. Connectivity 2: One Side Always P_2

In this section, we focus on the case that every 2-separation of $G \in \mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$ has one side that is P_2 (Subcase 2.2 of the outline given in the Introduction). We prove the following proposition, which says that, in this case, $G \in \mathcal{H} \cup \mathcal{Q}$.

Proposition 5.1. *If $G \in \mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$ is of connectivity 2 and for every 2-separation S of G over $\{x, y\}$ one side is P_2, then the following holds true:*

(1) *If $G - \{x, y\} \notin \mathcal{O}$ for some such S, then G is a member of the family $\mathcal{H}$;*

(2) *If $G - \{x, y\} \in \mathcal{O}$ for every such S, then G is a member of the family $\mathcal{Q}$.*

We define a few terms first. A graph H is *internally* 3-*connected* if it is 2-connected, and for every 2-cut $\{s, t\}$, $H - \{s, t\}$ has two connected components, one of which is a single vertex. We say that a vertex in H is *pendant* if its degree in H is 1. Similarly, we say that an edge in H is *pendant* if it is incident with a pendant vertex. Before presenting a proof of Proposition 5.1, we first establish some preliminary observations based on the hypotheses of Proposition 5.1, which will be used later in the proof.

It follows from the hypothesis of Proposition 5.1 that G is internally 3-connected. Let (L, R) be a 2-separation over vertices $\{x, y\}$ such that $L = P_2$. Let v be the third (middle) vertex of L. Since G is minor-minimal $\notin \mathcal{O}^*$, G/vy has an apex vertex a (i.e. a such that $(G/vy) - a \in \mathcal{O}$). Note that $a \neq y$ and $a \neq x$, for otherwise y (or x, respectively) is an apex vertex in G, a contradiction. Since $deg(v) = 2$, it follows that G/vy is also internally 3-connected. Hence, the only possible 1-separations in $(G/vy) - a$ are those that separate a pendant vertex. Call such 1-separations *trivial*. Therefore, $(G/vy) - a$ is 2-connected up to trivial 1-separations (pendant edges), and outerplanar.

Fix a planar embedding of G so that all the vertices of $(G/vy) - a \in \mathcal{O}$ and a are incident with the outer face (i.e. infinite face). Since $(G/vy) - a$ is 2-connected up to trivial 1-separations, it follows that all the vertices of $(G/vy) - a \in \mathcal{O}$ lie along a cycle C, except (possibly) for the vertices of degree 1 in $(G/vy) - a$ that are adjacent to some vertex of C. Note that such vertices have degree 2 in G/vy (and in G), and that no two of them are adjacent to the same vertex c of C, for otherwise G has a 2-separation (L', R') over $\{c, a\}$ such that $L' = C_4$ or $L' \notin \mathcal{O}$ and $R' \notin \mathcal{O}$, contradicting the hypothesis of Proposition 5.1. Since $v \in G - a \notin \mathcal{O}$, it follows that v is embedded in the interior of the disk bounded by C. We have

1. The edges of G are:

- edges of C;
- *chords* of C, that is, edges not in $E(C)$ with both endpoints in C (note that such edges are embedded in the interior of the disk bounded by C);
- edges xv and vy, with $x, y \in V(C)$;
- edges with one endpoint in C and the other at a (or such edges subdivided once).

Also note that there are no two consecutive vertices in C of degree 2, since such vertices and their neighbors would induce a P_3 or a C_4 in G giving rise to a 2-separation violating the hypothesis of Proposition 5.1 .

In this context, by a *neighbor of a*, we mean a vertex u in C such that au is actually an edge of G or an edge subdivided once. As usual, we denote by $N(a)$ the set of neighbors of a. Since $xy \notin E(G)$, it follows that G has vertices in both $C(x, y)$ and $C(y, x)$. Furthermore,

2. a must have a neighbor in both $C(x, y)$ and $C(y, x)$.

For otherwise, G has a 2-separation over $\{x, y\}$ contradicting the hypothesis of Proposition 5.1.

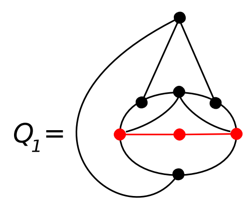
Note that a chord must have both of its endpoints in $C[x, y]$ or $C[y, x]$. We say that two chords $c := c_1c_2$ and $d := d_1d_2$ are *non-overlapping* if their endpoints satisfy $c_1 < c_2 \leqslant d_1 < d_2$ in the cyclic order of C, and are said to be *nested* if $c_1 \leqslant d_1 < d_2 \leqslant c_2$ or $d_1 \leqslant c_1 < c_2 \leqslant d_2$. It follows from **1** that:

3. If $c := c_1c_2$ is a chord with $c_1 < c_2$ (in the clockwise order restricted to $C[x, y]$ or $C[y, x]$), then a has a neighbor in $C(c_1, c_2)$.

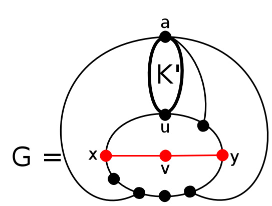
For otherwise, G has a 2-separation over $\{c_1, c_2\}$ contradicting the hypothesis of Proposition 5.1. Also,

4. Within a single segment $C[x, y]$ or $C[y, x]$, there are no non-overlapping chords (or equivalently, all the chords are nested).

Suppose that the chords $c := c_1c_2$ and $d := d_1d_2$ are non-overlapping with $c_1 < c_2 \leqslant d_1 < d_2$ within, say $C[x, y]$. Then, by **3**, a has a neighbor in $C(c_1, c_2)$ and in $C(d_1, d_2)$, and by **2**, it has a neighbor in $C(y, x)$. Then, G contains the following graph as a minor, which we label Q_1, and which can easily be verified to belong to $\mathbf{ob}(\mathcal{O}^*)$. This is a contradiction, since G is minor-minimal $\notin \mathcal{O}^*$.



5.1. **Proof of (1).** Let $G - \{x, y\} \notin \mathcal{O}$ for a 2-separation $S = (L, R)$ of G over $\{x, y\}$. Hence $G - \{x, y\}$ contains a K-subdivision as a subgraph, call it K'. By **1**, it follows that a is a cut-vertex in $G - \{x, y\}$, hence, without loss of generality, K' is a subgraph of $G - C[y, x]$. Let C' be the outer cycle of K'. Then, $|V(C') \cap C(x, y)| \geqslant 2$, for otherwise if $u := V(C') \cap C(x, y)$, then it follows by **1** that G has a 2-separation (L', R') over $\{a, u\}$ such that $L' \notin \mathcal{O}$ and $R' \notin \mathcal{O}$, contradicting the hypothesis that one side of (L', R') must be P_2 (and so in $\mathcal{O}$).



Let $s, t \in V(C') \cap C(x, y)$ be the first and last vertices, respectively, of $V(C') \cap C(x, y)$ in the clockwise order of $C(x, y)$. Note that $s \neq x$ and $t \neq y$. Also, since G does not contain two disjoint K-graphs, it follows that:

5. G does not have a chord with one endpoint in $C[x, s)$ and the other in $C(t, y]$.

It is straightforward to verify that the graphs in Figure 7 are minor-minimal $\notin \mathcal{O}^*$ satisfying the hypothesis of (1) and the initial hypothesis of Proposition 5.1. We label them H_1, H_2, H_3, H_4, H_5. Hence $H_i \in \mathbf{ob}(\mathcal{O}^*)$ for $i = 1, \ldots, 5$.

Therefore, if a has at least two neighbors in $C(y, x)$, or one neighbor $z \in C(y, x)$ and $C(y, z) \neq \varnothing$ and $C(z, x) \neq \varnothing$, then it is easy to verify that G contains an H_i-minor for some $i = 1, \ldots, 5$ (see figure below). Hence, let z be the only neighbor of a in $C(y, x)$. We only need to consider two cases: either both $C(y, z)$ and $C(z, x)$ are empty, or one of them is empty, say $C(y, z)$, and the other is not.

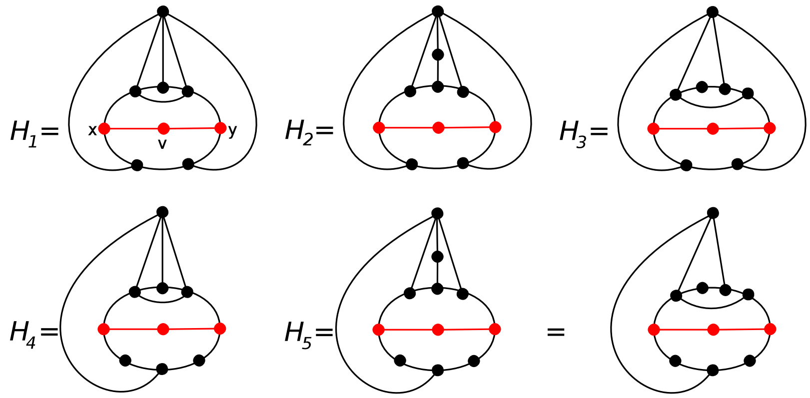
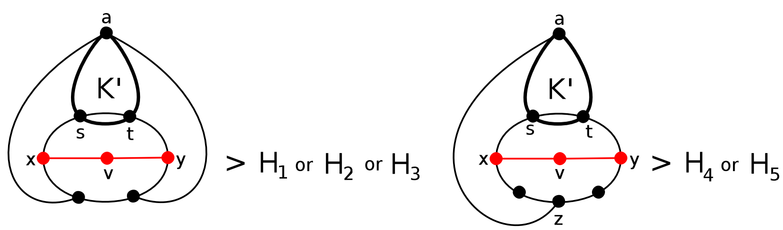
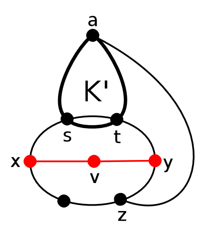


FIGURE 7. $\mathcal{H}$ family



First, suppose that $C(y, z) = \varnothing$ and $C(z, x) \neq \varnothing$. So $yz \in E(G)$. Then G has the following structure as a subgraph:



6. In G/yz, the only apex vertex is s.

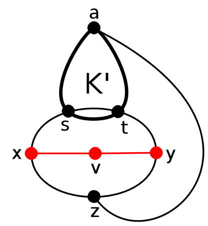
This is because an apex vertex in G/yz must destroy both K' and the $K_{2,3}$-subdivision with outer cycle C. Hence it must be a vertex in $V(C') \cap C(x, y)$. If $u \in C(s, t]$ is apex, then since s, t, and a all lie on C', it follows that in $G/yz - u$ there is a path P' in C' from a to s; this path, combined with the (possibly subdivided) edge ay $(= az)$ and the path along C from y to s form an outer cycle of a $K_{2,3}$-subdivision with inner path x, v, y. Hence, $G/yz - u$ contains a $K_{2,3}$-subdivision, a contradiction. This proves **6**.

