3-CONNECTED GRAPHS OF PATH-WIDTH AT MOST THREE

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Abstract. It is known that the list of excluded minors for the minor-closed class of graphs of path-width $\leqslant 3$ numbers in the millions. However, if we restrict the class to 3-connected graphs of path-width $\leqslant 3$, then we can characterize it by five excluded minors.

1. Introduction

The concepts of tree-width and path-width were introduced by Robertson and Seymour in [6] and [7]. Let G be a graph, T a tree, and let $\mathcal{V} = \{V_t\}_{t \in V(T)}$ be a family of vertex sets $V_t \subseteq V(G)$. The pair $(T, \mathcal{V})$ is called a *tree-decomposition* of G if it satisfies the following two conditions:

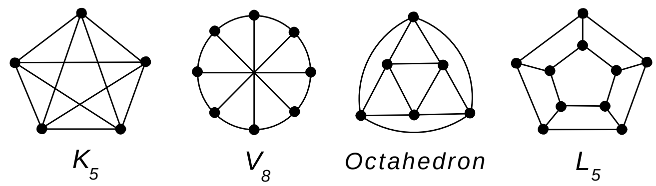
(T1) $V(G) = \bigcup_{t \in V(T)} V_t$, and every edge of G has both ends in some V_t;
(T2) for every $v \in V(G)$, the subgraph induced by those t for which $v \in V_t$ is connected.

The elements of $\mathcal{V}$ are called *bags*. The *width* of a tree-decomposition $(T, \mathcal{V})$ is $\max_{t \in V(T)}\{|V_t| - 1\}$. The *tree-width* of G, denoted by $\mathbf{tw}(G)$, is the minimum width over all possible tree-decompositions of G. Similarly, if the underlying structure is a path P, that is if $T = P$, then the pair $(P, \mathcal{V})$ is called a *path-decomposition* of G if again it satisfies (T1) and (T2). And, analogously, the *width* of a path-decomposition $(P, \mathcal{V})$ is $\max_{t \in V(P)}\{|V_t| - 1\}$, and the *path-width* of G, denoted by $\mathbf{pw}(G)$, is the minimum width over all possible path-decompositions of G. Since a path-decomposition of G is also a tree-decomposition of G, it follows from the definitions that $\mathbf{tw}(G) \leqslant \mathbf{pw}(G)$ for every graph G.

Given graphs H and G, H is a *minor* of G, denoted by $H \preceq G$, or $G \succeq H$, if H can be obtained from a subgraph of G by contracting edges. If H is not a minor of G, we say that G is *H-free*, and denote it by $H \npreceq G$, or $G \nsucceq H$. A class $\mathcal{C}$ of graphs is *minor-closed* if for every $G \in \mathcal{C}$ all the minors of G are also in $\mathcal{C}$. Some examples of minor-closed classes are: planar graphs, outerplanar graphs, series-parallel graphs, and graphs embeddable in a fixed surface. Also, it is easy to check that, for a fixed positive integer k, the following classes of graphs are minor-closed: $\mathcal{T}_k := \{G : \mathbf{tw}(G) \leqslant k\}$, $\mathcal{P}_k := \{G : \mathbf{pw}(G) \leqslant k\}$. Equivalently, tree-width and path-width are monotone under taking minors, namely if $H \preceq G$, then $\mathbf{tw}(H) \leqslant \mathbf{tw}(G)$ and $\mathbf{pw}(H) \leqslant \mathbf{pw}(G)$. Finally, since having loops or parallel edges has no impact on the tree-width or path-width of a graph, all graphs in this paper are considered to be simple.

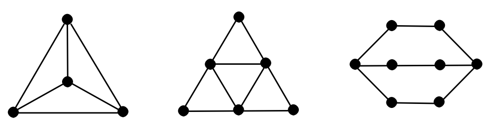
Note that graphs of tree-width $= 1$ are exactly forests (or equivalently, K_3-free graphs), and graphs of tree-width $\leqslant 2$ are exactly series-parallel graphs (or equivalently, K_4-free graphs). The following theorem due to Arnborg et. al. [1], and independently to Satyanarayana et. al. [10], characterizes the class $\mathcal{T}_3$ in terms of its excluded minors.

Theorem 1.1. [1], [10] *For a graph G, $\mathbf{tw}(G) \leqslant 3$ if and only if G does contain any of the following graphs as a minor: K_5, V_8, Oct, L_5.*



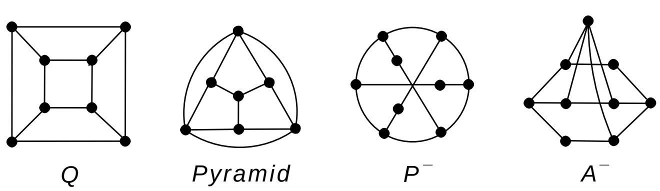
Similarly, graphs of path-width $= 1$ are exactly disjoint unions of paths (or equivalently, $\{K_3, K_{1,3}\}$-free graphs). And in [5], Kinnersley and Langston provide a complete list of 110 excluded minors for $\mathcal{P}_2$. By restricting this class to only 2-connected graphs, Barát et. al. [2] obtained the following theorem.