7. y $(= z)$ is a cut-vertex in $G/yz - s$.

Note that there are no edges (or edges subdivided once) from a to $C(z, x)$ in G/yz, since z is the only neighbor of a in $C(y, x)$ in G. Also, note that there are no edges (or edges subdivided once) from a to $C[x, s)$ in G/yz, for otherwise $G/yz - s$ contains a $K_{2,3}$-subdivision, contradicting **6**. Finally, there are no chords from $C[x, s)$ to $C(s, t]$ in G/yz, for otherwise $G/yz - s$ contains a $K_{2,3}$-subdivision. These facts combined with **5** imply **7**.

Therefore, it follows by **7** that after uncontracting edge yz in $G/yz - s$, the resulting graph $G - s$ is also outerplanar, a contradiction since $G \notin \mathcal{O}^*$.

Now consider the other case that both $C(y,z)$ and $C(z,x)$ are empty (so that $yz, zx \in E(G)$). Recall that z is the only neighbor of a in $C(y,x)$. Then G has the following structure as a subgraph:



Similary to **6**, we obtain the following fact.

8. In $G \backslash az$, the only possible apex vertices are s and t.

We use the above to prove the following key fact.

9. One or both of the following hold:

(i) xs is an edge of G (or an edge subdivided once) and $deg(x) = 3$;

(ii) yt is an edge of G (or an edge subdivided once) and $deg(y) = 3$.

Note that if a vertex in $C(x,s)$ or $C(t,y)$ has degree $\geqslant 3$, then it is a neighbor of a or an endpoint of a chord. Similarly, if $deg(x) \geqslant 4$ or $deg(y) \geqslant 4$, then x, respectively y, is a neighbor of a or an endpoint of a chord. To prove **9**, we first note that a does not have neighbors in both $C[x,s)$ and $C(t,y]$, for otherwise $G \backslash az$ has no apex vertex (since neither s nor t is apex in $G \backslash az$), a contradiction. Hence, by symmetry, we may assume that a has no neighbors in $C[x,s)$. Then, by **3**, there are no chords with both endpoints in $C[x,s]$. If a has a neighbor in $C(t,y]$, then there are no chords with one endpoint in $C[x,s)$ and the other in $C(s,t]$, for otherwise $G \backslash az$ has no apex vertex (note that the other endpoint cannot lie in $C(t,y]$ by **5**), and thus (i) holds. And if a has no neighbors in $C(t,y]$ then, again by **3**, there are no chords with both endpoints in $C[t,y]$. Therefore, the only chords in G are those with one endpoint in $C[x,s)$ and the other in $C(s,t]$ (in which case (ii) holds), or those with one endpoint in $C[s,t)$ and the other in $C(t,y]$ (in which case (i) holds), but not both, since two such chords would either cross or would be non-overlapping, violating **4**. This proves **9**.

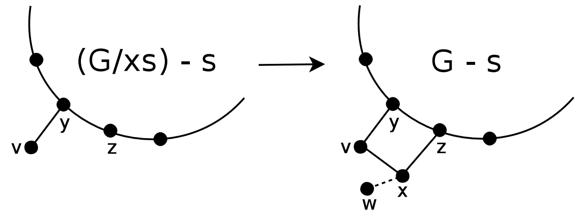
By symmetry, we may assume that (i) holds in **9**, so that xs is an edge of G (or an edge subdivided once, in which case denote the subdividing vertex by w). In the remainder of the proof, by G/xs we mean the graph obtained from G by contracting the path (of length 1 or 2) along C from s to x.

Similarly to **6** and **8**, we obtain:

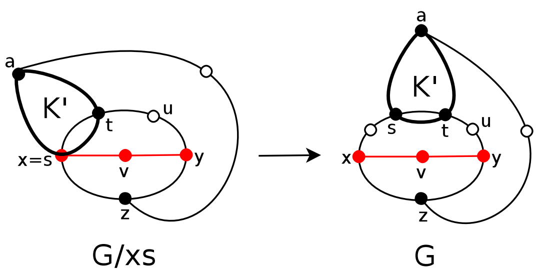
10. In G/xs, the only apex vertex is s $(= x)$, unless (ii) in **9** also holds, then t may also be apex.

If (ii) does not hold, then either a has a neighbor in $C(t,y]$ or G has a chord with one endpoint in $C[s,t)$ and the other in $C(t,y]$. And in either case t is not apex in G/xs.

Note that $(G/xs) - s = G - \{x,s\}$ (or possibly $(G/xs) - s = G - \{x,w,s\}$ if xs is subdivided). Re-embed the graph $(G/xs) - s \in \mathcal{O}$ (if necessary), so that all of its vertices are incident with the outer face. In $(G/xs) - s$, $deg(z) = 2$ and $deg(v) = 1$, hence edges zy and vy are also incident with the outer face. Since yz is a simple edge, by putting x (and possibly w) back in, we obtain an embedding of $G - s$ in which all the vertices are still incident with the outer face, hence $G - s$ is outerplanar, a contradiction (see figure below).



Finally, if t is also apex in G/xs, then by the above, (ii) in **9** also holds, so that yt is an edge of G (or an edge subdivided once, in which case denote the subdividing vertex by u) and $deg(y) = 3$. Since $(G/xs) - t \in \mathcal{O}$, there is a face f in the current embedding incident with all the vertices of $(G/xs) - t$. Since the path (of length 1 or 2) from s to x can be uncontracted along C, it follows that f is also incident with all the vertices of $G - t$, a contradiction since $G \notin \mathcal{O}^*$ (see figure below).



This concludes the proof of (1) in the case that both $C(y,z)$ and $C(z,x)$ are empty, as well as the proof of (1) of Proposition 5.1.

5.2. **Proof of (2).** It is straightforward to verify that the graphs in Figure 8 are minor-minimal $\notin \mathcal{O}^*$ satisfying the initial hypothesis and the hypothesis of (2) of Proposition 5.1. We label them Q_1, Q_2, Q_3, Q_4, Q_5. Hence $Q_i \in \mathbf{ob}(\mathcal{O}^*)$ for $i = 1, \ldots, 5$.

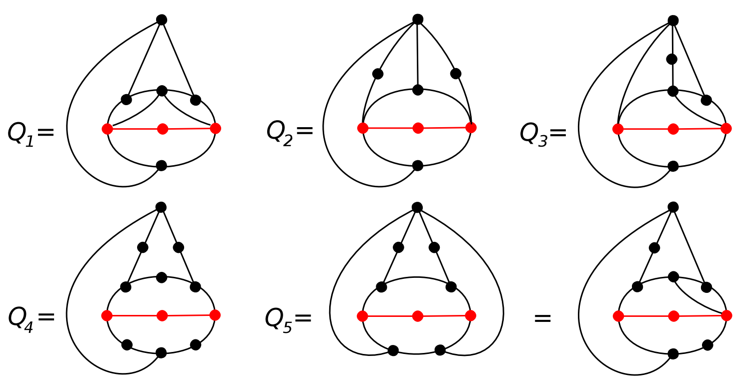


FIGURE 8. $\mathcal{Q}$ family

In the remainder of the proof, we assume that $G \notin \{Q_1, Q_2, Q_3, Q_4, Q_5\}$. Observe that now the vertex a from **2** satisfies

11. $deg(a) \geqslant 3$.

For otherwise, if $deg(a) = 2$, then let the two neighbors of a be a_1 and a_2 (in $C(x,y)$ and $C(y,x)$, respectively, by **2**). Note that there is a chord with one endpoint in $C[x, a_1)$ and the other in $C(a_1, y]$, for otherwise, it follows by **1** that a_1 is apex in G, a contradiction. Similarly, there is a chord with one endpoint in $C[y, a_2)$ and the other in $C(a_2, x]$, for otherwise, it follows by **1** that a_2 is apex in G, a contradiction. Since $deg(a) = 2$, it follows that G has a 2-separation over $\{a_1, a_2\}$ such that $G - \{a_1, a_2\}$ contains a $K_{2,3}$-subdivision, contradicting the hypothesis of (2) that $G - \{a_1, a_2\} \in \mathcal{O}$. This proves **11**.

Case 1. *G has no chords.*

Subcase 1.1. $|N(a) \cap C(x, y)| = 1$ and $|N(a) \cap C(y, x)| = 1$.

Then, there is a subdivided edge ay, for otherwise x is apex. Also, there is a subdivided edge ax, for otherwise y is apex, and hence $G \geqslant_m Q_2$, a contradiction.

Subcase 1.2. $|N(a) \cap C(x, y)| = 2$ and $|N(a) \cap C(y, x)| = 1$.

First suppose that $x, y \notin N(a)$. Let $a_1 \in N(a) \cap C(y, x)$ and $a_2, a_3 \in N(a) \cap C(x, y)$ in the clockwise order around C. Then, there is a vertex in $C(a_2, a_3)$, for otherwise a_1 is apex. Edge aa_3 is subdivided, for otherwise x is apex. Edge aa_2 is subdivided, for otherwise y is apex. There is a vertex in $C(y, a_1)$, for otherwise a_2 is apex. Finally, there is a vertex in $C(a_1, x)$, for otherwise a_3 is apex, and hence $G \geqslant_m Q_4$.

Next, suppose that $x \in N(a)$, but $y \notin N(a)$. Then, edge aa_3 is subdivided, for otherwise x is apex. Edge ax is not subdivided, for otherwise $G \geqslant_m Q_2$. Edge aa_2 is subdivided, for otherwise y is apex. Finally, there is a vertex in $C(a_1, x)$, for otherwise a_3 is apex, and hence $G \geqslant_m J_1$.

Finally, suppose that $x, y \in N(a)$. Then, at least one of aa_3, ay is subdivided, for otherwise x is apex. Also, at least one of aa_2, ax is subdivided, for otherwise y is apex. If aa_2 and aa_3 are, then $G \geqslant_m J_1$. If ax and ay are, then $G \geqslant_m Q_2$. Finally, if ax and aa_3 are, or a_2 and ay are, then again $G \geqslant_m Q_2$, a contradiction.

Subcase 1.3. $|N(a) \cap C(x, y)| \geqslant 3$ and $|N(a) \cap C(y, x)| = 1$.

Let $a_1 \in N(a) \cap C(y, x)$ and $a_2, a_3 \in N(a) \cap C(x, y)$ be such that a_2 is the vertex in $N(a) \cap C(x, y)$ closest to x, and a_3 is the vertex in $N(a) \cap C(x, y)$ closest to y. Note that if $u \in N(a) \cap C(a_2, a_3)$, then edge au is not subdivided, for otherwise $G - \{x, y\}$ contains a $K_{2,3}$-subdivision, contradicting the hypothesis that $G - \{x, y\} \in \mathcal{O}$.

Therefore, at least one of aa_3, ay (if $ay \in E(G)$) is subdivided, for otherwise x is apex. Also, at least one of aa_2, ax (if $ax \in E(G)$) is subdivided, for otherwise y is apex. Hence, $G \geqslant_m Q_2$, a contradiction.

Subcase 1.4. $|N(a) \cap C(x, y)| \geqslant 2$ and $|N(a) \cap C(y, x)| \geqslant 2$.

Let $a_1, a_2 \in N(a) \cap C(y, x)$ and $a_3, a_4 \in N(a) \cap C(x, y)$ be such that a_1 and a_4 are the two neighbors of a closest to y, and a_2 and a_3 are the two neighbors of a closest to x. Note that if $u \in N(a) \cap (C(a_1, a_2) \cup C(a_3, a_4))$, then edge au is not subdivided, for otherwise $G - \{x, y\}$ contains a $K_{2,3}$-subdivision, contradicting the hypothesis that $G - \{x, y\} \in \mathcal{O}$.

Therefore, at least one of aa_1, aa_4, ay (if $ay \in E(G)$) is subdivided, for otherwise x is apex. Also, at least one of aa_2, aa_3, ax (if $ax \in E(G)$) is subdivided, for otherwise y is apex. Hence, it follows from these two facts that if $ay \in E(G)$ and it is subdivided, then $G \geqslant_m Q_2$, a contradiction. Similarly, if $ax \in E(G)$ and it is subdivided, then $G \geqslant_m Q_2$, a contradiction. Hence, if $ax \in E(G)$ or $ay \in E(G)$, then they are not subdivided. Finally, if aa_1 and aa_2 are, or if aa_3 and aa_4 are, then $G \geqslant_m Q_5$, a contradiction. And if aa_1 and aa_3 are, or if aa_2 and aa_4 are, then $G \geqslant_m Q_2$, a contradiction. This concludes the proof of (2) of Proposition 5.1 in Case 1.

Case 2. *G has a chord.*

We first strengthen **3** to the following:

12. If $c := c_1c_2$ is a chord with $c_1 < c_2$ (in the clockwise order restricted to $C[x, y]$ or $C[y, x]$), then a has a neighbor in $C(c_1, c_2)$. Furthermore, for any such neighbor w, the edge aw is not subdivided.

For otherwise, G would have a 2-separation over $\{a, w\}$ such that $G - \{a, w\}$ has $K_{2,3}$-subdivision contradicting the hypothesis that $G - \{a, w\} \in \mathcal{O}$.

The following two claims greatly limit the structure of G.

Claim 1. Let $c = c_1c_2$, with $c_1, c_2 \in C(x, y)$ in the clockwise order around C, be an innermost chord of G (in the sense that there are no other chords with both endpoints in $C[c_1, c_2]$). Then a does not have two neighbors in $C(c_1, c_2)$.

Pf. Suppose that a does have two neighbors $a_1, a_2 \in C(c_1, c_2)$. By **12**, edges aa_1 and aa_2 are not subdivided. Also, a does not have any other neighbors in $C(x, y)$, for otherwise $G - \{x, y\}$ would contain a K_4-subdivision, violating the hypothesis of (2) that $G - \{x, y\} \in \mathcal{O}$. Also, $C(a_1, a_2) = \varnothing$, for otherwise $G - \{x, y\}$ would contain a $K_{2,3}$-subdivision, violating the hypothesis of (2). Note that possibly, edges c_1a_1 and a_2c_2 are subdivided once, but since c is an innermost chord, there are no other vertices in $C(c_1, c_2)$. If a has at least two neighbors in $C(y, x)$, then $G \geqslant_m Q_5$, a contradiction. Hence, let z be the only neighbor of a in $C(y, x)$.

We let u be an apex vertex in $G \backslash a_1a_2$, and we assume that the graph $(G \backslash a_1a_2) - u \in \mathcal{O}$ is embedded in the plane with all of its vertices incident with the outer face. Note that $u \in \{z, c_1, c_2\}$, for otherwise: if $u \in \{a_1, a_2\}$, then clearly u is apex in G, a contradiction; if $u \in \{a\} \cup C(c_1, a_1) \cup C(a_2, c_2)$, then $(G \backslash a_1a_2) - u$ contains a $K_{2,3}$-subdivision; and if $u \in \{v\} \cup C(c_2, z) \cup C(z, c_1)$, then $(G \backslash a_1a_2) - u$ contains a K_4-subdivision.

If $u = z$, then the only neighbors of a are a_1, a_2 and z (because if x or y is a neighbor of a then $(G \backslash a_1a_2) - z$ contains a $K_{2,3}$-subdivision). Then, in $(G \backslash a_1a_2) - z$, $deg(a) = 2$, hence edges aa_1 and aa_2 are incident with the outer face, and by putting the edge a_1a_2 back in, we obtain an embedding of $G - z$ in which all the vertices are still incident with the outer face, hence $G - z$ is outerplanar, a contradiction.

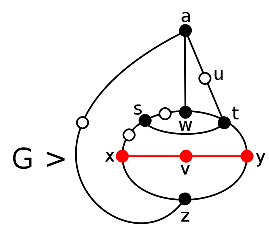
Finally, suppose that $u = c_1$ (the case $u = c_2$ is symmetric). If c_1a_1 is subdivided once, then let b be the subdividing vertex. Then, in $(G \backslash a_1a_2) - c_1$, then $deg(a_1) = 1$ (except if c_1a_1 is subdivided by b, then $deg(a_1) = 2$, but a_1 is adjacent to b with $deg(b) = 1$, that is a_1b is a pendant edge), and $deg(a_2) = 2$. Hence edges aa_2 and aa_1 (and possibly a_1b) are incident with the outer face, and since aa_2 is a simple edge, we can put edge a_1a_2 back in to obtain an embedding of $G - c_1$ in which all the vertices are still incident with the outer face, a contradiction. This proves Claim 1.

Claim 2. G does not have a chord with both endpoints distinct from x and y.

Pf. Suppose that G does have a chord with endpoints $s, t \in C(x, y)$ in the clockwise order around C. We may assume, without loss of generality, that st is the innermost chord, in the sense that there are no other chords with both endpoints in $C[s, t]$. By **12**, there is a vertex $w \in N(a) \cap C(s, t)$ and the edge aw is not subdivided. Also, by Claim 1, $N(a) \cap C(s, t) = \{w\}$. Also, a does not have neighbors in both $C(x, s]$ and $C[t, y)$, for otherwise $G - \{x, y\}$ would contain a K_4-subdivision, violating the hypothesis that $G - \{x, y\} \in \mathcal{O}$. Also, by **4**, G does not have chords with both endpoints in $C[x, s]$ or both in $C[t, y]$. Let $z \in N(a) \cap C(y, x)$. First, we show Claim 2a and then Claim 2b. They are needed for the proof of Claim 2.

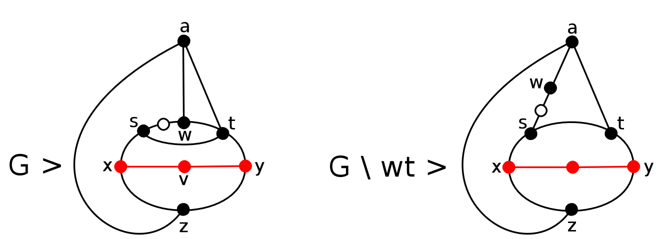
Claim 2a. Neither s nor t can be a neighbor of a.

Pf. By symmetry, we may assume that t is a neighbor of a, so that s is not. Then, $C(x, s] \cap N(a) = \varnothing$. Also, $C(w, t) = \varnothing$, for otherwise $G - \{x, y\}$ would contain a $K_{2,3}$-subdivision. Also, edges sw and ta are possibly subdivided once, but by choice of chord c, there are no other vertices in $C(s, t)$. Hence G contains the following subgraph:



First, suppose that edge ta is subdivided by vertex u. Then $C(t, y] \cap N(a) = \varnothing$, for otherwise $G \geqslant_m Q_3$. For the same reason, we have that $(C(y, z) \cup C(z, x)) \cap N(a) = \varnothing$. Hence, the only neighbor of a other than z, w and t is possibly x. Furthermore, if $ax \in E(G)$ then it is not subdivided for otherwise $G \geqslant_m Q_2$. Also, note that the remaining chords whose endpoints lie in $C[x, y]$ must have one of their endpoints at t, and the other in $C[x, s)$, for otherwise **4** is violated, or the subdivided edge ta violates **12**. It follows from all of the above that if $C(z, x) = \varnothing$, then t is apex in G, a contradiction. Hence $C(z, x) \neq \varnothing$. Then, if $ax \in E(G)$, then $G \geqslant_m J_1$ (by contracting z to y, contracting s to x, and deleting ws). Thus $ax \notin E(G)$. Therefore, since $C(w, t) = \varnothing$, if G has no chords with one endpoint in $C[y, z)$ and the other in $C(z, x]$, then z is apex in G, a contradiction. Hence, G does have at least one such chord c. If c has one endpoint in $C(z, x)$ and the other in $C[y, z)$, then $G \geqslant_m Q_3$ (by contracting z to a, and s to x). Hence, c has one endpoint at x and the other in $C(y, z)$, but then again $G \geqslant_m Q_2$ (by deleting st, contracting z to a, contracting s to x, and contracting t to y), a contradiction. Thus we have shown that ta is not subdivided, that is $ta \in E(G)$.

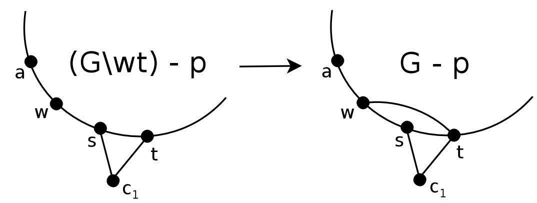
We let p be an apex vertex in $G \backslash wt$, and we assume that the graph $(G \backslash wt) - p \in \mathcal{O}$ is embedded in the plane with all of its vertices incident with the outer face. Note that $p \notin \{w, t\}$, for otherwise p is apex in G. In fact, it is easy to see that if $p \notin \{z\} \cup C[x, s]$, then p is not apex in $G \backslash wt$, a contradiction. G and $G \backslash wt$ contain the following subgraphs, respectively:



Suppose that $p = z$. Then, a has no neighbors other than w, t, and z, for otherwise $(G \backslash wt) - p$ contains a K_4-subdivision. Therefore, in the graph $(G \backslash wt) - p$, $deg(a) = 2$, hence edges aw and at are incident with the outer face, and we can put edge wt back in, to obtain an embedding of

$G - z$ in which all the vertices are still incident with the outer face, hence $G - z$ is outerplanar, a contradiction.