Theorem 1.2. [2] *For a 2-connected graph G, $\mathbf{pw}(G) \leqslant 2$ if and only if G does contain any of the following graphs as a minor.*



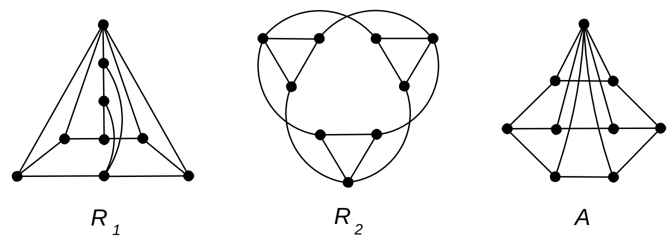
The class of graphs of path-width at most three is known to have at least 122 million excluded minors [5], and the complete list is not known. However, we prove that if we restrict the class to 3-connected graphs of path-width $\leqslant 3$ (as asked by the authors of [2]), then we can characterize it by five excluded minors and two exceptions. The following is the main result of this paper.

Theorem 1.3. *For a 3-connected graph G, $\mathbf{pw}(G) \leqslant 3$ if and only if $G \notin \{V_8, Q\}$ and G does contain any of the following graphs as a minor: K_5, Oct, Pyr, P^-, A^-.*



The graph P^- is obtained from the Petersen graph P by deleting any one vertex, hence its label. The graph A^- is obtained from the graph A (see next figure) by deleting two edges. The graph A, in turn, is obtained from the third graph in Theorem 1.2 by joining all of its degree-two vertices to a newly added vertex. Note that P^- and A^- are not 3-connected. Alternatively, if we would like all of the excluded minors for our class to be 3-connected, then we can characterize it by six excluded minors and two exceptions. The graphs R_1 and R_2 in the following Corollary each contain P^- and A^- as subgraphs.

Corollary 1.4. *For a 3-connected graph G, $\mathbf{pw}(G) \leqslant 3$ if and only if $G \notin \{V_8, Q\}$ and G does contain any of the following graphs as a minor: $K_5, Oct, Pyr, R_1, R_2, A$.*



Remark. A Θ-*graph* is one with two fixed vertices and at least three internally-vertex-disjoint paths between them, and with at least three such paths of length at least three. For example, the third graph in Theorem 1.2 is the smallest Θ-graph. Let $\mathcal{C}$ be the class of Θ-graphs, and $\mathcal{C}^*$ be the class of graphs that contain a vertex whose deletion results in a Θ-graph. For example, $A, A^- \in \mathcal{C}^*$. Then, in Theorem 1.3, we can reduce the number of excluded minors by one, by increasing the number of exceptions, namely: for a 3-connected graph G, $\mathbf{pw}(G) \leqslant 3$ if and only if $G \notin \mathcal{C}^* \cup \{V_8, Q\}$ and G does contain any of the following graphs as a minor: K_5, Oct, Pyr, P^-. The statement follows from the fact that a 3-connected $\{K_5, P^-\}$-free graph containing A^- is in $\mathcal{C}^*$. The proof of this fact follows from Lemma 3.7 and Seymour's splitter theorem [8] and is straightforward and thus ommitted.

2. Unavoidable minors

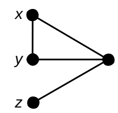
In this section we prove the following Lemma, which is key in proving the converse implication of Theorem 1.3.

Lemma 2.1. *If G is 3-connected and $\mathbf{pw}(G) \geqslant 4$, then G contains one of the following graphs as a minor: $V_8, Q, K_5, Oct, Pyr, P^-, A^-$.*

Before we prove it, we state the necessary definition and lemmas.

Definition 2.2. *Let $x, y, z \in V(G)$. A 3-separation of G over $\{x, y, z\}$ is a pair of induced subgraphs (L, R) of G such that: $E(L) \cup E(R) = E(G)$, $V(L) \cup V(R) = V(G)$, $V(L) \neq V(G)$, $V(R) \neq V(G)$, and $V(L) \cap V(R) = \{x, y, z\}$. Note that in such case $\{x, y, z\}$ is necessarily a 3-vertex-cut.*

Lemma 2.3. *Let H be a 3-connected graph with 3-separation (L, R) over $\{x, y, z\}$. If R does not contain the graph F as a minor (with vertices x, y, z preserved), then $R - z$ is a path from x to y.*

$F =$ 

Proof. If $R - z$ has a cycle C, then since H is 3-connected, it follows by Menger's Theorem that H has three vertex-disjoint paths: P_1, P_2, P_3 from $V(C)$ to $\{x, y, z\}$. Let the endpoints of P_1 be x and x_1, the endpoints of P_2 be y and y_1, and the endpoints of P_3 be z and z_1. Then, by contracting x_1 to x along P_1, y_1 to y along P_2, and contracting P_3 to a single edge, we obtain an F-minor in R, a contradiction.

Therefore $R - z$ is a forest. Since H is 3-connected, it follows that every vertex in $R - z$ (except possibly x and y) has degree $\geqslant 2$. Therefore, $R - z$ is a path from x to y. $\square$

The following basic lemma about 3-connected graphs can be found in [3].

Lemma 2.4. *If G is 3-connected and $|V(G)| \geqslant 5$, then G has an edge e such that G/e is again 3-connected.*

Such an edge is called *contractible*. Furthermore, Halin in [4] shows the following.

Theorem 2.5. [4] *If G is 3-connected with $|V(G)| \geqslant 5$ and $v \in V(G)$ has deg$(v) = 3$, then one of the three edges incident with v is contractible.*

Proof of Lemma 2.1. Suppose that G does not contain any of the following graphs as a minor: V_8, Q, K_5, Pyr, Oct, P^-, A^-. We will show that $\mathbf{pw}(G) \leqslant 3$.

Since $Q \preceq L_5$, it follows by Theorem 1.1 that $\mathbf{tw}(G) \leqslant 3$. Let $(T, \mathcal{V})$ be a tree-decomposition of G of width $\leqslant 3$. We may assume, without loss of generality, that:

(a) for all distinct $t, t' \in V(T)$, $V_t \nsubseteq V_{t'}$;

As a consequence of (a), we obtain:

(b) for all distinct $t, t' \in V(T)$, $V_t \neq V_{t'}$;

(c) for all edges $tt' \in E(T)$, $V_t \cap V_{t'}$ is a vertex-cut of G;

(d) for all $t \in V(T)$, $|V_t| = 4$.