Therefore $p \in C[x, s]$. Recall from above that $C(x, s] \cap N(a) = \varnothing$. Note that there are no chords with one endpoint in $C[x, p)$ and the other in $C[t, y]$, for otherwise $(G \backslash wt) - p$ contains a $K_{2,3}$-subdivision. Also, if a chord has one endpoint in $C(p, s]$, then its other endpoint is t, for otherwise $(G \backslash wt) - p$ contains a K_4-subdivision. For simplicity, assume that $c = c_1 t$ is the only such chord with $c_1 \neq s$. If there is more than one such chord, the argument is similar. Also, note that edges pc_1, $c_1 s$, and sw may be subdivided once, but the subdividing vertices can be ignored for the purposes of this argument, as will be apparent soon. So for simplicity, we assume that pc_1, $c_1 s$, and sw are simple edges. By the observations above, it follows that in $(G \backslash wt) - p$, $deg(w) = 2$, and $deg(c_1) = 2$, hence edges wa, ws, $c_1 s$ and $c_1 t$ are incident with the outer face, which implies that edge st is not. Therefore, since in $(G \backslash wt) - p$, $deg(s) = 3$, it follows that we can put edge wt back in, to obtain an embedding of $G - p$ in which all of the vertices are still incident with the outer face, hence $G - p$ is outerplanar, a contradiction (see figure below). Finally, note that if edges pc_1, $c_1 s$, and sw are subdivided once, then its subdividing vertices are still incident with the outer face in the above embedding of $G - p$, since in the above argument edges $c_1 s$ and sw are incident with the outer face. This proves Claim 2a.

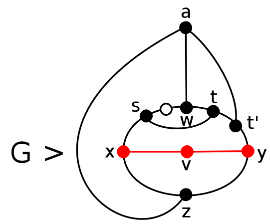


Therefore, neither s nor t is a neighbor of a. We now show furthermore:

Claim 2b. a does not have a neighbor in $C(x, s) \cup C(t, y)$.

Pf. By symmetry, suppose that $N(a) \cap C(t, y) \neq \varnothing$, so that $N(a) \cap C(x, s) = \varnothing$, and let $t' \in N(a) \cap C(t, y)$. Then, all the chords that have an endpoint in $C(t, y)$ have the other endpoint at x, for otherwise **4** is violated, or $G - \{x, y\}$ contains a K_4-subdivision. Also, $C(w, t) = \varnothing$, for otherwise $G - \{x, y\}$ would contain a $K_{2,3}$-subdivision, violating the hypothesis of (2) that $G - \{x, y\} \in \mathcal{O}$.

First, suppose that edge $t'a$ is subdivided by vertex u. Then, $C(t, t') \cap N(a) = \varnothing$, for otherwise $G \geqslant_m Q_2$. Also, $C(t', y] \cap N(a) = \varnothing$, for otherwise $G \geqslant_m Q_3$. For the same reason, we have that $(C(y, z) \cup C(z, x)) \cap N(a) = \varnothing$. Hence, the only neighbor of a other than z, w and t' is possibly x. Furthermore, if $ax \in E(G)$ then it is not subdivided for otherwise $G \geqslant_m Q_2$. Now consider what the remaining chords within $C[x, y]$ are. Note that a chord cannot have an endpoint in $C(t', y]$, since it would violate either **4** or **12**. And it cannot have an endpoint at t, since the other endpoint would be in $C[x, s)$, and G would contain a Q_2-minor; and similarly it cannot have an endpoint at $C(t, t')$ (and hence the other at x). Hence, all the remaining chords whose endpoints lie in $C[x, y]$ have an endpoint at t'. It follows from all of the above that if $C(z, x) = \varnothing$, then t' is apex in G, a contradiction. Hence $C(z, x) \neq \varnothing$, and so $G \geqslant_m Q_5$ (by contracting s to x and deleting all the chords incident with t'), a contradiction. Thus we have shown that $t'a$ is not subdivided, that is $t'a \in E(G)$.



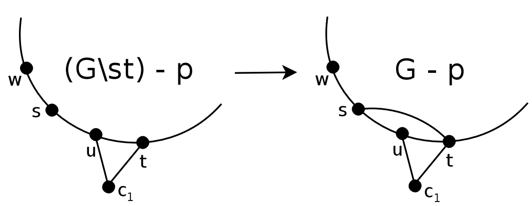
We will now proceed to show, in a sequence of steps, that the only possible chords with both endpoints in $C[x,y]$ other than st are the ones with one endpoint at x and the other in $C[t',y)$. Recall from above that:

A. All chords that have an endpoint in $C(t,y)$ have the other endpoint at x.

We prove

B. There is no chord with one endpoint at t and the other in $C[x,s)$.

For otherwise, let $u \in C[x,s)$ be the other endpoint of such a chord, and choose u to be the closest to s, in the sense that there are no chords with both endpoints in $C[u,t]$ other than st and ut. Note that, us and sw are either edges of G or edges subdivided once, but again we may assume, without loss of generality, that us and sw are just simple edges. Let p be an apex vertex in $G \backslash st$. It is easy to see that $p \in C[x,u]$. If $p = u$, then there are no more chords with an endpoint at t, for otherwise $(G \backslash st) - u$ contains a $K_{2,3}$-subdivision. Hence, in $(G \backslash st) - u$, $deg(t) = 2$ and $deg(s) = 1$, hence edge wt and the pendant edge ws are incident with the outer face. Therefore, since $C(w,t) = \varnothing$ (equivalently, $wt \in E(G)$), we can put edge st back into this embedding to obtain an outerplanar embedding of $G - u$, a contradiction. Therefore, we must have $p \in C[x,u)$. Also, if a chord has one endpoint in $C(p,u]$, then its other endpoint is t, for otherwise if the other endpoint is y, then $(G \backslash st) - p$ contains a K_4-subdivision. For simplicity, assume that $c = c_1 t$ is the only such chord with $c_1 \neq s$. If there is more than one such chord, the argument is similar. Also, note that edges pc_1, $c_1 u$, us and sw may be subdivided once, but again we may assume, without loss of generality, that they are all just simple edges (since they will turn out to be incident with the outer face in $(G \backslash st) - p$). By the observations above, it follows that in $(G \backslash st) - p$, $deg(s) = 2$, and $deg(c_1) = 2$, hence edges su, sw, $c_1 u$ and $c_1 t$ are incident with the outer face, which implies that edge ut is not. Therefore, since in $(G \backslash st) - p$, $deg(u) = 3$, it follows that by putting edge st back in, we can embed $G - p$ so that all the vertices are still incident with the outer face, hence $G - p$ is outerplanar, a contradiction (see figure below). This proves **B**.



C. There is no chord with one endpoint in $C(t,t')$ and the other at x.

Suppose the contrary, and let $u \in C(t,t')$ be the endpoint of such a chord. By **B**, there is no chord with one endpoint at t and the other in $C[x,s)$, hence xs is an edge, or an edge subdivided once. Note that, xs and sw are either edges of G or edges subdivided once, but again we may assume, without loss of generality, that xs and sw are just simple edges. Let p be an apex vertex in $G\backslash st$. It is easy to see that $p = x$. Hence, in $(G\backslash st) - p$, $deg(t) = 2$ and $deg(s) = 1$, hence edge wt and the pendant edge ws are incident with the outer face. Therefore, by putting edge st back into this embedding, we obtain an outerplanar embedding of $G - x$, a contradiction. This proves **C**.

D. There is no chord with one endpoint at y and the other in $C(x, s]$.

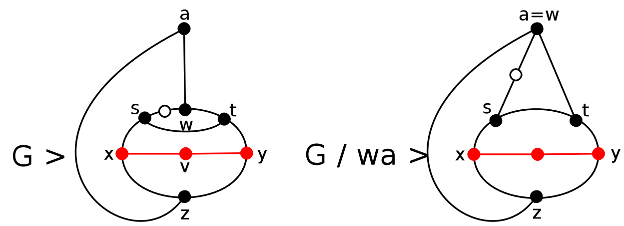
Suppose the contrary, and let $u \in C(x, s]$ be the endpoint of such a chord, and choose u to be the closest to s, in the sense that there is no other chords with one endpoint at y and the other in $C(u, s]$. Therefore, us and sw are either edges of G or edges subdivided once, but again we may assume, without loss of generality, that us and sw are just simple edges. It is easy to see that u is the only possible apex vertex in $G\backslash wt$. First, if $u \in C(x, s)$, then in $(G\backslash wt) - u$, $deg(s) = 2$, hence edges sw and st are incident with the outer face. Therefore, by putting edge wt back into this embedding, we obtain an outerplanar embedding of $G - u$, a contradiction. Finally if $u = s$, then in $(G\backslash wt) - s$, $deg(t') = 3$ and $deg(t) = 1 = deg(w)$, hence edge $t'a$ and pendant edges tt', aw are incident with the outer face. Therefore, since at' is a simple edge, by putting edge wt back into this embedding, we obtain an outerplanar embedding of $G - s$, a contradiction. This proves **D**.

It follows by **A** - **D** that:

E. The only possible chords with both endpoints in $C[x, y]$ other than st are the ones with one endpoint at x and the other in $C[t', y)$.

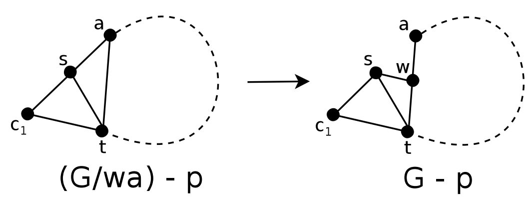
Hence, xs and sw are either edges of G or edges subdivided once, but again we may assume, without loss of generality, that xs and sw are just simple edges. In the remainder of the proof of Claim 2b, by G/xs we mean the graph obtained from G by contracting the path (of length 1 or 2) along C from s to x. Let p be an apex vertex in G/xs. It is easy to see that $p = x$ or $p = t'$. If $p = x$, then in $(G/xs) - x$, $deg(w) = 2 = deg(t)$, hence edge wt is incident with the outer face. Therefore, by putting edges ws and st back into this embedding, we obtain an outerplanar embedding of $G - x$, a contradiction. And if $p = t'$, then observe the following facts. First, there are no chords with one endpoint at x and the other in $C(t', y)$, therefore, by **E**, the only possible chord with both endpoints in $C[x, y]$ other than st is xt'. Second, a has no other neighbors, except possibly x, for otherwise $(G/xs) - t'$ contains a K_4-subdivision. And if $x \in N(a)$, then xa is not subdivided. Third, $C(z, x) = \varnothing$, and the only edges left in G are chords from x to $C(y, z)$. These facts account for all the edges of G. Hence t' is apex in G, a contradiction. This concludes the proof of Claim 2b.