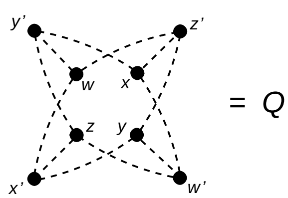
To see (d), note that since G is 3-connected, it follows by (c) that for all edges $tt' \in E(T)$, $|V_t \cap V_{t'}| \geqslant 3$. Therefore by (a) it follows that for all $t \in V(T)$, $|V_t| \geqslant 4$, but since the width of $(T, \mathcal{V})$ is at most three, we have $|V_t| \leqslant 4$, and so $|V_t| = 4$ for all $t \in V(T)$.

For every $t \in V(T)$, we call each of the four 3-element subsets of V_t a *triple* of V_t. A 3-element subset $W \subseteq V(G)$ is called a *bag intersection* if there exists an edge $st \in E(T)$

such that $W = V_s \cap V_t$. Hence we can think of bag intersections as *labels* on the edges of T. Note that it follows from (c) and (d) that every bag intersection is a triple (of V_s and V_t) and a 3-vertex-cut in G.

Observe that for each V_t, not all four of its triples are bag intersections. For otherwise, suppose that $V_t := \{w, x, y, z\}$ is such a bag. Then the labels on the edges incident with t in T are the following triples $\{w, x, y\}$, $\{w, x, z\}$, $\{w, y, z\}$, and $\{x, y, z\}$. Let T_1 be the subtree of T rooted at t consisting of the branches of T that are incident with t by edges with label $\{w, x, y\}$. Similarly, let T_2, T_3, and T_4 be the subtrees of T rooted at t consisting of the branches of T that are incident with t by edges with label $\{w, x, z\}$, $\{w, y, z\}$, and $\{x, y, z\}$, respectively. Note that T can be obtained by identifying the trees T_1, T_2, T_3, and T_4 at the vertex t. Let R_{wxy} be the subgraph of G induced by $\bigcup_{s \in T_1} V_s - \{z\}$. Similarly, let R_{wxz}, R_{wyz}, and R_{xyz} be the subgraphs of G induced by $\bigcup_{s \in T_2} V_s - \{y\}$, by $\bigcup_{s \in T_3} V_s - \{x\}$, and by $\bigcup_{s \in T_4} V_s - \{w\}$, respectively. Let $z' \in R_{wxy}$, $y' \in R_{wxz}$, $x' \in R_{wyz}$, and $w' \in R_{xyz}$.

Since G is 3-connected, it follows by Menger's Theorem that there are three internally-vertex-disjoint paths from z' to w, x, and y in R_{wxy}. Similarly, there are three internally-vertex-disjoint paths from y' to w, x, and z in R_{wxz}; three such paths from x' to w, y, and z in R_{wyz}; and three such paths from w' to x, y, and z in R_{xyz}. Note that the twelve paths are also pairwise internally vertex disjoint, because any two of the graphs: R_{wxy}, R_{wxz}, R_{wyz}, R_{xyz} only meet in V_t. Therefore, contracting these twelve paths to simple edges, we obtain a Q-minor of G, a contradiction.



For $t \in V(T)$, we call V_t *good* if at most two of its four triples are bag intersections, and we call V_t *bad* if exactly three of its four triples are bag intersections. We now show that:

$(*)$ G has a tree-decomposition $(T', \mathcal{V}')$ such that every bag in $\mathcal{V}'$ is good.

Suppose that $(T, \mathcal{V})$ has a bag $V_t := \{w, x, y, z\}$ that is bad, where all triples of V_t except $\{x, y, z\}$ are bag intersections. We will construct a new tree-decomposition $(T', \mathcal{V}')$ of G satisfying (a) such that the number of bad bags in $\mathcal{V}'$ is one less than the number of bad bags in $\mathcal{V}$.

Since V_t is bad and $\{x, y, z\}$ is not a bag intersection, it follows that the labels on the edges incident with t in T are the following triples $\{w, x, y\}$, $\{w, x, z\}$, and $\{w, y, z\}$. Let T_1 be the subtree of T rooted at t consisting of the branches of T that are incident with t by edges with label $\{w, x, y\}$. Similarly, let T_2 and T_3 be the subtrees of T rooted at t consisting of the branches of T that are incident with t by edges with label $\{w, x, z\}$ and $\{w, y, z\}$, respectively.