We now finish the proof of Claim 2. Note that edges sw and wt are possibly subdivided, but again we may assume, without loss of generality, that they are simple edges. It follows from Claims 2a and 2b that a does not have neighbors in $C(x, s] \cup C[t, y)$. Also, by **4**, there are no chords with both endpoints in $C[x, s]$ or both in $C[t, y]$. Again, we let p be an apex vertex in G/wa. It follows from **11** that besides w and z, a has another neighbor (in $C[y, x]$). Therefore $p \neq z$, since $(G/wa) - z$ contains a K_4-subdivision. In fact, it is easy to check that $p \in C[x, s] \cup C[t, y]$, for otherwise $(G/wa) - p$ contains a K-subdivision.



By symmetry, we may assume that $p \in C[x,s]$. First, if $p = s$, then all the chords whose endpoints lie in $C[x,y]$ have an endpoint at s, for otherwise $(G/wa) - s$ contains a $K_{2,3}$-subdivision. Thus, in $(G/wa) - s$, $deg(t) = 2$, hence edge ta is incident with the outer face. Therefore, in the current embedding of $(G/wa) - s$, we can subdivide edge ta by w to obtain an embedding of $G - s$ in which all the vertices are still incident with the outer face, hence $G - s$ is outerplanar, a contradiction.

Therefore, $p \in C[x,s)$. Then, by Claims 2a and 2b, a has no neighbors in $C(p,s]$. If a chord has an endpoint in $C(p,s]$, then its other endpoint is t, otherwise $(G/wa) - p$ contains a K_4-subdivision. For simplicity, assume that $c = c_1 t$ is the only such chord with $c_1 \neq s$. If there is more than one such chord, the argument is similar. Again, the edges pc_1 and $c_1 s$ may be subdivided once, but the subdividing vertices can be ignored for the purposes of this argument. So for simplicity, we assume that pc_1 and $c_1 s$ are simple edges. By the observations above, it follows that in $(G/wa) - p$, $deg(c_1) = 2$, hence edges $c_1 s$ and $c_1 t$ are incident with the outer face, which implies that edge st is not. Therefore, since in $(G/wa) - p$, $deg(s) = 3$, it follows that sa is also incident with the outer face (and hence edge at is not, for otherwise the edges of the cycle a, t, c_1, s, a are all incident with the outer face, which implies that those are all the vertices in $(G/wa) - p$, since $(G/wa) - p$ has no non-trivial 1-separations, a contradiction). Therefore, it follows that in the current embedding of $(G/wa) - p$, we can delete edge sa, subdivide edge at by vertex w, and add edge ws and obtain an embedding of $G - p$ in which all the vertices are still incident with the outer face, hence $G - p$ is outerplanar, a contradiction (see figure below). This concludes the proof of Claim 2.



We now finish the proof Case 2 ("G has a chord") and thus the entire proof of (2) of Proposition 5.1. By **4** and Claim 2, it follows that within each of the two segments $C[x,y]$ and $C[y,x]$ all the chords have an endpoint at x or all the chords have an endpoint at y. We have three subcases (of Case 2: "G has a chord"):

Subcase 2.1. There are chords within $C[x,y]$ and within $C[y,x]$, and the ones within $C[x,y]$ have an endpoint at y, and the ones within $C[y,x]$ have an endpoint at x.

Let $c_1 y$ and $d_1 x$ be innermost chords within $C[x,y]$ and $C[y,x]$, respectively. By **12**, a has a neighbor $w \in C(c_1, y)$, and a neighbor $z \in C(d_1, x)$, and edges aw and az are not subdivided.

First, suppose that a has a neighbor u such that edge au is subdivided. Then, by **12**, $u \notin C(c_1, y) \cup C(d_1, x)$. If $u \in C(x, c_1]$ or $u \in C(y, d_1]$, then $G \geqslant_m Q_3$ (by contracting za or wa, respectively). Therefore, $u \in \{x, y\}$, so by symmetry $u = x$. Since $G \not\geqslant_m Q_2$, it follows that

$N(a) \cap (C(x,w) \cup C(w,y)) = \varnothing$, $C(w,y) = \varnothing$, and if $y \in N(a)$, then ay is not subdivided. Therefore, x is apex in G, a contradiction.

Therefore, for all neighbors u of a, au is a simple edge. Note that if a has no neighbors in $C(x,w) \cup C(w,y)$ and $C(w,y) = \varnothing$, then x is apex in G. Similarly, if a has no neighbors in $C(y,z) \cup C(z,x)$ and $C(z,x) = \varnothing$, then y is apex in G, a contradiction. Therefore, either $N(a) \cap (C(x,w) \cup C(w,y)) \neq \varnothing$ or $C(w,y) \neq \varnothing$; and either $N(a) \cap (C(y,z) \cup C(z,x)) \neq \varnothing$ or $C(z,x) \neq \varnothing$. It can easily be seen that any one of the four combination yields a Q_2-minor in G, a contradiction.

Subcase 2.2. There are chords within $C[x,y]$ and within $C[y,x]$, and all chords of G have an endpoint at y.

Let c_1y and d_1y be innermost chords within $C[x,y]$ and $C[y,x]$, respectively. By **12**, a has a neighbor $w \in C(c_1,y)$, and a neighbor $z \in C(y,d_1)$, and edges aw and az are not subdivided.

Note that a has a neighbor $u \neq y$ such that au is subdivided, for otherwise y is apex in G, a contradiction. Then, by **12**, $u \notin C(c_1,y) \cup C(y,d_1)$, hence $u \in C[x,c_1] \cup C[d_1,x]$. By symmetry, we only need to consider $u \in C[x,c_1]$. First, if $u = x$, then since $G \not\geq_m Q_2$, it follows that $N(a) \cap (C(x,w) \cup C(w,y) \cup C(y,z) \cup C(z,x)) = \varnothing$, $C(w,y) \cup C(y,z) = \varnothing$, and if $y \in N(a)$, then ay is not subdivided. Therefore, x is apex in G, a contradiction. Second, if $u \in C(x,c_1)$, then since $G \not\geq_m Q_3$, it follows that $N(a) \cap C(z,u) = \varnothing$. Also, since $G \not\geq_m Q_2$, it follows that $N(a) \cap C(u,w) = \varnothing$, and $C(w,y) \cup C(y,z) = \varnothing$, and if $y \in N(a)$, then ay is not subdivided. Therefore, u is apex in G, a contradiction. Therefore we must have $u = c_1$. Again, since $G \not\geq_m Q_3$, it follows that $N(a) \cap C(z,u) = \varnothing$. And, since $G \not\geq_m Q_2$, it follows that $C(y,z) = \varnothing$, and if $y \in N(a)$, then ay is not subdivided. Therefore, u is apex in G, a contradiction.

Subcase 2.3. All the chords of G lie within $C[x,y]$ and they all have an endpoint at y.

Let c_1y be an innermost chord within $C[x,y]$. By **12**, a has a neighbor $w \in C(c_1,y)$, and edge aw is not subdivided.

Note that a has a neighbor $u \neq y$ such that au is subdivided, for otherwise y is apex in G, a contradiction. Then, by **12**, $u \notin C(c_1,y)$. Let $z \in C(y,x)$ be the neighbor of a closest to y, in the sense that yz is an edge of G or an edge subdivided once. Then $u \in C(z,c_1]$, for otherwise y is apex in G. First, if $u \in C(z,x]$, then $N(a) \cap C(u,x) = \varnothing$, for otherwise $G - \{x,y\}$ contains a $K_{2,3}$-subdivision. Also, since $G \not\geq_m Q_2$, it follows that $N(a) \cap C(x,w) = \varnothing$, $C(w,y) = \varnothing$, and if $y \in N(a)$, then ay is not subdivided. Therefore, x is apex in G, a contradiction. Second, if $u \in C(x,c_1)$, then, by **12**, there are no chords with an endpoint in $C(x,u)$. Also, since $G \not\geq_m Q_3$, it follows that $N(a) \cap C(z,u) = \varnothing$, and since $G \not\geq_m Q_5$, we have that $C(y,z) = \varnothing$. Also, since $G \not\geq_m Q_2$, it follows that $N(a) \cap C(u,w) = \varnothing$, and $C(w,y) = \varnothing$, and if $y \in N(a)$, then ay is not subdivided. Therefore, u is apex in G, a contradiction. Therefore, we must have $u = c_1$. Hence, by **12**, c_1y is the only chord in G. Again, since $G \not\geq_m Q_3$, it follows that $N(a) \cap C(z,u) = \varnothing$. Hence, zx and xc_1 $(= xu)$ are either edges of G or edges subdivided once. Also, since $G \not\geq_m Q_2$, it follows that if $y \in N(a)$, then ay is not subdivided. Hence, $C(y,z) \neq \varnothing$, for otherwise u is apex in G. Finally, since $G \not\geq_m J_1$, it follows that $C(u,w) = \varnothing$ and $N(a) \cap C(w,y] = \varnothing$, and hence z is apex in G, a contradiction.

This concludes the proof of Case 2 in (2), and the entire proof of (2) of Proposition 5.1. □

6. Connectivity 3

In this section, we focus on the case that $G \in \mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$ has connectivity three (recall from Lemma 2.2 that $G \in \mathbf{ob}(\mathcal{O}^*) - \mathcal{S}$ is not 4-connected, and thus K_5 and Oct are the only 4-connected members of $\mathbf{ob}(\mathcal{O}^*)$). Here, we rely on the existence of contractible edges in 3-connected graphs (Lemma 6.2) and the minor-minimality of G to prove the following proposition, which says that such a G does not exist.

Proposition 6.1. *There are no* 3*-connected graphs in* $\mathbf{ob}(\mathcal{O}^*) - \{K_5, K_{3,3}, Oct, Q\}$*. In other words, the only graphs of connectivity* 3 *in* $\mathbf{ob}(\mathcal{O}^*)$ *are* $K_{3,3}$ *and* Q*.*

Lemma 6.2 (see [3])**.** *If* G *is* 3*-connected and* $|V(G)| \geqslant 5$*, then* G *has an edge* e *such that* G/e *is also* 3*-connected.*

Such an edge is called *contractible*. We denote by v_{xy} the new vertex obtained by contracting edge xy in a graph.