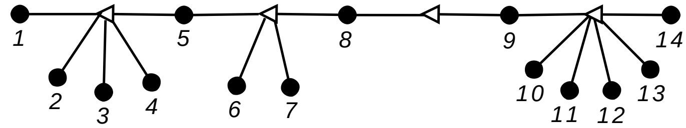
Note that T can be obtained by identifying the trees T_1, T_2, and T_3 at the vertex t. Let R_{wxy} be the subgraph of G induced by $\bigcup_{s\in T_1} V_s - \{z\}$. Similarly, let R_{wxz} and R_{wyz} be the subgraphs of G induced by $\bigcup_{s\in T_2} V_s - \{y\}$ and by $\bigcup_{s\in T_3} V_s - \{x\}$, respectively. Let L_{wxy} be the graph induced by $(V(G) - V(R_{wxy})) \cup \{w, x, y\}$. Similarly, let L_{wxz} and L_{wyz} be the graphs induced by $(V(G) - V(R_{wxz})) \cup \{w, x, z\}$ and by $(V(G) - V(R_{wyz})) \cup \{w, y, z\}$. Then, G has the following three 3-separations: (L_{wxy}, R_{wxy}), (L_{wxz}, R_{wxz}), and (L_{wyz}, R_{wyz}). If each one of R_{wxy}, R_{wxz}, and R_{wyz} contains an F-minor (as defined in Lemma 2.3, where in each case we choose w to be the vertex of degree one in F), then $G \succeq Pyr$, a contradiction. Therefore, by symmetry, R_{wxy} does not contain an F-minor (with vertices w, x, and y preserved), and thus by Lemma 2.3, $R_{wxy} - w$ is a path P from x to y. Let $a_0, a_1, \ldots, a_n := P$ with $a_0 = x$ and $a_n = y$. Note that $n > 1$, since (L_{wxy}, R_{wxy}) is a 3-separation over $\{w, x, y\}$. Since G is 3-connected, we have $wa_i \in E(G)$ for all i except possibly $i = 0$ and $i = n$.

Let $G_2 := R_{wxz}$ and $G_3 := R_{wyz}$. Let $\mathcal{V}_2 := \{V_s \in \mathcal{V} : s \in V(T_2)\}$, modifying the bag $V_t \in \mathcal{V}_2$ to be just $\{w, x, z\}$, and let $\mathcal{V}_3 := \{V_s \in \mathcal{V} : s \in V(T_3)\}$, modifying the bag $V_t \in \mathcal{V}_3$ to be just $\{w, y, z\}$. Then clearly $(T_2, \mathcal{V}_2)$ is a tree-decomposition of G_2, and $(T_3, \mathcal{V}_3)$ is a tree-decomposition of G_3.

For $i = 1, 2, \ldots, n$, let $V_{t_i} = \{a_{i-1}, a_i, w, z\}$. We construct the following tree T': relabel $t \in V(T_2)$ by t_1, relabel $t \in V(T_3)$ by t_n, and connect the two trees T_2 and T_3 by the path $t_1, t_2, \ldots, t_n$. Let $\mathcal{V}' = \{V_s\}_{s\in V(T')}$. Note that the bag $\{w, x, z\}$ from $\mathcal{V}_2$ got replaced by $V_{t_1} = \{a_1, w, x, z\} \in \mathcal{V}'$, and the bag $\{w, y, z\}$ from $\mathcal{V}_3$ got replaced by $V_{t_n} = \{a_{n-1}, w, y, z\} \in \mathcal{V}'$. Then clearly $(T', \mathcal{V}')$ is a tree-decomposition of G satisfying (a). Furthermore, for all i, the bags $V_{t_i} = \{a_{i-1}, a_i, w, z\}$ are good, because the triples $\{a_{i-1}, a_i, w\}$ and $\{a_{i-1}, a_i, z\}$ are not bag intersections (because $R_{wxy} - w$ is a path from x to y). Also, note that $\{w, x, y, z\} \notin \mathcal{V}'$ (because $n > 1$), and $\mathcal{V}' - \bigcup_{i=1}^{n} \{V_{t_i}\} \subseteq \mathcal{V}$, therefore the number of bad bags in $\mathcal{V}'$ is one less than the number of bad bags in $\mathcal{V}$. This proves $(*)$.

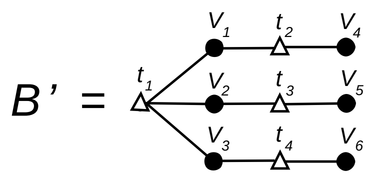
So we may assume that in the tree-decomposition $(T, \mathcal{V})$ of G every bag of $\mathcal{V}$ is good. This gives rise to the following tree structure of G. Let $\mathcal{T}$ denote the set of all bag intersections of $(T, \mathcal{V})$. We then have a natural bipartite graph B on $\mathcal{V} \cup \mathcal{T}$ where the edges of B join bag intersections in $\mathcal{T}$ to the bags of $\mathcal{V}$ to which they belong. Since T is a tree and the subgraph of T induced by the edges of a given label is a subtree of T, it follows that B is a tree. By definition of B, every vertex in $V(B) \cap \mathcal{V}$ has degree at most two, and all the leaves of B are elements of $\mathcal{V}$.

If all the vertices of $\mathcal{T}$ lie on a path in B, then all the vertices of $\mathcal{V}$ either also lie on the path or are leaves of B. Thus B has the structure as illustrated in the following example.



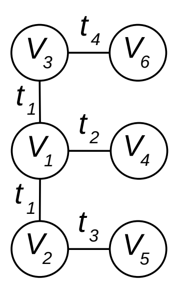
In this case, $\{V_i\}_{i=1,2,\ldots,n}$ is a path-decomposition of G of width $\leqslant 3$, where $n := |\mathcal{V}|$, and each V_i consists of the vertices of a single element of $\mathcal{V}$. The V_i's are indexed in the natural order as in the figure above. Hence $\mathbf{pw}(G) \leqslant 3$.

Finally, if the vertices of $\mathcal{T}$ do not all lie on a path of B, we will show that we achieve a contradiction. In this case B contains the following subgraph B'.



We will show that we can reduce B to B' by contractions in G in such a way that the resulting graph G' is still 3-connected. Let L be a leaf of B such that $L \in \mathcal{V} \cap (V(B) - V(B'))$. Let $t \in \mathcal{T}$ be the neighbor of L in B. Let $t = \{v_1, v_2, v_3\}$, and let $L = \{v_1, v_2, v_3, v_4\}$. Since L is a leaf of B, it follows that t is a 3-vertex-cut that separates v_4 from the rest of the graph. Since G is 3-connected, it follows that v_4 is adjacent to v_1, v_2, and v_3, hence $deg(v_4) = 3$. Hence, by Theorem 2.5, one of the edges v_4v_1, v_4v_2, v_4v_3 is contractible. Therefore, by contracting it we obtain a 3-connected minor of G whose corresponding tree is $B - \{L, t\}$. By repeating this process we can obtain a 3-connected minor G' of G and correspondingly reduce B to B'.

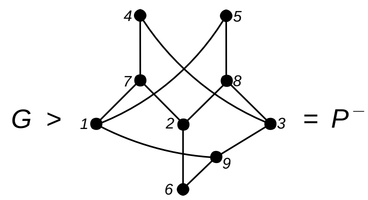
Therefore, we may assume that $G = G'$ and show that G contains either a P^-- or A^--minor, obtaining a contradiction. It follows from the above that G has a tree-decomposition $(T'', \mathcal{V}'')$ with $|V(T'')| = 6$ satisfying (a) such that every bag in $\mathcal{V}''$ is good. Let $\mathcal{V}'' := \{V_1, \ldots, V_6\}$ with triples $t_1, \ldots t_4$ as in the figures above and below. Hence, we have that $|V(G)| = 9$.



Let $V(G) := \{1, \ldots, 9\}$, and let $t_1 = \{1, 2, 3\}$, $V_1 = \{1, 2, 3, 4\}$, $V_2 = \{1, 2, 3, 5\}$, and $V_3 = \{1, 2, 3, 6\}$. Note that since $t_1 \notin \{t_2, t_3, t_4\}$, it follows that $4 \in t_2 \subseteq V_4$, $5 \in t_3 \subseteq V_5$, and $6 \in t_4 \subseteq V_6$, and each of t_2, t_3, t_4 must contain exactly one of the subsets $\{1, 2\}$, $\{2, 3\}$, $\{1, 3\}$. Let 7, 8, and 9 be the remaining vertices in V_4, V_5, and V_6, respectively. By symmetry, we have the following three cases:

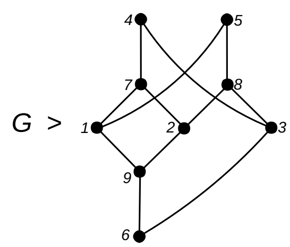
<u>Case 1:</u> $\{1, 2\} \subseteq t_2$, $\{2, 3\} \subseteq t_3$, $\{1, 3\} \subseteq t_4$.

Therefore, $t_2 = \{1,2,4\}$, $t_3 = \{2,3,5\}$, $t_4 = \{1,3,6\}$, and $V_4 = \{1,2,4,7\}$, $V_5 = \{2,3,5,8\}$, $V_6 = \{1,3,6,9\}$. Since G is 3-connected and vertex 7 only belongs to bag V_4, it follows that the degree of 7 in G is three, and $71, 72, 74 \in E(G)$. Similarly, $82, 83, 85, 91, 93, 96 \in E(G)$. Also, since t_1 separates vertices 4 and 7 from the rest of the graph, it follows from the 3-connectivity of G and Menger's Theorem that G has three internally-vertex disjoint paths from 4 to the vertices 1, 2, and 3. But, since $73 \notin E(G)$, it follows that $43 \in E(G)$. Similarly, $51, 62 \in E(G)$. Therefore G contains the following subgraph, which is isomorphic to P^-, a contradiction.



<u>Case 2:</u> $\{1,2\} \subseteq t_2 \cap t_4$, $\{2,3\} \subseteq t_3$.

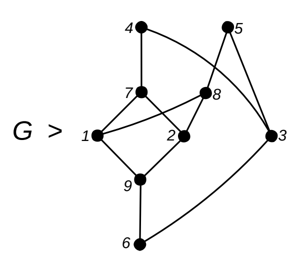
Hence in this case $t_2 = \{1,2,4\}$, $t_3 = \{2,3,5\}$, $t_4 = \{1,2,6\}$, and $V_4 = \{1,2,4,7\}$, $V_5 = \{2,3,5,8\}$, $V_6 = \{1,2,6,9\}$. Then, similarly to the argument in Case 1, G contains the following subgraph.



Also, since G is 3-connected, it follows (similarly to the argument in Case 1) that either $14 \in E(G)$ or $24 \in E(G)$. In the first case, G contains P^- as a subgraph (by deleting edge 17), a contradiction. And in the second case G contains A^- as a subgraph, again a contradiction.

<u>Case 3:</u> $\{1,2\} \subseteq t_2 \cap t_3 \cap t_4$,

Hence in this case $t_2 = \{1,2,4\}$, $t_3 = \{1,2,5\}$, $t_4 = \{1,2,6\}$, and $V_4 = \{1,2,4,7\}$, $V_5 = \{1,2,5,8\}$, $V_6 = \{1,2,6,9\}$. Then, similarly to the argument in Case 1, G contains the following subgraph.



Since G is 3-connected, it follows (similarly to the argument in Case 1) that either $14 \in E(G)$ or $24 \in E(G)$, and either $15 \in E(G)$ or $25 \in E(G)$. By symmetry, we only need to consider two cases. If $14, 15 \in E(G)$, then G contains A^- as a subgraph (by deleting edge 17), a contradiction. And if $14, 25 \in E(G)$, then G contains P^- as a subgraph (by deleting edges 17 and 28), a contradiction. This concludes the proof of Lemma 2.1. □

3. Proof of the main theorem and corollary

We first verify that all seven graphs: V_8, Q, K_5, Oct, Pyr, P^-, A^- have path-width at least four, which helps establish the forward implication of Theorem 1.3. For this we need the following structural lemma about 3-connected graphs of path-width at most three.