The proof of Proposition 6.1 follows from Lemma 6.2 and two lemmas which are stated and proved below.

Lemma 6.3. *There is no* 3*-connected graph* G *in* $\mathbf{ob}(\mathcal{O}^*) - \{K_5, K_{3,3}, Oct, Q\}$ *that has a contractible edge* xy *such that* v_{xy} *is not apex vertex in* G/xy*.*

Proof. Suppose otherwise that there exists a 3-connected graph G in $\mathbf{ob}(\mathcal{O}^*) - \{K_5, K_{3,3}, Oct, Q\}$ that has a contractible edge xy such that v_{xy} is not apex vertex in G/xy, and hence there is an apex vertex $a \neq v_{xy}$ in G/xy. Then, $(G/xy) - a \in \mathcal{O}$ is 2-connected. Since G is 3-connected (and simple and planar), it has a unique planar embedding by the well-known theorem of Whitney from 1933 (see [2]). Since $(G/xy) - a \in \mathcal{O}$ is 2-connected, it follows that restricting this embedding to $(G/xy) - a$, we have that all the vertices of $(G/xy) - a$ lie on a cycle C' and are incident with the outer face. This is so because, by Whitney's theorem, it follows that every simple 2-connected outerplanar graph has a unique outerplanar embedding. Since $G - a \notin \mathcal{O}$, it follows that x or y, say x, is embedded in the interior of the disk bounded by C, where $C \subseteq G$ is the cycle isomorphic to C', and the corresponding isomorphism $\phi : V(C') \to V(C)$ is the identity map on $V(C') - v_{xy}$ and $\phi(v_{xy}) = y$.

Let $u_1, u_2, \ldots, u_n \in V(C)$ $(n \geqslant 3)$ be the neighbors of x in the clockwise order around C. For $i = 1, \ldots, n$, let $S_i := C[u_i, u_{i+1}]$, where S_n is understood to be $C[u_n, u_1]$. We call the S_i's the *segments* of C. We call u_i's the *endpoint* vertices of the segments and the vertices in $C(u_i, u_{i+1})$ for $i = 1, \ldots, n$, the *interior* vertices of the segments. Two segments of S_i and S_j are said to be *consecutive* if $|i - j| = 1$, or $\{i, j\} = \{1, n\}$. We observe the following facts.

1. The edges of G are:

- edges of C;
- edges xu_i for $i = 1, \ldots, n$;
- *chords* of C, that is, edges not in $E(C)$ with both endpoints in a single segment of C (note that such edges are embedded in the interior of the disk bounded by C);
- edges with one endpoint in C and the other at a.

It follows by the above that:

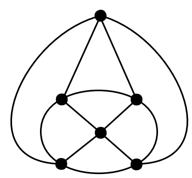
2. Interior vertices of the segments are either endpoints of chords or neighbors of a.

3. For every chord c_1c_2 in G with $c_1 < c_2$ (in the clockwise order of C restricted to the segment containing c_1c_2), a has a neighbor (in the usual sense, as opposed to the one from Section 5) in $C(c_1, c_2)$ (by 3-connectedness of G).

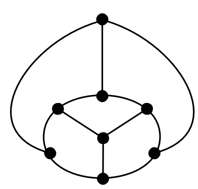
Let $N(a) := N_G(a)$. We now prove:

4. *$N(a)$ is covered by exactly two consecutive segments of C.*

Pf. First, we show that $N(a)$ is covered by exactly two segments of C. If there are four internally disjoint paths from a to x, then the subgraph of G formed from the union of those paths and C contains an Oct-minor, a contradiction.

$Oct =$ 

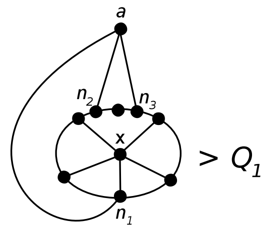
Therefore, by Menger's theorem and the fact that G is 3-connected, it follows that G has a 3-cut separating a and x. By **1** above, it follows that this 3-cut is a subset of $V(C)$, and therefore at least one of a or x has degree 3. Let $u \in \{a, x\}$ be such that $deg_G(u) = 3$, and let $v \in \{a, x\} - \{u\}$. The three neighbors of u divide C into three segments. If all three segments contain interior vertices that are in $N(v)$, then G contains a Q-minor, a contradiction

$Q =$ 

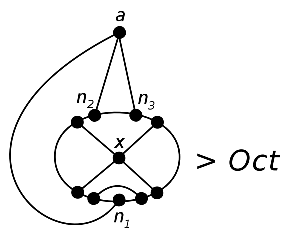
Hence, one segment does not contain any interior vertices that are in $N(v)$. Then, if $u = x$ then we are done. And similarly, if $u = a$ then we are done. Hence, we have shown that $N(a)$ is covered by exactly two segments of C.

Furthermore, the two segments that cover $N(a)$ are consecutive. Suppose not, and let S_i and S_j be the two segments that cover $N(a)$ with $|i - j| > 1$. If both of them contain at least two neighbors of a, then two of those neighbors in each segment can be contracted to four distinct endpoint vertices and thus $G \geqslant_m Oct$, a contradiction. Hence, one of them, say S_i, contains only one neighbor of a, call it n_1. Since $deg(a) \geqslant 3$, S_j must contain at least two neighbors of a: let n_2 be the closest one to u_j, and n_3 be the closest one to u_{j+1}.

Suppose n_1 is an endpoint vertex, so that $n_1 = u_i$ or u_{i+1}. Note that in this case $deg(x) \geqslant 5$, for otherwise two consecutive segments cover $N(a)$. Then, since $G \notin \mathcal{O}^*$, it follows that $C(n_2, n_3) \neq \varnothing$ (for otherwise n_1 is an apex vertex). But then, $G \geqslant_m Q_1$, a contradiction (by deleting edge n_1x and contracting n_2 to u_j, and n_3 to u_{j+1}).



Therefore, n_1 must be an interior vertex, so $n_1 \in C(u_i, u_{i+1})$. Again, since $G \notin \mathcal{O}^*$, there is a vertex in $C(n_2, n_3)$, or there is a chord with one endpoint in $C[u_i, n_1)$ and the other in $C(n_1, u_{i+1}]$ (for otherwise n_1 is an apex vertex). In the first case, $G \geqslant_m Q_1$ (just like above) while in the second, $G \geqslant_m Oct$ (by contracting edge $n_1 a$), a contradiction. This proves **4**.



We now show that C actually has exactly three segments.

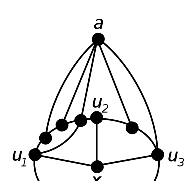
5. *C has exactly three segments, or equivalently $deg(x) = 3$, or equivalently $n = 3$.*

Pf. By **4**, we may assume that $N(a)$ is covered by S_1 and S_2. Since interior vertices are either endpoints of chords or neighbors of a, it follows by **2** and **3** that $C(u_i, u_{i+1}) = \varnothing$ for $i = 3, 4, \ldots, n$ (where $u_{n+1} = u_1$).

Suppose that $n \geqslant 4$. By **4**, it follows that a has neighbors in $C[u_1, u_2)$ and in $C(u_2, u_3]$. Therefore, in the graph $G \backslash xu_4$, none of the vertices a, u_2, x, u_4 can be apex (since the deletion of any one of them still leaves a $K_{2,3}$-subdivision as a subgraph). Let s be an apex vertex in $G \backslash xu_4$. Then $s \in V(C)$. Therefore, the unique embedding of G restricted to the graph $(G \backslash xu_4) - s \in \mathcal{O}$ is an embedding in which all the vertices (including x) are incident with the outer face. By adding edge xu_4 to this embedding, we obtain an embedding of $G - s$ in which all the vertices are incident with the outer face, a contradiction.

We have shown that for $i = 3, 4, \ldots, n$ $xu_i \notin E(G)$ which, by 3-connectivity of G, implies that C has exactly three segments and proves **5**.

By **5**, G has the following general structure:



Therefore, let S_1 and S_2 cover $N(a)$. It follows by **2**, and **3** that $C(u_3, u_1) = \varnothing$ (that is $u_3 u_1 \in E(G)$). Also, similarly to **4** from the proof of Proposition 5.1, since $G \not\geqslant_m Q_1$, we have:

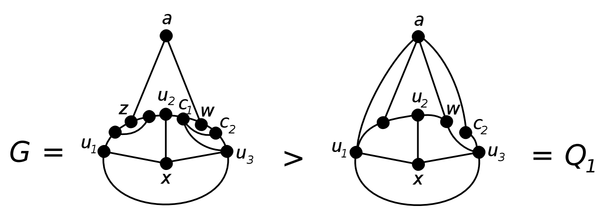
6. Within a single segment S_1 or S_2, there are no non-overlapping chords (or equivalently, all the chords are nested).

We say that segment S_1 (respectively S_2) is of *type-one*, if $\{z\} := N(a) \cap C[u_1, u_2)$ with $z \neq u_1$, and $C(z, u_2) \neq \varnothing$ (respectively, $\{w\} := N(a) \cap C(u_2, u_3]$ with $w \neq u_3$, and $C(u_2, w) \neq \varnothing$). And we say that S_1 (respectively S_2) is of *type-two*, if $|N(a) \cap C[u_1, u_2)| \geqslant 2$ (respectively $|N(a) \cap C(u_2, u_3]| \geqslant 2$). Note that if S_1 (respectively S_2) is not of type-one nor type-two, then $\{z\} := N(a) \cap C[u_1, u_2)$ and $zu_2 \in E(C)$ (respectively $\{w\} := N(a) \cap C(u_2, u_3]$ and $u_2w \in E(C)$). Finally, note that at least one of S_1 or S_2 is of type-one or type-two, for otherwise u_2 is apex in G.

There are two cases to consider.

Case 1. Each of S_1 and S_2 is of type-one or type-two.

Suppose that one of the segments, say S_2 is of type-one. Then, $\{w\} := N(a) \cap C(u_2, u_3]$ with $w \neq u_3$, and $C(u_2, w) \neq \varnothing$. Hence, it follows by **2**, that there is a chord with one endpoint $c_1 \in C(u_2, w)$, and the other $c_2 \in C(w, u_3]$. Choose c_1 and c_2 so that the chord c_1c_2 is innermost. Then by **6**, all other chords in S_2 have one endpoint in $C[u_2, c_1]$ and the other in $C[c_2, u_3]$. However, since S_1 is of type-one or type-two, we either have $\{z\} := N(a) \cap C[u_1, u_2)$ with $z \neq u_1$, and $C(z, u_2) \neq \varnothing$ (which by **2** implies that there is a chord with one endpoint in $C[u_1, z)$ and the other in $C(z, u_2)$), or $|N(a) \cap C[u_1, u_2)| \geqslant 2$. This implies that the only other chords in S_2 that do not have an endpoint at u_2 (that is, those that do have an endpoint in $C(u_2, c_1]$) have an endpoint at c_2, for otherwise $G \geqslant_m Q_1$ (by contracting wa and za if necessary, see figure below).