Lemma 3.1. *Let G be a connected graph with $n := |V(G)| > 4$ and $\mathbf{pw}(G) \leqslant 3$. Then for each $k = 1, 2, \ldots, n-4$, G has 3-vertex-cut separating k vertices from $(n-3)-k$ vertices.*

Proof. Let $\{V_i\}_{i=1,2,\ldots,m}$ be a path-decomposition of G of width $\leqslant 3$. We may assume, without loss of generality, that:
(a) for all i, $|V_i| = 4$ (by adding vertices to V_i if necessary);
(b) for all distinct i, j, $V_i \nsubseteq V_j$, hence $V_i \neq V_j$;
(c) for all i, $|V_i \cap V_{i+1}| = 3$ (by inserting new bags between V_i and V_{i+1} if necessary).

Therefore, it follows that for $k = 1, 2, \ldots, n-4$, $V_k \cap V_{k+1}$ is a 3-vertex-cut separating k vertices from $(n-3)-k$ vertices. □

The following lemma helps to establish the forward implication of Theorem 1.3.

Lemma 3.2. *If $G \in \{V_8, Q, K_5, Oct, Pyr, P^-, A^-\}$, then $\mathbf{pw}(G) \geqslant 4$.*

Proof. If $G \in \{V_8, K_5, Oct\}$, then it follows from Theorem 1.1 that $\mathbf{pw}(G) \geqslant \mathbf{tw}(G) \geqslant 4$. Now let $G \in \{Q, Pyr, P^-, A^-\}$ and suppose that $\mathbf{pw}(G) \leqslant 3$. Then, by Lemma 3.1, it follows that G has a 3-vertex-cut separating $\lfloor \frac{n-3}{2} \rfloor$ vertices from $\lceil \frac{n-3}{2} \rceil$ vertices, where $n = |V(G)|$. But in each case we have a contradiction since 3-cuts in P^- and in A^- can only separate one vertex from five, or two from four, and 3-cuts in Q and in Pyr can only separate a single vertex from the rest of the graph. □

To finish the proof of Theorem 1.3, we will need the following theorem of Wagner [9] and the following lemmas.

Theorem 3.3. [9] *If G is a 3-connected K_5-free graph containing a V_8-minor, then $G = V_8$.*

Actually, the above theorem can also be proved directly using Seymour's splitter theorem [8].

Lemma 3.4. *If G is a 3-connected $\{K_5, Pyr\}$-free graph containing a Q-minor, then $G = Q$.*

Proof. Suppose that $G \succeq Q$ and $G \neq Q$. Then, since both G and Q are 3-connected it follows from Seymour's splitter theorem [8], that $G \succeq Q + e$ or $G \succeq Q + f$, where $Q + e$, and $Q + f$ respectively, is the graph obtained from Q by adding an edge e between two vertices at distance two from each other, and an edge f between two vertices at distance three, respectively. But this is a contradiction since $Q + e \succeq Pyr$ and $Q + f \succeq K_5$. □

Proof of Theorem 1.3. Since path-width is monotone under taking minors, Lemma 3.2 establishes the forward implication of Theorem 1.3.

Conversely, suppose that $G \notin \{V_8, Q\}$ and G is $\{K_5, Oct, Pyr, P^-, A^-\}$-free. Then, from Lemmas 3.3 and 3.4, it follows that G is $\{V_8, Q, K_5, Oct, Pyr, P^-, A^-\}$-free. Therefore, by Lemma 2.1 it follows that $\mathbf{pw}(G) \leqslant 3$. This proves Theorem 1.3. □

Finally, Corollary 1.4 follows from the following three lemmas. By $H + v_1v_2$, we mean the graph obtained from H by adding edge v_1v_2 to H for non-adjacent vertices $v_1, v_2 \in V(H)$. For any edge $e := uv \in E(H)$ with $deg_H(v) \geqslant 3$, the operation of *uncontracting* vertex v *relative to* edge e is defined to be that of deleting v, adding two new adjacent vertices v_1 and v_2 each adjacent to u, and joining each old neighbor of v (in H), other than u, by an edge to exactly one of v_1 or v_2 in such a way that both v_1 and v_2 have degree at least three in the new graph.

Lemma 3.5. *Let H be a 2-connected minor of a 3-connected graph G. Let $u \in V(H)$ with $deg_H(u) = 2$, and let u_1 and u_2 be its two neighbors with $deg_H(u_i) \geqslant 3$ for $i = 1, 2$. Then $G \succeq H'$, where H' is obtained from H by one of the following operations:*

(1) $H' = H + uv$ for some $v \in V(H) - \{u, u_1, u_2\}$;

(2) uncontracting u_i relative to uu_i for some $i \in \{1, 2\}$.

Proof. Since H is a minor of G, it follows that G has a subgraph G' that is a union of pairwise vertex-disjoint trees $\mathcal{V} := \{T_v\}_{v \in V(H)}$, and pairwise internally-vertex-disjoint paths $\mathcal{E} := \{P_e\}_{e \in E(H)}$ that are also internally-vertex disjoint from the trees in $\mathcal{V}$, such that for each $vw \in E(H)$ the two endpoints of P_{vw} are a vertex in T_v and a vertex in T_w. To obtain H from G' we contract all of the trees in $\mathcal{V}$ to single vertices and all of the paths in $\mathcal{E}$ to single edges. We choose the trees to be as small as possible (by possibly making the paths longer). From this choice it follows that for every $v \in V(H)$, every leaf l of T_v is the endpoint of at least two paths P_{vw} and $P_{vw'}$ for some $w, w' \in V(H) - \{v\}$ (if $T_v = K_1$ then the only vertex of T_v is considered to be its leaf). Clearly this is true if $T_v = K_1$ by the 2-connectivity of H; also, if $T_v \neq K_1$ and l is a leaf of T_v and l is the endpoint of only one such path or none, then in the first case, by adding l to the path and discarding it from the tree T_v, we can make T_v smaller; and in the second case, by simply discarding l from T_v we can make T_v smaller, in both cases a contradiction. Also, if $d := deg_H(v) \leqslant 3$ then the vertices of T_v are the endpoints of exactly d paths P_{vw} for some $w \in V(H) - \{v\}$, hence by the above it follows that T_v has only one leaf, thus $T_v = K_1$.