Therefore, u_2 is apex in G, a contradiction.

Similarly, suppose that for one of the segments, say S_1, is of type-two. Then, $|N(a) \cap C[u_1, u_2)| \geqslant 2$. If there are chords with endpoints distinct from u_2 in S_1, then let d_1d_2, with $d_1 < d_2$ in the cyclic order of C, be an innermost chord of S_1 with $d_2 \neq u_2$, and let $z \in N(a) \cap C(d_1, d_2)$. Then again, since S_2 is of type-one or type-two, we either have $\{w\} := N(a) \cap C(u_2, u_3]$ with $w \neq u_3$, and $C(u_2, w) \neq \varnothing$ (which by **2** implies that there is a chord with one endpoint in $C(u_2, w)$ and the other in $C(w, u_3]$), or $|N(a) \cap C(u_2, u_3]| \geqslant 2$. This implies that the only other chords in S_1 that do not have an endpoint at u_2 (that is, those that do have an endpoint in $C[d_2, u_2)$) have an endpoint at d_1, for otherwise $G \geqslant_m Q_1$ as above. Furthermore, $N(a) \cap (C(d_1, z) \cup C(z, u_2)) = \varnothing$, for otherwise $G \geqslant_m Q_1$ as above. Therefore again, u_2 is apex in G, a contradiction.

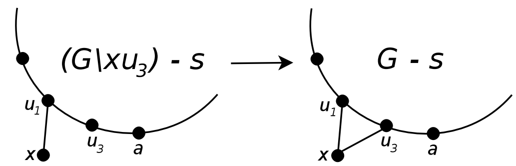
Case 2. Exactly one of the segments S_1 or S_2 is of type-one or type-two.

By symmetry, suppose that S_2 is not of type-one nor type-two, and that S_1 is. Then, $\{w\} := N(a) \cap C(u_2, u_3]$ and $u_2w \in E(C)$. We divide this case into two subcases depending on whether u_1u_2 is an edge of G.

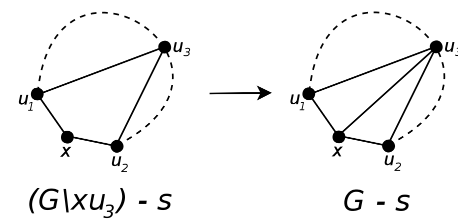
Subcase 2.1. $u_1u_2 \notin E(G)$.

Let s be an apex vertex in $G \backslash xu_3$, and we assume that the graph $(G \backslash xu_3) - s \in \mathcal{O}$ is embedded in the plane with all of its vertices incident with the outer face. Clearly, $s \neq x$ and $s \neq u_3$, for otherwise x or u_3 is apex in G, a contradiction. Also, $s \neq a$, since $(G \backslash xu_3) - a$ contains a $K_{2,3}$-subdivision (because $C(u_1, u_2) \neq \varnothing$, since S_1 is of type-one or type-two).

First, suppose that $w = u_3$. Then $u_2u_3 \in E(C)$ (that is, $C(u_2, u_3) = \varnothing$). If $s = u_2$ (or by symmetry, if $s = u_1$), then in $(G\backslash xu_3) - s$, $deg(u_3) = 2$ and $deg(x) = 1$ hence edges u_3u_1, u_3a, and xu_1 are also incident with the outer face. Since u_3u_1 is a simple edge, by putting the edge xu_3 back in, we can embed $G - s$ so that all the vertices are still incident with the outer face, hence $G - s$ is outerplanar, a contradiction (see figure below).



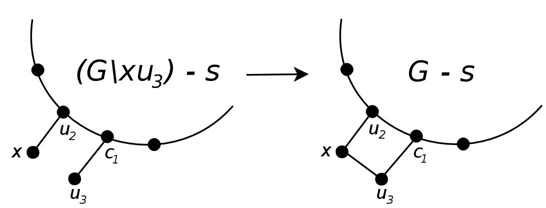
Therefore, $s \notin \{u_1, u_2, u_3, x, a\}$, so that $s \in C(u_1, u_2)$. Then, in $(G\backslash xu_3) - s$, $deg(x) = 2$, and so $(G\backslash xu_3) - s$ has an outerplanar embedding such that edges xu_1 and xu_2 are incident with the outer face. Also, note that x, u_1, u_3, u_2 is a 4-cycle in $(G\backslash xu_3) - s$. Therefore, since $u_1u_2 \notin E(G)$, we can put the edge xu_3 back in to obtain an embedding of $G - s$ in which all the vertices are still incident with the outer face, hence $G - s$ is outerplanar, a contradiction (see figure below).



Therefore, $w \neq u_3$ and so $w \in C(u_2, u_3)$. Since $u_2w \in E(C)$, the only possible chords in S_2 have one endpoint at u_2 and the other in $C(w, u_3]$. Note that by Case 2 hypothesis, $u_3a \notin E(G)$.

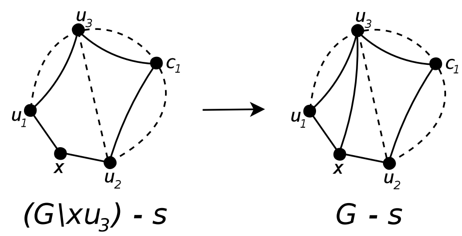
If $s = u_2$, then in $(G\backslash xu_3) - s$, $deg(u_3) = 2$ and $deg(x) = 1$, hence edge u_3u_1 and the other edge incident with u_3, as well as the pendant edge xu_1 are all incident with the outer face. Since u_3u_1 is a simple edge, by putting the edge xu_3 back in, we obtain an embedding of $G - s$ in which all the vertices are still incident with the outer face, hence $G - s$ is outerplanar, a contradiction.

Now suppose $s = u_1$. If u_2u_3 is a chord of C, then in $(G\backslash xu_3) - s$, $deg(u_3) = 2$ and $deg(x) = 1$, hence edges u_3u_2, u_3w, and xu_2 are incident with the outer face. Since u_3u_2 is a simple edge, by putting the edge xu_3 back in, we can embed $G - s$ so that all the vertices are still incident with the outer face, hence $G - s$ is outerplanar, a contradiction. Hence u_2u_3 is not a chord of C. If G has a chord $c = u_2c_1$ with $c_1 \in C(w, u_3)$, then choose c_1 closest to u_3, so that $c_1u_3 \in E(C)$. And if there is no such chord, then let $c_1 := w$. Then, in $(G\backslash xu_3) - s$, $deg(x) = 1$, and $deg(c_1) = 3$, but c_1 is adjacent to u_3 with $deg(u_3) = 1$, hence edge u_2c_1 and the pendant edges xu_2 and c_1u_3 are all incident with the outer face. Since u_2c_1 is a simple edge (even if $c_1 = w$), by putting the edge xu_3 back in, we can embed $G - s$ so that all the vertices are still incident with the outer face, hence $G - s$ is outerplanar, a contradiction (see figure below).



Similarly, if $s = w$, then G has no chords with one endpoint at u_2 and the other in $C(w, u_3]$, for otherwise $(G\backslash xu_3) - s$ contains a $K_{2,3}$-subdivision (because $C(u_1, u_2) \neq \varnothing$, since S_1 is of type-one or type-two). Hence, $C(w, u_3) = \varnothing$. Therefore, in $(G\backslash xu_3) - s$, $deg(x) = 2$ and $deg(u_3) = 1$, hence edges xu_1, and u_1u_3 are incident with the outer face. Since xu_1 is a simple edge, by putting the edge xu_3 back in, we can embed $G - s$ so that all the vertices are still incident with the outer face, hence $G - s$ is outerplanar, a contradiction.

Therefore, $s \notin \{u_1, u_2, u_3, x, a, w\}$, and so $s \in C(u_1, u_2)$ (by **1**). Again, if G has a chord $c = u_2c_1$ with $c_1 \in C(w, u_3)$, then choose c_1 closest to u_3, so that $c_1u_3 \in E(G)$. And if there is no such chord, then let $c_1 := w$. Then, in $(G\backslash xu_3) - s$, $deg(x) = 2$, hence edges xu_2 and xu_1 are incident with the outer face. Also, note that x, u_1, u_3, c_1, u_2 is a 5-cycle in $(G\backslash xu_3) - s$. Therefore, since $u_1u_2 \notin E(G)$ (the Subcase 2.1 hypothesis) and $u_1c_1 \notin E(G)$ (by **1**), we can put the edge xu_3 back in (even if $u_2u_3 \in E(G)$) to obtain an embedding of $G - s$ in which all the vertices are still incident with the outer face, hence $G - s$ is outerplanar, a contradiction (see figure below).



Subcase 2.2. $u_1u_2 \in E(G)$.

If all chords within S_1 have an endpoint at u_1 or all have an endpoint at u_2, then u_1, or u_2 respectively, is apex in G, a contradiction. Hence, there is a chord with both endpoints in $C(u_1, u_2)$. Let $c_1c_2 \in E(G)$ be the innermost chord with $c_1, c_2 \in C(u_1, u_2)$ (in the sense that there are no other chords with both endpoints in $C[c_1, c_2]$), and let $a_1 \in N(a) \cap C(c_1, c_2)$.

Suppose that $a_2 \neq a_1$ is another neighbor of a in $C(c_1, c_2)$. Then by choice of c_1c_2, we have that $deg(a_1) = 3 = deg(a_2)$. Note that a has no other neighbors in $C(u_1, u_2)$, for otherwise G contains two disjoint K-graphs, a contradiction. Let s be an apex vertex in $G\backslash a_1a_2$, and we assume that the graph $(G\backslash a_1a_2) - s \in \mathcal{O}$ is embedded in the plane with all of its vertices incident with the outer face. It is easy to see that $s = w$ (regardless of whether $w = u_3$), for otherwise: if $s \in \{a\} \cup C(u_1, u_2)$, then $(G\backslash a_1a_2) - s$ contains a K_4-subdivision; and if $u \in \{u_1, u_2\} \cup C(w, u_3]$, then $(G\backslash a_1a_2) - s$ contains a $K_{2,3}$-subdivision. Therefore $s = w$, and hence the only neighbors of a are a_1, a_2 and w (because if u_1 or u_2 is a neighbor of a then $(G\backslash a_1a_2) - s$ contains a $K_{2,3}$-subdivision). Then, in $(G\backslash a_1a_2) - s$, $deg(a) = 2$, hence edges aa_1 and aa_2 are incident with the outer face, and by putting the edge a_1a_2 back in, we obtain an embedding of $G - s$ in which all the vertices are still incident with the outer face, hence $G - s$ is outerplanar, a contradiction.