Let x be the only vertex in $V(T_u)$, and let $x_1 \in V(T_{u_1})$ and $x_2 \in V(T_{u_2})$ be the other endpoints of the paths P_{u_1u} and P_{uu_2}, respectively. Let P be the concatenation of P_{u_1u} and P_{uu_2} at x. Since G is 3-connected, there is a path Q in G internally vertex disjoint from the trees in $\mathcal{V}$ and the paths in $\mathcal{E}$, with one endpoint q_1 in the interior of P and the other $q_2 \in V(G') - V(P)$. Also, in the case that $u_1u_2 \in E(H)$ and the endpoints of $P_{u_1u_2}$ are x_1 and x_2, then q_1 is in the interior of P or the interior of $P_{u_1u_2}$ and $q_2 \in V(G') - (V(P) \cup V(P_{u_1u_2}))$. Since in this case P and $P_{u_1u_2}$ are symmetric, we may assume, without loss of generality, that q_1 is in the interior of P.

If q_2 is a vertex of T_w for some $w \notin \{u, u_1, u_2\}$, then clearly G contains a minor H' obtained from H by (1).

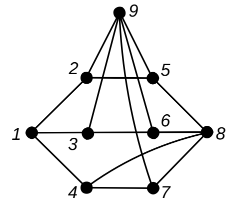
If $q_2 \in V(T_{u_i})$ for some $i \in \{1, 2\}$, say for $i = 2$, then $q_2 \neq x_2$. Let x_2' be the neighbor of x_2 on the unique x_2q_2-path in T_{u_2} (note that possibly $x_2' = q_2$). Deleting edge x_2x_2' from T_{u_2} divides it into two trees, call them T_{x_2} and $T_{x_2'}$, the first containing x_2 and the second x_2' and q_2. Then, we replace T_{u_2} by T_{x_2} and $T_{x_2'}$ and add the path $P_{x_2x_2'}$ consisting of the single edge x_2x_2'. Also, we replace T_u by the single vertex q_1, the path P_{u_1u} by the subpath of P from x_1 to q_1, the path P_{ux_2} by the subpath of P from q_1 to x_2, and the path $P_{ux_2'}$ by Q. Then, since every leaf l of T_v was the endpoint of at least two paths P_{vw} and $P_{vw'}$ for some $w, w' \in V(H) - \{u\}$, the new minor $H' \preceq G$ obtained by contracting the new trees is clearly obtained from H by (2).

Therefore, q_2 lies in the interior of the path $P_{ww'}$ for some $w, w' \in V(H)$. If at least one of w or w' is different from u_1 and u_2, then clearly G contains a minor H' obtained from H by (1). Hence $\{w, w'\} = \{u_1, u_2\}$. Let $y_1 \in T_{u_1}$ and $y_2 \in T_{u_2}$ be the endpoints of $P_{ww'} = P_{u_1u_2}$. Then we must have that either $y_1 \neq x_1$ or $y_2 \neq x_2$ and, as in the previous paragraph, we obtain a minor H' obtained from H by (2). ☐

Remark 3.6. *If in Lemma 3.5, $deg_H(u_i) = 3$ for $i = \{1, 2\}$, then G contains a minor H' obtained from H by (1), since the unique graph obtained from uncontracting u_i relative to uu_i, contains as a minor a graph obtained from H by (1).*

Lemma 3.7. *If G is a 3-connected K_5-free graph containing an A^--minor, then $G \succeq R_1$, or $G \succeq R_2$, or $G \succeq A$.*

Proof. Label the vertices of A^- as in the figure below.

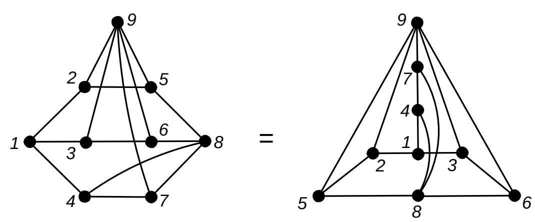
$A^- =$ 

Since G is 3-connected and A^- is 2-connected, it follows by Lemma 3.5 and Remark 3.6 that $G \succeq G'$, where G' is obtained from A^- by adding one of the edges in $\{42, 43, 45, 46, 48, 49\} \cup \{57, 56, 54, 53, 51, 59\}$. However:

- ▶ $A^- + 45 \succeq K_5$, by contracting edges 13, 25, 47, and 68;
- ▶ $A^- + 42 \succeq K_5$, by contracting edges 13, 25, 47, and 68; and symmetrically $A^- + 57 \succeq K_5$;
- ▶ $A^- + 43 \succeq K_5$, by contracting edges 12, 47, 58, and 68; and symmetrically $A^- + 56 \succeq K_5$;
- ▶ $A^- + 46 \succeq K_5$, by contracting edges 13, 47, 25, and 58; and symmetrically $A^- + 53 \succeq K_5$;