Hence, we have shown that a_1 is the only neighbor of a in $C(c_1, c_2)$.

We now show furthermore that $N(a) \cap (C(u_1, c_1] \cup C[c_2, u_2)) = \varnothing$. For suppose otherwise, and let $c_3 \in C(u_1, c_1]$ (the argument for $c_3 \in C[c_2, u_2)$ is similar). Then $N(a) \cap C[c_2, u_2) = \varnothing$ (for otherwise $G \geqslant_m 2K_4$). Let s be an apex vertex in $G\backslash c_1a_1$. Then clearly $s \in \{u_2, w\}$. If $s = w$, then since w is apex in $G\backslash c_1a_1$ we have that: $c_3 = c_1$; $N(a) \cap (C[u_1, c_1) \cup \{u_2\}) = \varnothing$; and G does not have any chords with one endpoint at u_2 and the other in $C(w, u]$ (in the case that $w \neq u_3$).

Therefore w is apex in G, a contradiction. If $s = u_2$, then since u_2 is apex in $G\backslash c_1a_1$, it follows that G has no chords with one endpoint in $C[u_1, c_1)$ and the other in $C[c_2, u_2)$. Hence all chords of G have one endpoint at c_1 or at u_2. Therefore u_2 is apex in G, a contradiction.

Hence, we have shown that $N(a) \cap (C(u_1, a_1) \cup C(a_1, u_2)) = \varnothing$. Thus the only possible neighbors of a (other than a_1 and w) are u_1 and u_2. In fact, at least one of them is a neighbor of a since $deg_G(a) \geqslant 3$. Let s be an apex vertex in G/aa_1. Then clearly $s \in \{u_1, u_2\}$. Suppose that $s = u_2$ (the argument for $s = u_1$ is similar). Since u_2 is apex in G/aa_1, it follows that G has no chords with one endpoint in $C[u_1, c_1)$ and the other in $C[c_2, u_2)$. Hence all chords of G have one endpoint at c_1 or at u_2. Therefore u_2 is apex in G, a contradiction. This concludes the proof of Lemma 6.3. □

Lemma 6.4. *There is no 3-connected graph G in $\mathbf{ob}(\mathcal{O}^*) - \{K_5, K_{3,3}, Oct, Q\}$ with the property that for every contractible edge xy in G the vertex v_{xy} is apex in G/xy.*

Proof. Suppose otherwise that there exists a 3-connected graph G in $\mathbf{ob}(\mathcal{O}^*) - \{K_5, K_{3,3}, Oct, Q\}$ with the property that for any contractible edge xy in G the vertex v_{xy} is an apex vertex in G/xy. The following claim provides a way of testing whether an edge in a 3-connected graph is contractible.
1. Let G be a 3-connected graph with edge xy. Then, G/xy is 3-connected if and only if $G - \{x, y\}$ $(= (G/xy) - v_{xy})$ is 2-connected.
Pf. If G/xy is 3-connected, then clearly $(G/xy) - v_{xy}$ is 2-connected. Now, suppose that $G - \{x, y\}$ $(= (G/xy) - v_{xy})$ is 2-connected and that G/xy is not 3-connected, so that G/xy has a 2-cut. Since G is 3-connected, it follows that v_{xy} is one of the vertices in that 2-cut (for otherwise, this 2-cut would also be a 2-cut in G). Therefore, $(G/xy) - v_{xy}$ has a cut-vertex, a contradiction which proves **1**.

Let xy be a contractible edge in G. Then, by **1**, $(G/xy) - v_{xy} \in \mathcal{O}$ is 2-connected. Since G is 3-connected it has a unique planar embedding. Restricting this embedding to $(G/xy) - v_{xy}$, we have that all the vertices of $(G/xy) - v_{xy}$ lie on a cycle C and are incident with the outer face.

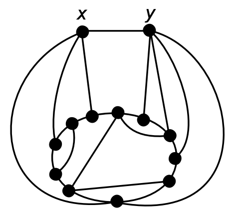
Let $x_1, x_2, \ldots, x_m \in V(C)$ $(m \geqslant 2)$ be the neighbors of x in the clockwise order around C. And let $y_1, y_2, \ldots, y_n \in V(C)$ $(n \geqslant 2)$ be the neighbors of y in the clockwise order around C. Note that $x_i \notin C(y_1, y_n)$ for all i and $y_j \notin C(x_1, x_m)$ for all j, for otherwise G would contain a $K_{3,3}$-minor. Also, note that possibly $x_m = y_1$ or $y_n = x_1$.
2. The edges of G are:
- edges of C;
- edges xx_i for $i = 1, \ldots, m$, and yy_j for $j = 1, \ldots, n$;
- *chords* of C, that is, edges not in $E(C)$ with both endpoints in C (note that such edges are embedded in the interior of the disk bounded by C);
- edge xy.

Just as in the proof of Lemma 6.3, it follows from **2** that:
3. The vertices of C are either endpoints of chords or neighbors of x or y.
4. For every chord c_1c_2 in G with $c_1 < c_2$ (in the clockwise order restricted to the segment containing c_1c_2), there is a neighbor of x or y in $C(c_1, c_2)$.

Also, since neither y nor x is apex in G, it follows, respectively, that:

5. $C(x_1, x_m) \neq \varnothing$ and $C(y_1, y_n) \neq \varnothing$.

Hence G has the following general structure:



Before we proceed, we prove a claim regarding the structure of G.

6. G does not have a chord with both endpoints in $C[y_n, y_1]$. And by symmetry, the same statement holds for $C[x_m, x_1]$.

Pf. Let c_1c_2 be a chord of G with both endpoints in $C[y_n, y_1]$. Without loss of generality, we may assume that c_1c_2 is the innermost such chord, in the sense that there are no other chords with both endpoints in $C[c_1, c_2]$. By **4**, it follows that x has a neighbor s in $C(c_1, c_2)$. Note that x does not have another such neighbor t in $C(c_1, c_2)$, for otherwise edge st is contractible (because $G - \{s, t\}$ is 2-connected), but $(G/st) - v_{st} \notin \mathcal{O}$ (because it contains a $K_{2,3}$-subdivision, since $C(y_1, y_n) \neq \varnothing$), a contradiction because v_{st} by the assumption of the proof is an apex vertex in G/st. So, the only vertex in $C(c_1, c_2)$ is s. But then, edge xs is contractible (because $G - \{x, s\}$ is 2-connected), and $(G/xs) - v_{xs} \notin \mathcal{O}$ (because it contains a $K_{2,3}$-subdivision, since $C(y_1, y_n) \neq \varnothing$), a contradiction. This proves **6**.

By **6**, we have:

7. The only chords in G have one endpoint in $C(x_1, x_m)$ and the other in $C(y_1, y_n)$.

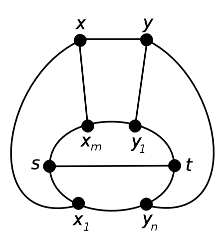
The following claim further tightens up the structure of G.

8. There is exactly one vertex in $C(x_1, x_m)$ and exactly one in $C(y_1, y_n)$.

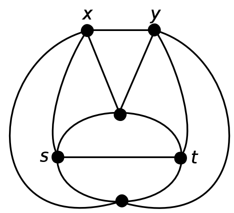
Pf. Suppose that $C(x_1, x_m)$ has two vertices s and t. Then, by **3** it follows that both s and t are neighbors of x, or endpoints of chords whose other endpoints lie in $C(y_1, y_n)$ by **7**, or both. Note that st is contractible (by **1**, because $G - \{s, t\}$ is 2-connected), and $(G/st) - v_{st} \notin \mathcal{O}$ (because it contains a K_4-subdivision, consisting of the cycle formed by edge xx_m, the clockwise path along C from x_m to x_1, and edge x_1x; and the three spokes from y to this cycle), violating the hypothesis of Lemma 6.4. This proves **8**.

With the structure of G restricted by **6** and **8**, we are ready to finish the proof of the lemma. Let s and t be the unique vertices in $C(x_1, x_m)$ and $C(y_1, y_n)$, respectively. Note that $st \in E(G)$, for otherwise any one of x_1, x_m, y_1, y_n is apex in G, a contradiction. Also, it follows by **2** and **7** that $C(x_m, y_1) = \varnothing$ and $C(y_n, x_1) = \varnothing$.

If $x_m \neq y_1$ and $y_n \neq x_1$, then $G \geqslant_m Q$, a contradiction (see figure below).

$Q =$ 

Hence, by symmetry, we have either the case that $x_m \neq y_1$ and $y_n = x_1$, or that $x_m = y_1$ and $y_n = x_1$. In either case, we cannot have that both $sx, ty \in E(G)$, for otherwise $G \geqslant_m Oct$ (see figure below).

$Oct =$ 

Hence, by symmetry, $sx \notin E(G)$, and it follows that x_m is apex, a contradiction. This concludes the proof of the lemma. □

References

[1] I. Adler. Open problems related to computing obstruction sets. *Pre-print*, see www2.informatik.hu-berlin.de/~adler/publications/obstructions-open-problems.pdf.

[2] J.A. Bondy, U.S.R. Murty. *Graph Theory*, Springer, 2008.

[3] R. Diestel. *Graph Theory*, 2nd Edition, Springer-Verlag, New York, 2000.

[4] S. Dziobiak. Excluded-minor characterization of apex-outerplanar graphs, The 23rd Cumberland Conference on Combinatorics, Graph Theory, and Computing, University of Mississippi, Oxford, MS, May 20-22, 2010.

[5] R. Halin, H.A. Jung. Über minimalstrukturen von graphen, insbesondere von n-fach zusammenhängenden graphen, *Math. Ann.* **152** (1963), 75-94.

[6] A. Kezdy. Personal communication.

[7] N. Robertson, P. Seymour. Graph minors. XX. Wagner's conjecture. *J. Comb. Theory Ser. B* **92** (2004), 325-357.

[8] N. Robertson, P. Seymour, R. Thomas. Hadwiger's conjecture for K_6-free graphs. *Combinatorica* **13** (3) (1993) 279-361.

[9] G. E. Turner. Personal communication.

Department of Mathematics, Louisiana State University, Baton Rouge, Louisiana
E-mail address: ding@math.lsu.edu

Department of Mathematics, University of Mississippi, Oxford, Mississippi
E-mail address: smdziobi@olemiss.edu