Therefore, G' is obtained from A^- by adding one of the edges in $\{48, 49\}$ (or symmetrically, one of the edges in $\{51, 59\}$) . First, suppose that $G' = A^- + 49$. Then, Lemma 3.5 and Remark 3.6 applied to the graphs G and G' yield another minor $G'' \preceq G$ obtained from G' by adding one of the edges $\{51, 59\}$. Therefore G'' is one of the following two graphs:

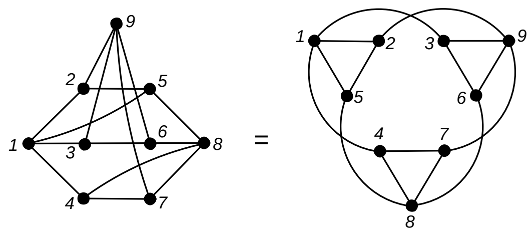
- ▶ $G'' = A^- + 49 + 51 = R_1$ (or symmetrically $G'' = A^- + 48 + 59 = R_1$);



$= R_1$

- ▶ or $G'' = A^- + 49 + 59 = A$;

Finally, if $G' = A^- + 48$, then if vertex 8 is uncontracted relative to 58 according to Lemma 3.5, then G contains one of the following graphs as a minor: $A^- + 56$, $A^- + 57$, or $A^- + 45$, each of which contains a K_5-minor (as above), a contradiction. Therefore, it follows from Lemma 3.5 that $G \succeq G''$, where $G'' = A^- + 48 + 59 = R_1$ (as above), or $G'' = A^- + 48 + 51 = R_2$ as illustrated below.

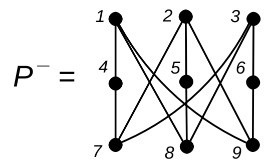


$= R_2$

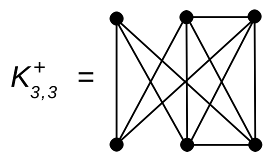
□

Lemma 3.8. *If G is a 3-connected K_5-free graph containing a P^--minor, then $G \succeq A^-$.*

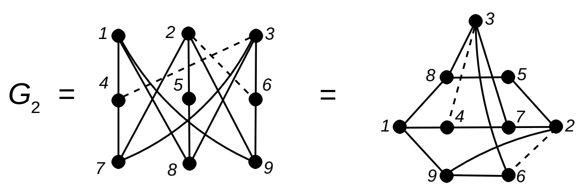
Proof. Suppose that $G \not\succeq A^-$. Label the vertices of P^- as in the figure below.



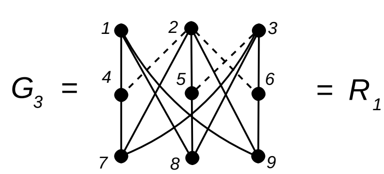
First, note that $G \not\succeq K_{3,3}^+$, where $K_{3,3}^+$ is the graph obtained from $K_{3,3}$ by adding one edge to each of the two bipartitions (see figure below). This is because $K_{3,3}^+ \succeq K_5$ by contracting the edge connecting the two vertices of degree three.



Since G is 3-connected and P^- is 2-connected, it follows by Lemma 3.5 and Remark 3.6 that $G \succeq G_1$, where G_1 is obtained from P^- by adding one of the edges $\{61, 62, 64, 65, 67, 68\}$. First, suppose that the added edge is not incident with 4 nor 5, so by symmetry let $G_1 = P^- + 62$. Then, the same Lemma and Remark applied to the graphs G and G_1 yield another minor $G_2 \preceq G$ obtained from G_1 by adding one of the edges $\{42, 43, 45, 46, 48, 49\}$. However, since $G \not\succeq K_{3,3}^+$, it follows that the only choices are $\{42, 43, 46\}$. If $G_2 = P^- + 62 + 43$, then we get a contradiction because $G_2 \succeq A^-$.



If $G_2 = P^- + 62 + 46$, then $G_2 \succeq K_5$, by contracting edges 19, 25, 38, 47, a contradiction. And if $G_2 = P^- + 62 + 42$, then if vertex 2 is uncontracted relative to 52 according to Lemma 3.5, then G contains either $K_{3,3}^+$ or $P^- + 62 + 46 \succeq K_5$ (as above), a contradiction. Therefore, it follows from Lemma 3.5 and the fact that $G \not\succeq K_{3,3}^+$, that $G \succeq G_3$, where G_3 is obtained from G_2 by adding one of the edges $\{51, 53\}$. By symmetry, we may assume the added edge is 53. But then, $G_3 = R_1 \succeq A^-$, a contradiction.



Therefore we have shown that $G \succeq G_1$, where G_1 is obtained from P^- by adding one of the edges $\{64, 65\}$, by symmetry, say 65. Then, applying the Lemma and Remark to the graphs G and G_1, and using the fact that $G \not\succeq K_{3,3}^+$, we obtain another minor $G_4 \preceq G$ obtained from G_1 by adding one of the edges $\{45, 46\}$, by symmetry, say 45. But $G_4 \succeq K_5$, by contracting edges 14, 27, 38, and 69, a contradiction. □

Proof of Corollary 1.4. The forward direction of the Corollary follows from Theorem 1.3 since both R_1 and R_2 contain P^- as a subgraph, and A contains A^- as a subgraph.

For the converse direction, since G is $\{K_5, Oct, Pyr, R_1, R_2, A\}$-free, it follows from Lemma 3.7 that $G \not\succeq A^-$. Therefore, it follows from Lemma 3.8 that $G \not\succeq P^-$. Hence, G is $\{K_5, Oct, Pyr, P^-, A^-\}$-free, and so by Theorem 1.3, $\mathbf{pw}(G) \leqslant 3$. □

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